THC (Semana 11): Filtración de Hodge algebraica y base de Griffiths

X var. alg suave$/\mathbb{C}$. Entonces, $\mathbb{H}^k(X,\Omega_X^\bullet) \simeq H^k_{dR}(X)$

· Si X var proy suave, $Y \subseteq X$ sección hiperplana, $\mathcal{U} = X \setminus Y$, $\mathcal{U} \subseteq X$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\mathcal{U}) \arrow[r] & H_n(X) \arrow[r] \arrow[dr, "\mathrm{int}"'] & H_n(X,\mathcal{U}) \arrow[r] \arrow[d, phantom, "\wr"] & H_{n-1}(\mathcal{U}) \arrow[r] & \cdots \\
 & & & H_{n-2}(Y) \arrow[ur] & &
\end{tikzcd}$$

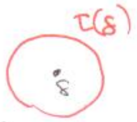
Sucesión de Leray-Thom-Gysin:

$$\cdots \to H^{n-1}_{dR}(\mathcal{U}) \xrightarrow{Res} H^{n-2}_{dR}(Y) \to H^n_{dR}(X) \to H^n_{dR}(\mathcal{U}) \to \cdots \quad (\star)$$

$$\eta \longmapsto \eta|_{\mathcal{U}}$$

$$\langle \omega, \tau(\delta) \rangle = \langle \tau^*(\omega), \delta \rangle$$

$$\int_{\tau(\delta)} \omega = \int_\delta Res(\omega)$$



¿Secuencia corta de complejos que induce $(\star)$ en hipercohom.?

Aquí: $H^n_{dR}(X) = \mathbb{H}^n(X,\Omega^\bullet_X)$ y $H^n_{dR}(\mathcal{U}) = \mathbb{H}^n(X, i_*\Omega^\bullet_{\mathcal{U}})$

$\mathcal{U} \overset{i}{\hookrightarrow} X$

Luego: $0 \to \Omega^\bullet_X \to i_*\Omega^\bullet_{\mathcal{U}} \to \dfrac{i_*\Omega^\bullet_{\mathcal{U}}}{\Omega^\bullet_X} \to 0$

Preg: $i_*\Omega^\bullet_{\mathcal{U}}/\Omega^\bullet_X$ da $H^{*-1}_{dR}(Y)$? Ni idea ... ⇝ No es el buen complejo! Induce una filtración trivial.

Deligne: Hay que considerar la sucesión de residuos de Poincaré:

$0 \to \Omega^\bullet_X \to \Omega^\bullet_X(\log Y) \to \Omega^{\bullet-1}_Y \to 0$ con $\Omega^p_X(\log Y) = \ker\left(\Omega^p_X(Y) \xrightarrow{d} \dfrac{\Omega^p_X(2Y)}{\Omega^p_X(Y)}\right)$

Localmente: $\omega \in (\Omega^p_X(\log Y))(\mathcal{U})$ y $\omega = \dfrac{dF}{F} \wedge \alpha + \beta$, $\alpha \in \Omega^{p-1}_X(\mathcal{U})$, $\beta \in \Omega^p_X(\mathcal{U})$

con $d\omega = \dfrac{dF}{F} \wedge d\alpha + d\beta$ (ie, orden del polo no aumenta!) $\quad Y \underset{\text{loc a } \mathcal{U}}{=} \{F=0\}$

(cf. $d\left(\dfrac{\alpha}{F}\right) = \dfrac{d\alpha}{F} - \alpha \wedge \dfrac{dF}{F^2}$).

Se define: $Res(\omega) := \alpha|_Y$ ("a_{-1}")

⚠ Si $Res(\omega) = 0$, $\alpha = F \cdot \tilde{\alpha}_1 + dF \cdot \tilde{\alpha}_2$

ie, $\omega = dF \wedge \tilde{\alpha}_1 + \tilde{\beta}$ ← es cualquier forma holom. en X.

