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Kodaira dimension of moduli spaces of polarized abelian varieties

Tesis para optar al Grado de Magíster en Matemáticas

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Introduction

The study of moduli spaces of abelian varieties is a profound and intricate topic within algebraic geometry, providing significant insights into the structure and classification of these mathematical objects. Abelian varieties, which are higher-dimensional generalizations of elliptic curves, play a crucial role in various branches of mathematics, including number theory, complex analysis, and certainly algebraic geometry. This work delves into the essential groundwork needed to elucidate insights on the Kodaira dimension of these moduli spaces.

The concept of Kodaira dimension serves as a fundamental tool in the classification theory of algebraic varieties, offering a measure of the complexity of the geometry of a variety. For moduli spaces of abelian varieties, determining the Kodaira dimension helps in understanding the geometric properties and potential applications of these spaces. In this endeavor, the appearance of modular forms and their generalizations is almost natural, following a rich history of both classical and modern approaches; starting with elliptic curves and traditional complex-analytic modular forms back with Taniyama in 1950, and continuing with hyperkähler surfaces and global sections of the canonical divisor for a smooth model of a variety, as explored by Barros et al. in [3].

Building upon foundational concepts, we begin with an overview of abelian varieties, their moduli, and their associated properties. We then delve into the theory of modular forms and modular curves to establish a strong background for understanding the relation between modular forms and complex tori. The final chapter focuses on the main topic, providing detailed steps for calculating the Kodaira dimension of A_g, supported by significant results from referential research.

This thesis not only offers a roadmap to a vantage point in the landscape of the geometry of moduli spaces—specifically the Kodaira dimension of moduli spaces of abelian varieties—but also showcases the type of mathematical machinery, both existing and developed, needed to build the roads themselves. We intend for this work to serve as a beacon in this vast and complex landscape, guiding future research and exploration.

Chapter 0

Notation and preliminaries

In this short chapter we introduce some notation to be used throughout this work, and some preliminaries of importance.

0.1 Matrices

For a (square) matrix A, A^t will denote its transpose, A^{-t} its inverse transpose (since our rings are fields, therefore commutative), A^* its Hermitian transpose, and $|A| = \det(A)$ its determinant. A matrix named I will denote an (appropriately sized) identity matrix unless otherwise specified.

Definition 0.1.1. *A subgroup H of $G(\mathbb{Q})$ of a linear algebraic group G over $\mathbb{Q}$ is called **neat** if the image of H under some faithful represenation $G \to GL_n(\mathbb{Q})$ is neat. This latter neat means that the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of the elements in the subgroup of $GL_n(\mathbb{Q})$ is torsion free.*

0.2 Tori and lattices

Definition 0.2.1. *A **lattice** over a complex vector space V is a finite subgroup L with rank $\dim_{\mathbb{R}} V$.*

Definition 0.2.2. *A (complex) **torus** of dimension g is a quotient $X = V/L$ where V is a complex vector space, and L is a rank $2g$ lattice.*

Definition 0.2.3. *A **polarization** on a lattice L is a positive definite hermitian form ω with integer imaginary part (on L).*

Proposition 0.2.1. *(Riemann relations) A polarization for a lattice over a g-dimensional complex vector space is uniquely determined by a matrix*

$$W = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

in which D is a $g \times g$ positive-integer valued diagonal matrix such that, if P is a matrix with columns as basis vectors for L:

1. $PW^{-1}P^t = 0$,
2. $PW^{-1}P^*$ *is positive definite.*

A polarization as above in which $D = I$ is called **principal**.
The diagonal entries of D as a tuple is called the **type** of the polarization.

Remark 0.2.1. This construction is equivalent to that of the first Chern class of a positive definite line bundle on a complex torus.

Remark 0.2.2. A matrix P as above can be normalized as (I, Z), for which Z will be symmetric and its imaginary part positive definite. It is called the **period matrix**.

Definition 0.2.4. *The dual torus* $X^\vee$ *of* $X = V/L$ *is defined as* $V^\vee/L^\vee$, *where* $V^\vee = Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$ *(the space of* $\mathbb{C}$*-antilinear forms) and* $L^\vee = \{l \in V^\vee : \langle l, L \rangle \subseteq \mathbb{Z}\}$ *with* $\langle \cdot, \cdot \rangle : V^\vee \times V \to \mathbb{R}$ *the canonical* $\mathbb{R}$*-bilinear form given by* $\langle l, v \rangle = Im\ l(v)$.

Definition 0.2.5. *The* ***Siegel upper half space*** H_g *is the space of all symmetric* $g \times g$ *matrices with entries in* $\mathbb{C}$ *and positive definite imaginary part.*

Chapter 1

Abelian varietes and their moduli

In this chapter we develop all the necessary theory to study the moduli spaces of (principally polarized) abelian varieties. We begin by defining them classically.

1.1 Abelian varieties

Definition 1.1.1. *An **abelian variety** (AV) is a complex torus X on which there exists a polarization ω. A **polarized** abelian variety is a pair (X, ω). A **principally polarized abelian variety (PPAV)** is a polarized abelian variety on which the polarization is principal.*

An important equivalence relation amongst AVs are *isogenies*, which are the morphisms of the **AbVar** category.

Definition 1.1.2. *A **morphism** f between abelian varieties $X = V/L$ and $Y = U/K$ is given by a $\mathbb{C}$-linear map $F : V \to U$ such that $F(L) \subseteq K$. f is called an **isogeny** if F is an isomorphism (of vector spaces), and an (abelian variety) **isomorphism** if furthermore $F(L) = K$*

Note 1.1.1. This definition prescinds from a polarization, so it is valid for common complex tori.

Note that isogenies are "isomorphisms everywhere but finite points".

