# Week 3: Moving Lemma
(+ some loose ends)

Recall: $f_* : A(X) \longrightarrow A(Y)$
why is it well-defined?

Idea: If $X \xrightarrow{f} Y$ generically finite
$\Rightarrow k(X) \supseteq k(Y)$ finite extension

Can define Norm $N_f : k(X) \longrightarrow k(Y)$
$$\varphi \mapsto \det m_\varphi$$

where $m_\varphi$ is the $k(Y)$-linear map
$$m_\varphi : N(X) \longrightarrow N(X)$$
$$\phi \mapsto \phi \cdot \varphi$$

Note: $N_f(\varphi) = \prod \det \varphi^\sigma$
$\varphi^\sigma$ = conjugates
on a Galois extension

Lemma (non-trivial!)
$X \xrightarrow{f} Y$ dominant, generically finite
$\varphi \in k(X) \Rightarrow f_*(\operatorname{div} \varphi) = \operatorname{div}(N_f(\varphi))$

Rmk the equivalence relation $\sim$ that defines $A(X)$ is generated by
$\operatorname{div} \varphi = W|_\infty - W|_0 \subseteq X \times \mathbb{P}^1$

Now let
$$\begin{tikzcd} W \arrow[d, "\pi"] & \subseteq & X \times \mathbb{P}^1 \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\ W' := \pi(W) & \subseteq & X & \mathbb{P}^1 \end{tikzcd}$$

Then $W|_\infty - W|_0 = \operatorname{div} \varphi$

& $\pi_*(\operatorname{div}\varphi) = \operatorname{div}(N_\pi(\varphi))$.

<u>Consequence</u>: $\sim$ is generated by divisors of rational functions on subvarieties $W' \subseteq X$

§ <u>Moving Lemma</u>

$X$ smooth, quasi-proj

Let $B = \sum m_i B_i$, $B_i \subseteq X$ subvarieties

$\alpha \in A(X)$

Then

a) $\exists$ $A = \sum n_j A_j$ s.t.

$[A] = \alpha$

& all $A_j, B_i$ intersect generically transversely

b) $\sum m_i n_j [A_j \cap B_i] \in A(X)$

is <u>independent</u> of choice of such $A$.

<u>Side note</u>:

i) $A, B$ transverse at $p \in A \cap B$ means
$$T_p A + T_p B = T_p X$$

ii) $A, B$ "generically transverse" if this is true on an open $U \subset A \cap B$

Main idea: (the proof is highly technical
cf. §5.2 of "3264")

"Cone construction":

Embed $X \hookrightarrow \mathbb{P}^N$ $\quad (\dim X = n)$

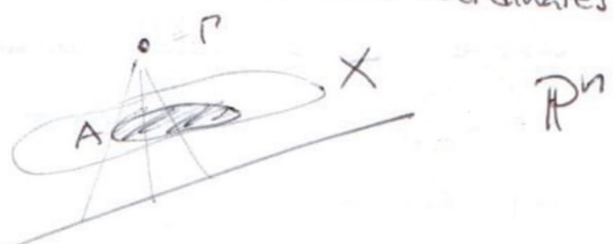
& take $\Gamma = \mathbb{P}^{N-n-1} \subseteq \mathbb{P}^N$

with $X \cap \Gamma = \phi$ (open condition)

$\rightarrow$ Project $\pi_\Gamma : X \longrightarrow \mathbb{P}^n$ away from $\Gamma$
finite map!

e.g. if $\Gamma = \{ (\underbrace{0 : \cdots : 0}_{n+1} : * : \cdots *) \}$

then $\pi_\Gamma$ = take first $n+1$ coordinates




$$D_\Gamma := \pi_\Gamma^{-1}(\pi_\Gamma(A))$$

$$= X \cap \overline{\Gamma, A} \supset A$$

where $\overline{\Gamma, A}$ = union of lines passing through both $\Gamma$ and $A$

Then can write
$$D_\Gamma = A + A'_\Gamma \quad \text{where} \quad A \not\subset A'_\Gamma$$

Recall $\Gamma \in G^* \subseteq Gr(N-n-1, \mathbb{P}^N)$

$$\begin{gathered} G^* \\ \parallel \\ \{\Gamma \mid \Gamma \cap X = \phi\} \end{gathered}$$

We want to choose $\Gamma \in G^*$ so that $D_\Gamma$, $A'_\Gamma$ are both generically transverse to $B$

- For $D_\Gamma$: it will be a cycle that comes "from the ambient"