Tomando hipercohomología obtenemos:

$$\begin{array}{ccccccccc}
\mathbb{H}^{n-1}(\Omega_Y^{\bullet-1}) & \to & \mathbb{H}^n(\Omega_X^\bullet) & \to & \mathbb{H}^n(\Omega_X^\bullet(\log Y)) & \to & \mathbb{H}^n(\Omega_Y^{\bullet-1}) & \to & \mathbb{H}^{n+1}(\Omega_X^\bullet) \to \dots \\
\wr\vert & & \wr\vert & & \downarrow \cong \text{ (Lema de los 5)} & & \Vert & & \wr\vert \\
H^{n-2}_{dR}(Y) & \to & H^n_{dR}(X) & \to & H^n_{dR}(\mathcal{U}) & \longrightarrow & H^{n-1}_{dR}(Y) & \longrightarrow & H^{n+1}_{dR}(X) \qquad (\star)
\end{array}$$

$$H^n_{dR}(\mathcal{U}) = \mathbb{H}^n(\Omega_{\mathcal{U}}^\bullet) \underset{\text{dy}}{=} \mathbb{H}^n(\Omega_X^\bullet(\ast Y))$$

Por otro lado, obtengo $0 \to \Omega_X^{\bullet\geq p} \to \Omega_X^{\bullet\geq p}(\log Y) \to \Omega_Y^{\bullet-1\geq p-1} \to 0$

induce $\mathbb{H}^{n-1}(\Omega_Y^{\bullet\geq p-1}) \to \mathbb{H}^n(\Omega_X^{\bullet\geq p}) \to \mathbb{H}^n(\Omega_X^{\bullet\geq p}(\log Y)) \to \mathbb{H}^n(\Omega_Y^{\bullet-1\geq p-1}) \to \dots$

Obs: Si $E_r^{p,q}$ es la sec. espectral de $\mathbb{H}^n(\mathcal{F}^\bullet)$ resp. a la filtración inocente

Entonces, degenera en $E_1 \underset{(\star)}{\overset{\text{"Ejer"}}{\Longleftrightarrow}} \mathbb{H}^n(\mathcal{F}^{\bullet\geq p}) \overset{\varphi}{\hookrightarrow} \mathbb{H}^n(\mathcal{F}^\bullet)$ es inyectivo

$$F^p\,\mathbb{H}^n(\mathcal{F}^\bullet) = \mathrm{Im}(\varphi)$$

Nos da (por dos comp. de Hodge):

$$F^{p-1}H^{n-2}_{dR}(Y) \to F^p H^n_{dR}(X) \to \boxed{F^p H^n_{dR}(\mathcal{U})} \to F^{p-1}H^{n-1}_{dR}(Y) \to \dots$$

Def: X var. alg suave y sea $X \subseteq \overline{X}$, $X = \overline{X} \setminus Y \leftarrow$ div. a cruces normales

$F^p\,\mathbb{H}^k(\overline{X}, \Omega_{\overline{X}}^\bullet(\log Y))$ es la filtración de Hodge algebraica.

Además (Deligne):
- $\mathbb{H}^k(\overline{X}, \Omega_{\overline{X}}^\bullet(\log Y)) \simeq H^k_{dR}(X)$
- La sec. espectral degenera en E_1, i.e.,
  $F^p\,\mathbb{H}^k(\overline{X}, \Omega_{\overline{X}}^\bullet(\log Y)) = \mathbb{H}^k(\overline{X}, \Omega_{\overline{X}}^{\bullet\geq p}(\log Y))$

Teo (Deligne): X proy suave, $Y \subseteq X$ sec. hiperplana suave, $\mathcal{U} = X \setminus Y$

~~Además~~, Entonces, $\Omega_X^\bullet(\log Y) \hookrightarrow \Omega_X^\bullet(\ast Y) = i_\ast \Omega_{\mathcal{U}}^\bullet$ induce isom.

$\mathbb{H}^k(\Omega_X^\bullet(\log Y)) \cong \mathbb{H}^k(\Omega_X^\bullet(\ast Y))$ $\ni \omega = \sum_{p+q=k} \omega^{p,q}$

$$\begin{array}{ccccc}
C^1(\mathcal{U}, \Omega_X^0(\ast Y)) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega_X^1(\ast Y)) & \xrightarrow{d} & \\
\uparrow & & \uparrow & & \\
C^0(\mathcal{U}, \Omega_X^0(\ast Y)) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega_X^1(\ast Y)) & \xrightarrow{d} & \\
\uparrow & & \uparrow & & \\
0 \to \Gamma(\Omega_X^0(\ast Y)) & \to & \Gamma(\Omega_X^1(\ast Y)) & \to & \Gamma(\Omega_X^2(\ast Y)) \to \dots
\end{array}$$

Lema (Carlson - Griffiths): Existe una operación $H : \bigoplus_{p+q=k} \dfrac{C^q(\mathcal{U}, \Omega_X^p(\ell Y))}{C^q(\mathcal{U}, \Omega_X^p((\ell-1)Y))}$

$\to \bigoplus_{r+s=k-1} \dfrac{C^s(\mathcal{U}, \Omega_X^r((\ell-1)Y))}{C^s(\mathcal{U}, \Omega_X^r((\ell-2)Y))}$ explícita para $Y = \{F = 0\} \cap X$

es tal que $DH + HD = \mathrm{Id}$.