Example 1.1.1. *Multiplication by an integer $n : V \to V$ given by $z \mapsto n \cdot z$ is an isomorphism only when $n = \pm 1$.*

We'll call abelian varieties **isogenous** if there is an isogeny between them, and isomorphic is there is an isomorphism in the above sense between them.

Remark 1.1.1. We can equivalently define a polarization on a complex torus as an isogeny $X \to X^\vee$ induced by a positive definite line bundle.

Proposition 1.1.1. *Let $X = V/L$ an abelian variety with polarization W. Then, it is isogenous to a principally polarized abelian variety.*

Proof: Let (X, W) as in the statement above and $(e_1, \ldots, e_g, h_1, \ldots, h_g)$ a basis of L such that W is determined by its type matrix $D = (d_1, d_2, \ldots, d_g)$. Let L' a lattice generated by $(e_1/d_1, \ldots, e_g/d_g, h_1, \ldots, h_g)$ In this basis, W is a principal polarization in $Y = V/L'$. So we can surject cannonically $s : X \to Y$, which is an isogeny since the d_i are integers making $s(L) \subseteq L'$. □

1.2 Their moduli

To start building the moduli space we want, let us consider the following construction:

Let $Z \in H_g$, i.e. a symmetric $g \times g$ matrix with positive definite imaginary part. We can construct a torus $X_Z = \mathbb{C}^g/(Z \cdot \mathbb{Z}^g + \mathbb{Z}^g)$, with principal polarization W_Z represented by Z^{-1} and period matrix (Z, I). So (X_Z, W_Z) is a principally polarized abelian variety. We can prove:

Lemma 1.2.1. *Let (X, W) be a g-dimensional PPAV, then there exists $Z \in H_g$ such that (X, W) is isomorphic to (X_Z, W_Z)*

Proof: As in remark 0.2.2 we can change the basis of the period matrix of X to get $P = (N, I)$ with $N \in H_g$. □

So, now we have a moduli space of PPAV **with a preferred basis**, so there could be isomorphic varieties represented as different points due to choice of basis. Of course, the next natural step is to find the rules to quotient out this redundancy. This means we have to find a way to change lattice basis respecting the polarization. With this aim, let's look at H_g as a symmetric space. The symplectic group of $2g \times 2g$ matrices is

$$Sp_{2g}(\mathbb{R}) = \left\{ M \in GL_{2g}(\mathbb{R}) : M^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}.$$

This group acts on H_g as follows: given $Z \in H_g$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$

$$M(Z) := (AZ + B)(CZ + D)^{-1} \in H_g.$$

Indeed, we note that A^tC and B^tD are symmetric and $A^tD - C^tB = I$. So

$$\begin{aligned} 0 &= (CZ + D)^t(M(Z) - M(Z)^t)(CZ + D) \\ &= (M(Z) - M(Z)^t) \end{aligned}$$

proving symmetry, and in the same fashion

$$(CZ + D)^t \mathrm{Im}(M(Z))(CZ + D) = \mathrm{Im}(Z) > 0$$

proving positive definiteness.

We can also prove:

Lemma 1.2.2. *The action of $Sp_{2g}(\mathbb{R})$ on H_g is transitive.*

Proof: Since the matrices we're working with are non-degenerate, we only need to check that the orbit of iI is H_g. Let $Z = X + iY \in H_g$. There's an $A \in \mathrm{GL}_g(\mathbb{R})$ such that $Y = AA^t$ (since Y is symmetric and pos. def.). Then, consider

$$\begin{aligned} Z = X + iY &= X + iAA^t \\ &= X(A^t)^{-1}A^t + iAA^t \\ &= (X(A^t)^{-1} + iA)(A^t) \\ &= (A(iI) + X(A^t)^{-1})(0(iI) + (A^t)^{-1})^{-1} \\ &= M(iI). \end{aligned}$$

So $M = \begin{pmatrix} A & X(A^t)^{-1} \\ 0 & (A^t)^{-1} \end{pmatrix}$ acts on iI mapping it to an arbitrary Z. □

We can also calculate the stabilizer of iI as

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : AB^t = BA^t, AA^t + BB^t = I \right\} \cong U_g$$

(the unitary group of degree g) where the block structure comes from

$$\begin{aligned} iI = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (iI) &= (B + iA)(D + iC)^{-1} \\ i(D + iC) &= B + iA \\ -C + iD &= B + iA \\ \iff A &= D, C = -B. \end{aligned}$$

and the presentation is inherited from $Sp_{2g}(\mathbb{R})$ (cf. last page). It is easily seen compact and isomorphic to the unitary group through $A + iB$.

With this, it is clear that $H_g \cong Sp_{2g}(\mathbb{R})/U_g$. We can now prove that isomorphisms of PPAV translate to the action of integer symplectic matrices on the Siegel upper half space.

Theorem 1.2.1. $X_Z \cong X_{Z'} \iff \exists M \in Sp_{2g}(\mathbb{Z}) \mid Z' = M(Z)$.

Proof: Proving the implication $\Rightarrow$, let $X_Z = \mathbb{C}^g/L, X_{Z'} = \mathbb{C}^g/L'$ PPAVs with period matrices (Z, I) and (Z', I) respectively. Suppose f is such that $f(X_Z) = X_{Z'}$, i.e.

$$Q(Z', I) = (Z, I)R, \tag{1.1}$$

with $Q \in M_g(\mathbb{C})$ representing f in $\mathbb{C}^g$ and $R \in M_{2g}(\mathbb{Z})$ representing it between the basis of the lattices. Assume

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t,$$

then, rewriting (1.1), $QZ' = ZA^t + B^t$ and $Q = ZC^t + D^t$. With this,

$$Z' = Z'^t = (A^t Z + B^t)^t (Q^{-1})^t = (AZ + B)(CZ + D)^{-1},$$

since A^t is invertible given that f is an isomorphism. Furthermore, thanks to the principal polarization of X_Z and $X_{Z'}$ $R^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} R = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, so $M = R^t \in Sp_{2g}(\mathbb{Z})$ is the matrix we were looking for.
Conversely, we can see that M^t is the representation of an isomorphism between X_Z and $X_{Z'}$ in terms of the lattice basis. □

With this result, we now know what to quotient out to get the desired moduli space.