  Recall: $D_\Gamma = X \cap \overline{\Gamma, A}$

  For generic $\Gamma$, $\overline{\Gamma, A}$ intersects $X$ generically

  $\Rightarrow D_\Gamma \sim d[L]$ for some $d$

  where $L$ = some linear section of $X \subseteq \mathbb{P}^N$

  We can choose $L$ (generic) so that it is generically transverse to all of the $B_i$'s (use Bertini)

- For $A'_\Gamma$:

  the key will be that "$A'_\Gamma$ meets $B$ more transversely than does $A$"

More precisely

Lemma

For general $\Gamma \in G^*$

(*) The irreducible components of $A'_\Gamma \cap B$ that are not contained in $A$ intersect $B$ generically transversely

(**) The irred. components of $A'_\Gamma \cap B$ that are contained in $A$ have dimension strictly $<$ than any of the components of $A \cap B$

From the Lemma, we do induction:

$$A = D_\Gamma - A'_\Gamma$$

where $D_\Gamma \cap B$ gen. transv. ✓

$A'_\Gamma \cap B$ ... gen. transv except maybe for bad components of smaller dimension.

In other words, (**) makes sure you reach the end after a finite number of steps, while (*) makes sure you don't screw up the progress made at the previous step ... :)

This will prove a) of the Moving Lemma

Just a few words on the proof of (*)

We can at least prove $A'_p$, $B$ are "dimensionally transverse" (outside of $A \cap B$)

i.e. let $B^* := B \setminus (A \cap B)$, then

We need $\operatorname{codim}(A'_p) + \operatorname{codim}(B^*) = \operatorname{codim}(A \cap B^*)$

or equivalently:

$$\dim(A'_p \cap B^*) = \dim A + \dim B - n$$

$$(\text{since } \dim A'_p = \dim A)$$

Let $\Psi := \{ (\Gamma, p, q) \in Gr(N-n-1, \mathbb{P}^N) \times A \times B^* \mid \Gamma \cap \overline{pq} \neq \phi \}$

the line thru $p$ & $q$

Then $\Psi \xrightarrow{\pi_2} A \times B^*$ projects

with $\underbrace{\dim \pi_2^{-1}(p,q)}_{\text{general fiber}} = \dim Gr(N-n-1, \mathbb{P}^N) - n$

$\Rightarrow \dim \Psi = \dim A + \dim B + \dim Gr - n$

While $\Psi \xrightarrow{\pi_1} Gr(N-n-1, \mathbb{P}^N)$ projects

with $\dim \pi_1^{-1}(\Gamma) = \dim A + \dim B - n$

& $\underbrace{\pi_1^{-1}(\Gamma)}_{\text{general fiber}} \twoheadrightarrow D_p \cap B^* = A'_p \cap B^*$

Therefore $\dim(A'_p \cap B^*) \leq \dim A + \dim B - n$

But also $\geq$ by subaditivity of codimension $\square$

In order to prove part b), one also uses the Cone Construction +

<u>Lemma</u>:

If $A \sim A'$ pure dimensional cycles, with $B$ generically transverse to both
$\Rightarrow A \cap B \sim A' \cap B$

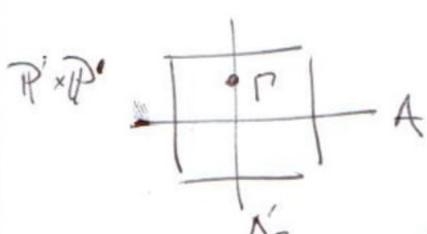
<u>Finally</u>.

Short example of the Cone Construction

Let $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ smooth quadric

$A = B = \mathbb{P}^1$ one of the rulings

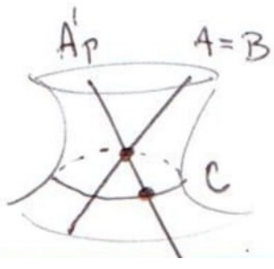
Then $\Gamma = pt \notin A$ & $D_p = \mathbb{P}^1 \cup \mathbb{P}^1$




Now move $D_p$ to a general linear section of $X = C$ a smooth conic

$$\Rightarrow A \sim D_p - A'_p \sim C - A'_p$$

$$\& \quad \left.\begin{array}{l} C \cap B = pt \\ A'_p \cap B = pt \end{array}\right\} A \cdot B = \underset{pt - pt}{1 - 1} = 0$$




$$\therefore A^2 = 0$$