Si $\omega \in \mathbb{H}^k(\mathcal{U}, \Omega_X^\cdot(*Y))$, entonces $\omega \in \mathbb{H}^k(\mathcal{U}, \Omega_X^\cdot(\ell Y))$ cierto ℓ.

$\Rightarrow \omega - DH\omega = H\underbrace{D\omega}_{=0} \equiv 0$ en el cociente : ω se reemplaza por $DH\omega$ ($= \omega$ en cohom) en $\mathbb{H}^k(\mathcal{U}, \Omega_X^\cdot((\ell-1)Y))$

$\vdots$

$(I - DH)^\ell \omega =: \eta \in \mathbb{H}^k(\mathcal{U}, \Omega_X^\cdot(Y))$

⚠ $D\eta = 0$ implica η polos logarítmicos : $\eta = \eta^{0,k} + \eta^{1,k-1} + \dots + \eta^{k,0}$

$\Rightarrow d(\eta^{k,0}) = 0$ y $\eta^{k,0}$ luego tiene polo log !

"polo simple"

$-d(\eta^{k-1,1}) + \delta(\eta^{k,0}) = 0 \iff d(\eta^{k-1,1}) = \underbrace{\delta(\eta^{k,0})}_{\text{polo simple !}} \Rightarrow \eta^{k-1,1}$ polos log

etc

$\Rightarrow \eta \in \mathbb{H}^k(\mathcal{U}, \Omega_X^\cdot(\log Y))$.

(Conclusión: $X \setminus Y = \mathcal{U}$ $\quad H^k_{dR}(\mathcal{U}) \cong \mathbb{H}^k(\Omega_X^\cdot(\log Y))$).

Teo (base de Griffiths): Sea $X = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ hip. suave de grado d.

Entonces, para $p+q = n$ se tiene :

$$H^{p,q}(X)_{\text{prim}} \cong R^F_{d(q+1)-n-2}$$

donde $R^F = \dfrac{\mathbb{C}[x_0, \dots, x_{n+1}]}{J^F}$ con $J^F = \left\langle \dfrac{\partial F}{\partial x_0}, \dots, \dfrac{\partial F}{\partial x_{n+1}} \right\rangle$, y donde

el isom. es dado por : si $\mathcal{U} := \mathbb{P}^{n+1} \setminus X$

$$P \in R^F_{d(q+1)-n-2} \longmapsto \frac{P\Omega}{F^{q+1}} \in H^{n+1}_{dR}(\mathcal{U})$$

$$\Omega = \sum_{i=0}^{n+1} (-1)^i x_i \, dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_{n+1}$$

$$\left(\begin{array}{c} H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}(n+2)) \\ \text{\rotatebox{90}{$\cong$}} \\ H^0(\mathcal{O}_{\mathbb{P}^{n+1}}) \cong \mathbb{C} \end{array} \right)$$

Además :

Obs: $\dfrac{P\Omega}{F^{q+1}} \in H^{n+1}_{dR}(\mathcal{U}) \xrightarrow{\sim} \operatorname{Res}\left(\dfrac{P\Omega}{F^{q+1}}\right) \in \dfrac{F^p H^n_{dR}(X)_{\text{prim}}}{F^{p+1} H^n_{dR}(X)_{\text{prim}}} \simeq H^{p,q}(X)_{\text{prim}}$

Dem: $\mathcal{U}$ cobertura jacobiana $\left\{\frac{\partial F}{\partial x_i} \neq 0\right\}$.