Lemma 1.2.3. *The isomorphism classes of principally polarized abelian varieties is in bijection with the elements of*

$$A_g = H_g / Sp_{2g}(\mathbb{Z}).$$

1.3 Structure and properties

The performed construction doesn't directly show us if there's convenient structure or algebro-geometric properties. We now turn our attention to the study of these and their implications.

Thanks to the following theorem by Cartan shown in [5] we can endow A_g with an analytic space structure inherited from $\mathbb{C}$.

Theorem 1.3.1. *Let X be an analytic space, G a group with a properly discontinuous action on X by biholomorphic transformations, and $\rho : X \to X/G$ the quotient projection. The structural ring sheaf $\mathcal{O}$ defined at every open set $U \subseteq X/G$ as*

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} | f \circ \rho \text{ is holomorphic in } \rho^{-1}(U)\}$$

defines an analytic space structure over X/G.

To apply this theorem, we only have to prove that the action of $Sp_{2g}(\mathbb{Z})$ on H_g is properly discontinuous. It turns out (and it's actually more useful) that this is true for any discrete subgroup of $Sp_{2g}(\mathbb{R})$

Proposition 1.3.1. *Any discrete subgroup $G \subseteq Sp_{2g}(\mathbb{R})$ acts properly and discontinuously on H_g.*

Proof: We have to show that for all compact $K_1, K_2 \subseteq H_g$ there are at most finitely many $M \in G$ such that $M(K_1) \cap K_2 \neq \emptyset$. This is immediately true since $H_g \cong Sp_{2g}(\mathbb{R})/U_g$ and the projection $p : Sp_{2g}(\mathbb{R}) \to Sp_{2g}(\mathbb{R})/U_g$ is a proper map because U_g is compact. □

With this we have

Theorem 1.3.2. *The normal analytic space A_g is a moduli space for principally polarized abelian varieties.*

This is satisfactory, but still we have a *coarse* moduli space. This means that there is no universal family over A_g. To solve this, a widely used approach is to consider additional structure or information on each variety to further refine the moduli space. In this case, we consider *level n-structures.*

Definition 1.3.1. *A* ***level*** n***-structure*** *on an abelian variety* $A = \mathbb{C}^g/L$ *is fixing a basis of*

$$H^1(A, \mathbb{Z}/n\mathbb{Z}) \cong Hom(L, \mathbb{Z}/n\mathbb{Z})$$

Remark 1.3.1. A level n-structure can be "forced" by dividing the period matrix by n.

This (for $n \geq 3$) collapses the nontrivial automorphisms that might appear, which would mean that there are no nontrivial isotropy groups.

Proposition 1.3.2. *For* $n \geq 3$ *and* X *PPAV, then the subgroup of* $Aut(X)$ *that fixes the lattice mod* n

$$\{\gamma \in Aut(X) : \gamma(x) \equiv x \ mod \ nL \ \forall x \in L\}$$

is trivial.

Proof: By contradiction, suppose $1 \neq \gamma \in Aut(X)$ that fixes the lattice mod n. Then, its order is finite and we can assume it a prime p simply by replacing γ with a power of itself. Then, $1 - \gamma = n\varphi$ with $\varphi \in End(X)$. If λ, η nontrivial corresponding eigenvalues of γ, φ respectively, λ is a pth root of unity and η integer in $\mathbb{Q}(\lambda)$ and we'll have $n\eta = 1 - \lambda$ which by taking the norm N in $\mathbb{Q}(\lambda)/\mathbb{Q}$ yields $n^{p-1}N(\eta) = (1-\lambda)(1-\lambda^2)\dots(1-\lambda^{p-1}) = p$ as integers, which is a contradiction since p prime and $n \geq 3$.

□

Since we've shown that H_g was a moduli space for PPAV with a basis, we'll now find a group with which to quotient in order to identify the isomorphic PPAVs with level n-structure.

Suppose $Z, Z' \in H_g$ and $\varphi : X_Z \to X_{Z'}$ an isomorphism of PPAVs respecting level n-structure. Then, as in the proof of thm. 1.2.1, we can represent it through Q and R relative to $\mathbb{C}^g$ and the lattices respectively. We know that $R \in Sp_{2g}(\mathbb{Z})$ but by the n-structure condition of the morphism, the equality with representations is now $Q(Z'/n, I/n) = (Z/n, I/n)R \equiv (Z/n, I/n)$ mod L, which is to say $R^t \equiv I$ mod n.

Now, let's take $Z \in H_g$ and an $R \in Sp_{2g}(\mathbb{Z})$ s.t. $R \equiv I$ mod n. As in the mentioned proof, we can interpret it as a representation of an isomorphism of PPAVs that respects the level structure.

The subgroup that contains these matrices is what we're looking for.

Definition 1.3.2. *For* $n > 1$ *the* ***principal congruence subgroup*** $\Gamma(n) \leq Sp_{2g}(\mathbb{Z})$ *is defined as*

$$\Gamma(n) := \{M \in Sp_{2g}(\mathbb{Z}) : M \equiv I \ mod \ n\} = \ker[Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/n\mathbb{Z})]$$

Since it is a discrete subgroup, we've proven the first half of

Theorem 1.3.3. *The normal analytic space* $A_g(n) := H_g/\Gamma(n)$ *is a moduli space for principally polarized abelian varieties with level n-structure. It is a manifold for* $n \geq 3$.

Proof: Since we've proved that $n \geq 3$ collapses the nontrivial automorphisms, the action of $\Gamma(n)$ is fixed point free, therefore $A_g(n)$ is a manifold. □

Lastly, we mention a very important fact crucial for numerous calculations

Theorem 1.3.4. A_g *is a quasiprojective variety.*

Since we've constructed A_g analytically, realizing this is quite difficult. Fortunately, Mumford gives a rather direct algebro-geometric construction of A_g in which he uses GIT methods to show that it is indeed quasiprojective.

Chapter 2

Modular forms and modular curves

We will now focus a little on the $g = 1$ case to develop theory on modular forms and modular curves that can be generalized to higher dimensions. We will loosely follow [6], from where we've taken some proofs.

2.1 Modular forms

Before we start, we have to define some important congruence subgroups that aren't principal, namely

Definition 2.1.1.

$$\Gamma_0(n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \ (mod\ n) \right\}$$

Definition 2.1.2.

$$\Gamma_1(n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \ (mod\ n) \right\}$$

Remark 2.1.1. These subgroups satisfy $\Gamma(n) \subset \Gamma_1(n) \subset \Gamma_0(n) \subset Sp_2(\mathbb{Z})$. More generally, a **congruence subgroup** Γ is a subgroup of $Sp_{2g}(\mathbb{Z})$ for which there is a finite n such that $\Gamma(n) \subset \Gamma$

Some useful definitions to speak about modular forms are

Definition 2.1.3. *The* ***automorphic factor*** $j : Sp_2(\mathbb{Z}) \times H_1 \to \mathbb{C}$ *is given by* $(\gamma, \tau) \mapsto c\tau + d$, *where* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 2.1.4. *The* ***weight*** k ***operator*** $[\gamma]_k$ *for functions* $f : H_1 \to \mathbb{C}$ *is given by* $f([\gamma]_k)(\tau) = j(\gamma, \tau)^{-k} f(\gamma(\tau))$.

In any dimension,

Definition 2.1.5. *Given a congruence subgroup* $\Gamma \subseteq Sp_{2g}(\mathbb{Z})$ *and an integer* k*, a* ***modular form of weight*** k ***with respect to*** Γ *is a holomorphic*

$$F : H_g \to \mathbb{C}$$

that for any $Z \in H_g$ *satisfies the automorphic condition*

$$F(MZ) = \det(CZ + D)^k F(Z) \; \forall M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

When $g = 1$ *it is also required that* F *is holomorphic at the* Γ *equivalence classes of* $\mathbb{Q} \cup \{\infty\}$*.*

Definition 2.1.6. *The* Γ *equivalence classes of* $\mathbb{Q} \cup \{\infty\}$ *in* H_1 *are called the* ***cusps*** *of* Γ*.*

Remark 2.1.2. Being holomorphic at the cusps is equivalent to $f[\alpha]_k(z)$ being bounded as $Im(z) \to \infty$ for all $\alpha \in Sp_2(\mathbb{Z})$

Definition 2.1.7. *A* ***cusp form*** *is a modular form for which* $f[\alpha]_k$ *has zero constant Fourier coefficient for all* α*.*

Definition 2.1.8. *The space of modular (resp. cusp) forms of weight k with respect to* Γ *is denoted* $\mathcal{M}_k(\Gamma)$ *(resp.* $\mathcal{S}_k(\Gamma)$*).*

Remark 2.1.3. Note that if $-I \in \Gamma$, the only odd weighted modular form is the 0 function.

2.2 Modular forms and elliptic curves

It is not irrelevant to define

Definition 2.2.1. *An* ***elliptic curve*** *is a nonsingular algebraic curve defined by the polynomial relation*

$$y^2 = x^3 + ax + b.$$

Equivalently,

Definition 2.2.2. *An* ***elliptic curve*** *is an abelian variety with* $g = 1$*. I.e., a complex torus* $X = \mathbb{C}/L$ *with* $L = \tau\mathbb{Z} + \mathbb{Z}$ *and* $\tau \in H_1$*.*

Clearly, we are interested in the latter definition, but the first part of this section is dedicated to proving this equivalence, and from there we will work towards relating these to modular forms.

Definition 2.2.3. *For even* $k > 2$*, the* ***Eisenstein series*** *is*

$$G_k(\tau) := \sum_{(c,d)\neq(0,0)} \frac{1}{(c\tau + d)^k}.$$

We also define $g_2(\tau) = 60G_4(\tau), g_3(\tau) = 140G_6(\tau)$*.*

It is rather direct to check that

Proposition 2.2.1. *The Eisenstein series is a modular form of weight k*

Proof: $G_k(\tau)$ is holomorphic since it is absolutely convergent, and uniformly convergent on compact subsets of H_1. Indeed, it is dominated by an absolutely convergent function, and to check for uniform convergence on H_1 consider

$$|c\tau + d|^2 = \|T \cdot (c,d)^t\|^2,\ T = \begin{pmatrix} \mathrm{Re}(\tau) & 1 \\ \mathrm{Im}(\tau) & 0 \end{pmatrix}$$

where $\|\cdot\|$ is the usual euclidian norm. Then, by singular value decomposition $T = VSW$, so for a radius r

$$\inf_{\|(c,d)\|=r} |c\tau + d| = \inf_{\|W(c,d)\|=r} \|VSW(c,d)^t\| = s_\tau$$

which is the first (smallest) singular value, which as a funtion of τ is continuous and does not vanish for $\tau \in H_1$. With this we can bound the series outside the radius as

$$\sum_{\|(c,d)\|>r} |c\tau + d|^{-k} \leq \sum_{\|(c,d)\|>r} s_\tau^{-k}(c^2 + d^2)^{-k/2}$$

so the remainders converge locally uniformly, therefore the series converges locally uniformly implying compact convergence and holomorphy.