$C^{n+1}(\mathcal{U}, \mathcal{O}_{\mathbb{P}^{n+1}}(*X))$

$\ddots$

$C^2(\mathcal{U}, \Omega^{n-1}_{\mathbb{P}^{n+1}}(*X))$

$C^1(\mathcal{U}, \Omega^n_{\mathbb{P}^{n+1}}(*X))$

$C^0(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(*X))$

Sabemos que $\frac{P\Omega}{F^{q+1}} \in F^{p+1} H^{n+1}_{dR}(\mathcal{U}) \longmapsto \mathrm{Res}\left(\frac{P\Omega}{F^{q+1}}\right) \in \frac{F^p H^n_{dR}(X)_{pri}}{F^{p+1} H^n_{dR}(X)_{pri}}$

Basta probar que esto es sobrey:

$$\frac{P\Omega}{F^{q+1}} \in F^{p+1} H^{n+1}_{dR}(\mathcal{U}) = \mathbb{H}^{n+1}(\Omega_X^{\bullet \geq p+1}(\log Y))$$

Además: $\frac{P\Omega}{F^{q+1}} \in C^0(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(*X)) \to 0$

$(\partial - DH)^q \omega$ tiene polos log. y

$$\eta^{p+1,q} + \eta^{p+2,q} + \dots + \eta^{n+1,0} \in \mathbb{H}^{n+1}(\Omega_X^{\bullet \geq p+q+1}(\log(Y))) \overset{def}{=} F^{p+1} H^{n+1}_{dR}(\mathcal{U})$$

Si supieramos que $\underbrace{H^k(\Omega^p_{\mathbb{P}^{n+1}}(*X))}_{\text{Anulación de Kodaira}} = 0 \ \forall k > 0$ podemos hacer el proceso inverso.

De hecho obtenemos un isom! Luego basta analizar el kernel de

$$P \in \mathbb{C}[x_0, \dots, x_{n+1}] \longmapsto \frac{P\Omega}{F^{q+1}} \in \frac{F^{p+1} H^{n+1}_{dR}(\mathcal{U})}{F^{p+2} H^{n+1}_{dR}(\mathcal{U})}$$

Si $\frac{P\Omega}{F^{q+1}} \in F^{p+2} H^{n+1}_{dR}(\mathcal{U}) \iff \frac{P\Omega}{F^{q+1}} \overset{\exists}{=} \frac{Q\Omega}{F^q}$

$$\iff \frac{(P - FQ)\Omega}{F^q} = d\gamma \quad (= 0 \in H^{n+1}_{dR}(\mathcal{U}))$$

con $\gamma \in H^0(\Omega^n_{\mathbb{P}^{n+1}}(qX)) \cong \Omega^n_{\mathbb{P}^{n+1}}(dq)$ $\underset{\text{Explícito}}{\Longrightarrow}$ $\gamma = \frac{\sum_{i=0}^{n+1} T_i \, L_{\frac{\partial}{\partial x_i}}(\Omega)}{F^q}$

$$\Rightarrow d\gamma = \sum_{i=0}^{n+1} T_i \frac{L_{\frac{\partial}{\partial x_i}}(\Omega) \wedge (-q)\, dF}{F^{q+1}} + \frac{\omega}{F^q} \underset{\text{hip}}{=} \frac{(P - FQ)\Omega}{F^q}$$

mult de Ω.

$\Rightarrow$ $L_{\frac{\partial}{\partial x_i}}(\Omega) \wedge dF = \frac{\partial F}{\partial x_i} \Omega$ (Euler) Igualdad de polinomios (eliminando Ω!)

$$\underbrace{P - FQ}_{\in J_F} = \underbrace{\sum_{i=0}^{n+1} -q \cdot T_i \frac{\partial F_i}{\partial x_i}}_{\in J_F} + \underbrace{\alpha F}_{\in J_F} \iff P \in J_F \quad \blacksquare$$

Ejemplo: $X = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \rightsquigarrow H^{1,1}(X)_{pri} = ?$

$R^F_{3\cdot(2)-4} = R^F_2$ con $R^F = \frac{\mathbb{C}[x_0, \dots, x_3]}{\langle x_0^2, \dots, x_3^2 \rangle} \rightsquigarrow R^F_2 = \langle x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3 \rangle \cong \mathbb{C}^6$

$\Rightarrow H^{1,1}(X) \cong \mathbb{C}^7$ $(\Leftarrow \rho(X) = 7) \Leftarrow$ cf. $X \cong Bl_{p_1, \dots, p_6}(\mathbb{P}^2)$!