Absolute convergence allows us to rearrange terms in the series, so the automorphic condition is easily checked. It also shows holomorphy at infinity since the absolute summands vanish. □

Definition 2.2.4. *The* ***Weierstrass $\wp$ function with respect to a lattice*** *L is a meromorphic L-periodic function given by*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \quad \forall z \in \mathbb{C} : z \notin L$$

with double poles at $z \in L$, and its derivative

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

Proposition 2.2.2. *The functions $\wp$ and $\wp'$, generate the the field of meromorphic functions of $\mathbb{C}/L$. I.e. its meromorphic function field is $\mathbb{C}(\wp, \wp')$.*

Remark 2.2.1. Since $\wp$ depends on a lattice L it is usual to write it as $\wp_L$ or $\wp_\tau$ when τ is the period of L.

Definition 2.2.5. *The* ***generalized Eisenstein function*** *is given by*

$$G_k(\Lambda) = \sum_{\omega \in \Lambda\setminus\{0\}} \frac{1}{\omega^k}.$$

The generalized functions g_2 and g_3 are defined similarly as before.

With these definitions we can state an important result that relates modular forms and elliptic curves, both in the complex-toric and polynomial sense.

Proposition 2.2.3. *Let $\wp_L$ as before. Then*

(a) The Laurent expansion of $\wp$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2,\, n \text{ even}}^{\infty} (n+1)G_{n+2}(L)z^n$$

for all z such that $0 < |z| < \inf\{|w| : w \in L \setminus \{0\}\}$.

(b) The functions $\wp$ and $\wp'$ satisfy the relation

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2(L)\wp(z) - g_3(L)$$

where $g_2(L) = 60G_4(L)$ and $g_3(L) = 140G_6(L)$.

(c) Let $L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and let $\omega_3 = \omega_1 + \omega_2$. Then the cubic equation satisfied by $\wp$ and $\wp'$, $y^2 = 4x^3 - g_2(L)x - g_3(L)$, can be written as

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3), \quad e_i = \wp(\omega_i/2) \text{ for } i = 1, 2, 3.$$

This equation is nonsingular, meaning its right side has distinct roots.

Proof: We will prove (b) as it is the most important part of the propositions for our endeavor.

Using (a) we can directly calculate $(\wp'(z))^2 - 4(\wp(z))^3 - g_2(L)\wp(z) - g_3(L) = O(z^2)$ but remember that these are holomorphic and L-periodic, which implies that it's bounded then constant, then 0 since $O(z^2) \to 0$ when $z \to 0$. □

This implies that we have a bijection $z \mapsto (\wp_L(z), \wp'_L(z))$ from $\mathbb{C} \setminus L$ points to elliptic curve points. We can extend it sending lattice points to the point at infinity. So in sum, for a lattice L we have a bijection $(\wp_L, \wp'_L)$ from the complex torus to an elliptic curve.

Remark 2.2.2. This map also transports the group law from the torus to the curve, and the classical construction of three points on secant or tangent lines to the curve adding to zero correlates to the torus addition.

We have shown that, given a lattice, we can find a related elliptic curve. Now, let us show that

Proposition 2.2.4. *Given a nonsingular elliptic curve*

$$E : y^2 = 4x^3 - a_2x - a_3, \quad a_2^3 - 27a_3^2 \neq 0$$

there exists a lattice L such that $a_2 = g_2(L)$ and $a_3 = g_3(L)$.

To prove this proposition, we will need to define

Definition 2.2.6. *The* ***modular invariant*** j *is defined as*

$$j : H_1 \to \mathbb{C}, \quad \tau \mapsto 1728 \frac{(g_2(\tau))^3}{(g_2(\tau))^3 - 27(g_3(\tau))^2}$$

And an important theorem

Theorem 2.2.1. *The j invariant is holomorphic, surjective, and invariant under the action of $Sp_2(\mathbb{Z})$.*

With an ingenious proof by Cox in [4] (thm. 11.2).

So, we prove prop. 2.2.4 as

Proof: Assuming, $a_2, a_3 \neq 0$, since j is surjective, there's $\tau \in H_1$ such that

$$j(\tau) = 1728 \frac{(g_2(\tau))^3}{(g_2(\tau))^3 - 27(g_3(\tau))^2} = 1728 \frac{a_2^3}{a_2^3 - 27a_3^2}$$

this means

$$\frac{a_2^3}{g_2(\tau)^3} = \frac{a_3^2}{g_3(\tau)^2}$$

So if we take $L = z_1\mathbb{Z} + z_2\mathbb{Z}$ with $z_1 = \tau z_2$, then

$$g_2(L) = z_2^{-4} g_2(\tau) \quad \text{and} \quad g_3(L) = z_2^{-6} g_3(\tau).$$

So we need z_2 such that

$$z_2^{-4} = \frac{a_2}{g_2(\tau)} \quad \text{and} \quad z_2^{-6} = \frac{a_3}{g_3(\tau)}.$$

Given the equality from the j invariant, we choose z_2 satisfying $z_2^{-12} = \frac{a_2^3}{g_2(\tau)^2}$ and $z_2^{-6} = \pm \frac{a_3}{g_3(\tau)}$ with the option to replace z_2 with iz_2.

In the case of $a_2 = 0$, g_2 vanishes at $e^{2\pi i/3}$ so the lattice is

$$\left(\frac{a_3}{g_3(e^{2\pi i/3})} \right) L_{e^{2\pi i/3}},$$

and if $a_3 = 0$, g_3 vanishes at i so the lattice is

$$\left(\frac{a_2}{g_2(i)} \right) L_i.$$

□

2.3 Modular curves

As before, we can construct moduli spaces purely for elliptic curves, and for *enhanced* elliptic curves with additional data. In this case, we employ the congruence subgroups $\Gamma_0(n), \Gamma_1(n)$ and $\Gamma(n)$. For this, we need

Definition 2.3.1. *The **Weil pairing** e_n on the torsion group $E[n] = \langle \omega_1/n + L, \omega_2/n + L\rangle$ is defined as*

$$e_n : E[n] \times E[n] \to \mu_n, \quad (P, Q) \mapsto e^{2\pi i|\gamma|/n}$$

where γ is such that

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \gamma \begin{bmatrix} \omega_1/n + L \\ \omega_2/n + L \end{bmatrix}$$

Remark 2.3.1. $e_n(P, Q) \in \mu_n = \{z \in \mathbb{C}\ z^n = 1\} \setminus \{1\}$ if $E[n] = \langle P, Q\rangle$

Definition 2.3.2. *We define:*

1. *An **enhanced elliptic curve for** $\Gamma_0(n)$ is a pair (E, C) of an elliptic curve and a cyclic subgroup of E of order n. Two pairs are equivalent when there's a morphism that respect these featured subgroups. The set of equivalence classes is denoted $S_0(n)$*
2. *An **enhanced elliptic curve for** $\Gamma_1(n)$ is a pair (E, Q) of an elliptic curve and a point of E of order n. Two pairs are equivalent when there's a morphism that respect these featured points. The set of equivalence classes is denoted $S_1(n)$*
3. *An **enhanced elliptic curve for** $\Gamma(n)$ is a pair $(E, (P, Q))$ of an elliptic curve and a pair of points of E that generates the n-torsion subgroup $E[n]$ with Weil pairing a primitive root of unity. Two pairs are equivalent when there's a morphism that respect these featured pairs of points. The set of equivalence classes is denoted $S(n)$*

On the other hand, we define

Definition 2.3.3. *For a congruence subgroup Γ the **modular curve** $Y(\Gamma)$ is the quotient space*

$$Y(\Gamma) := H_1/\Gamma.$$

We write $Y_0(n), Y_1(n), Y(n)$ for the modular curves for the noted respective congruence subgroups.

Note 2.3.1. Throughout this section we'll call the quotient projection $\pi : H_1 \to H_1/\Gamma = Y(\Gamma)$

Each of these defined moduli spaces and modular curves are in bijection, and proving this is merely checking that the extra data for the curves match the definition of the congruence subgroups.

The modular curves $Y(\Gamma)$ can be endowed with geometric structure becoming Riemann surfaces that can be compactified into $X(\Gamma)$. The topology is evident since we've proved that all discrete subgroups of $Sp_2(\mathbb{Z})$ act properly and discontinuously. So $Y(\Gamma)$ is naturally Hausdorff. Unfortunately this is not enough since there can be (finite) *elliptic points*

Definition 2.3.4. *Let Γ be a congruence subgroup. For each point $z \in H_1$, let Γ_z denote the isotropy subgroup (stabilizer) of z. A point is called **elliptic for** Γ if Γ_z is nontrivial as a transformation group, i.e., contains more matrices than just I and $-I$. Its corresponding point in the modular curve is also called elliptic.*

These points are in some sense dual to the cusps, since to construct charts on $Y(\Gamma)$ we have to specially treat elliptic points, and to compactify $Y(\Gamma)$ by adding the cusps and building charts on them we go through an analogous process. Let's start by the former with an important proposition.

Proposition 2.3.1. *The isotropy subgroups Γ_z are finite cyclic.*

This proposition is not trivial, in fact Diamond & Shurman in [6] devote an entire section (2.3) to proving it. This proof relies on realizing that any isotropy subgroup Γ_τ must be a subgroup of $Sp_2(\mathbb{Z})_\tau$, and that there's only 2 elliptic points for $Sp_2(\mathbb{Z})$, namely, the classes of i, and the primitive third root of unity.

From the proposition it makes sense to define

Definition 2.3.5. *The **period** h_z of z is given by the positive integer*

$$h_z := |\{\pm I\}\Gamma_z/\{\pm I\}|$$

With this, building neighborhoods for points is summarized as follows:

For a point $z \in H_1$, find a neighborhood U such that $\forall \gamma \in \Gamma$, if $\gamma(U) \cap U \neq \emptyset$ then $\gamma \in \Gamma_z$. Which exists by cor. 2.2.3 in [6]. Then,

1. Use the map $\delta_z = \left(\begin{smallmatrix} 1 & -z \\ 1 & -\bar{z} \end{smallmatrix}\right)$ that sends $z \mapsto 0$ and $\overline{z} \mapsto \infty$. The isotropy subgroup of 0 in the conjugated transformation group $(\delta_z\{\pm I\}\Gamma\delta_z^{-1})_0/\{\pm I\}$ is the conjugation $\delta_z\left(\{\pm I\}\Gamma_z/\{\pm I\}\right)\delta_z^{-1}$ so it is finite cyclic and can be identified with integer rotations about the origin, so δ_z can be interpreted as "straightening" neighbourhoods of z.

2. Use the map ρ_z given by $\rho_z(\tau) = \tau^{h_z}$ to send the "straightened" neighbourhoods to $\mathbb{C}$ respecting quotient projection π identifications. This can be interpreted as wrapping the circular sector around the origin through the h_z-th power map.

Since π and ρ identify the same points, there's an injection φ such that $\varphi \circ \pi = \psi := \rho \circ \delta_z$. φ must also surject, since ψ does, and furthermore φ is a homeomorphism since both π and ψ are open and continuous.

To check holomorphism of transition maps, for overlapping charts $(\pi(U_1), \varphi_1), (\pi(U_2), \varphi_2)$ and x in the intersection, checking holomorphy of transition maps reduces to checking holomorphy of $\varphi_{2,1} := \varphi_2 \circ \varphi_1^{-1}|_{V_{1,2}}$ in a neighbourhood of $\varphi_1(x)$ in $V_{1,2} := \varphi_1(\pi(U_1) \cap \pi(U_2))$. By the transitivity of the Γ action, there's a $\gamma \in \Gamma$ such that we can write $x = \pi(z_1) = \pi(z_2) = \pi(\gamma z_1)$ with $z_1 \in U_1, z_2 \in U_2$. If $U_{1,2} := U_1 \cap \gamma^{-1}U_2$, $\pi(U_{1,2})$ is a neighbourhood of x, and $\varphi_1(\pi(U_{1,2}))$ is a neighbourhood of $\varphi_1(x)$.

Then, assuming $\varphi_1(x) = 0$ we will apply $\varphi_{2,1}$ to $q := \varphi_1(x') = \varphi(\pi(z'))) = \psi(z') = (\delta_1 z')_1^h$ where h_1 is the period of z_1 for some $z \in U_{1,2}$ If z_2' is such that $\psi_2(z_2') = 0$ and h_2 its period, the image is

$$\begin{aligned}
\varphi_2(x') &= \varphi_2(\pi(\gamma(z'))) = \psi_2(\gamma(z')) \quad \text{which is defined since } \gamma(z') \in U_2 \\
&= (\delta_2(\gamma(z')))^{h_2} = ((\delta_2\gamma\delta_1^{-1})(\delta_1(z')))^{h_2} \\
&= ((\delta_2\gamma\delta_1^{-1})(q^{1/h_1}))^{h_2}.
\end{aligned}$$

So this could be possibly non holomorphic when τ_1 is elliptic, but in that case, τ_2 will be elliptic with the same period. The conjugation $(\delta_2\gamma\delta_1^{-1})$ above is a diagonal matrix with entries $\alpha, \beta \in \mathbb{C}$ so the map

$$q \mapsto \left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \left(q^{1/h}\right) \right)^h = (\alpha/\beta)^h q$$

is clearly holomorphic.

To proceed in the compactification process, recall that cusps are equivalence classes of $\mathbb{Q} \cup \{\infty\}$ so we will construct

$$X(\Gamma) := H_1^*/\Gamma$$

where $H_1^* = H_1 \cup \mathbb{Q} \cup \{\infty\}$. And we will denote the compactification of $Y_0(n), Y_1(n), Y(n)$ as $X_0(n), X_1(n), X(n)$ respectively.

As suggested before, this compactification requires additional treatment. Our first stop is adding open sets $\alpha(N_m \cup \infty)$ with $\alpha \in Sp_2(\mathbb{Z})$ and $N_m := \{\tau \in H_1 : \mathrm{Im}(\tau) > m\}$ and the resultant topology in H_1^* induces a quotient topology on $X(\Gamma)$ for which we can extend the projection into $\pi : H_1^* \to X(\Gamma)$. The modular curve with this topology is Hausdorff, connected, and compact (prop. 2.4.2 in [6]).

The before constructed charts don't change, so we are only missing charts for cusp neighborhoods. For a cusp s, we can choose a $\delta_s \in Sp_2(\mathbb{Z})$ that maps $s \to \infty$. We define the **width** h_s of s as

$$h_s := \left| Sp_2(\mathbb{Z})_\infty / (\delta_s\{\pm I\}\Gamma\delta_s^{-1})_\infty \right|.$$

The construction process in this case is as follows

1. Take δ_s which "rectifies" the neighbourhoods of s and separating equivalent points in equispaced strips.

2. Take $\rho(z) = e^{2\pi i z/h}$ which rolls the strip into an infinitely long cylinder and looking through it in perspective we get a disk with the infinity point at the center.

So, analogously, in this case $\varphi = \rho \circ \delta_s$ and there also exists a homeomorphism $\varphi : \pi(U) \to V$ where $U = \delta_s^{-1}(N_2 \cup \{\infty\})$ and $V = \mathrm{Image}(\psi)$. Still analogously, transition maps are holomorphic too.

With these constructions, we've now endowed the compact modular curve $X(\Gamma)$ with a Riemann surface structure.

2.4 Dimension formulas

In this section we look at several dimension formulas for modular curves, given their Riemann surface structure.

2.4.1 Genus

Theorem 2.4.1. *Let Γ be a congruence subgroup of $Sp_2(\mathbb{Z})$. Let $f : X(\Gamma) \to X(1)$ be the natural projection, and let d denote its degree. Let ϵ_2 and ϵ_3 denote the number of elliptic points of period 2 and 3 in $X(\Gamma)$, and ϵ_∞ the number of cusps of $X(\Gamma)$. Then the genus of $X(\Gamma)$ is*

$$g = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}.$$

Proof: (Sketch) we use the Riemann-Hurwitz formula for f knowing that the genus of $X(1)$ is 0. □

2.4.2 Meromorphic differentials

This subsection relates automorphic forms (a kind of modular form) to the differential structure of the modular curve.

Definition 2.4.1. *An* ***automorphic form*** *f* ***of weight*** *k* ***with respect to*** *Γ is a weight k Γ-modular form in which the holomorphism requirement is relaxed to meromorphism. The* ***order of*** *f* ***at a cusp*** *s is defined as $\nu_s(f) = \nu_\infty(f[\alpha]_k)$, $\alpha(\infty) = s$ where $\nu_z(f)$ at a non cusp z is the usual order of vanishing of f at z. The set of automorphic forms of weight k with respect to Γ is denoted $\mathfrak{A}_k(\Gamma)$*

Extending this definition of vanishing to the modular curves may not make much sense, so it requires some workaround. Viewed as a function on $X(\Gamma)$, the following definitions follow from examining f's Laurent expansion in local coordinates.

Definition 2.4.2. *For an automorphic form f of weight k with respect to Γ we define:*

1. *The order of f at a non-cusp $\pi(z)$ of period h is*

$$\nu_{\pi(z)}(f) = \frac{\nu_z(f)}{h}.$$

2. *The order of f at a cusp $\pi(s)$ of width h is, for any $\alpha \in Sp_2(\mathbb{Z})$ such that $\alpha(\infty) = s$:*

$$\nu_{\pi(s)}(f) = \begin{cases} \nu_s(f)/2 & \text{if } (\alpha^{-1}\Gamma\alpha)_\infty = \langle -\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\rangle \text{ and } k \text{ is odd,} \\ \nu_s(f) & \text{otherwise.} \end{cases}$$

Remark 2.4.1. One can see that these orders are integers for regular points, but may be half or third integers for elliptic points, and half integers for some cusps. Note that the condition for half-integrality does not depend on f, so cusps that meet this are of interest. Such cusps are called **irregular**, and those who don't are **regular**.

Definition 2.4.3. *Let V open in $\mathbb{C}$ and n a natural number. A* ***meromorphic differential on*** *V* ***of degree*** *n is an element of $\Omega^{\otimes n}(V) = \{f(q)(dq)^n : f \text{ meromorphic}\}$ with q the V-variable.*

Remark 2.4.2. This is a $\mathbb{C}$ vector space with expected sum and scalar multiplication. The direct sum over all degrees forms the ring $\Omega(V)$ under expected differential multiplication.

A chart change $\varphi : V_1 \to V_2$ induces a pullback of meromorphic differentials:

$$\varphi^* \left(f(q_2)(dq_2)^n \right) = f(\varphi(q_1))(\varphi'(q_1))^n (dq_1)^n$$

i.e., a change of variables. With this we can glue local differentials along a Riemann surface X so

Definition 2.4.4. *A **meromorphic differential on** X **of degree** n is a collection of degree n local meromorphic differentials*

$$(\omega_j)_{j\in J} \in \prod_{j\in J} \Omega^{\otimes n}(V_j)$$

which is compatible in the sense that the pullback of chart changes on each local differential gives the expected local differential in the changed chart. The set of meromorphic differentials on X of degree n is denoted $\Omega^{\otimes n}(X)$ is also a vector space and its sum over n is a ring.

With this, we see that the pullback of the natural projection $\pi : H_1 \to X(\Gamma)$ gives us a mapping of meromorphic differentials. This is given locally as

$$\pi^*(\omega)\big|_{U_j} = \psi_j^*(\omega_j\big|_{\psi_j(U_j)})$$

where U_j, ψ_j are as in the charts we've constructed before. Again gluing these differentials, we get the pullback differential $\pi^*(\omega) = f(\tau)(d\tau)^n$. One can calculate that this is well defined globally, with f *an automorphic form of weight 2n with respect to* Γ.

Through similar calculations we can ultimately prove that

Theorem 2.4.2. *Let $k \in \mathbb{N}$ be even and let Γ be a congruence subgroup of $Sp_2(\mathbb{Z})$. The map*

$$\begin{aligned} \omega : \mathfrak{A}_k(\Gamma) &\to \Omega^{\otimes k/2}(X(\Gamma)) \\ f &\mapsto (\omega_j)_{j\in J} \text{ where } (\omega_j) \text{ pulls back to } f(\tau)(d\tau)^{k/2} \in \Omega^{\otimes k/2}(H_1) \end{aligned}$$

is an isomorphism of complex vector spaces.

To prove this we again refer the reader to [6] (thm. 3.3.1), where this map is constructed.

This isomorphism is of much importance, and results of this kind are what are used in higher dimensions to relate moduli spaces and (analogues of) modular forms.

With these forms, we can now define (canonical) divisors to apply all of their rich theory, and in particular the Riemann-Roch theorem.

Theorem 2.4.3. *Let X be a compact Riemann surface of genus g. Let K be a canonical divisor of X. For any divisor D of degree 0, we have*

$$\ell(D) = \deg(D) - g + 1 + l(K - D),$$

where $\ell(D)$ is the dimension of the complete linear system of D.

We can immediately deduce the following

Corollary 2.4.1. *Let X, g, K, and D be as above. Then*

(a) $\ell(K) = g$.
(b) $\deg(K) = 2g - 2$.
(c) *If* $\deg(D) < 0$ *then* $\ell(D) = 0$.
(d) *If* $\deg(D) > 2g - 2$ *then* $\ell(D) = \deg(D) - g + 1$.

Using these properties with care, it can be proved that

Theorem 2.4.4. *Let k be an even integer. Let Γ be a congruence subgroup of $Sp_2(\mathbb{Z})$, g the genus of $X(\Gamma)$, ϵ_2 the number of elliptic points with period 2, ϵ_3 the number of elliptic points with period 3, and ϵ_∞ the number of cusps. Then*

$$\dim(\mathcal{M}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty & \text{if } k \geq 2, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases}$$

and

$$\dim(S_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \left(\frac{k}{2} - 1\right) \epsilon_\infty & \text{if } k \geq 4, \\ g & \text{if } k = 2, \\ 0 & \text{if } k \leq 0. \end{cases}$$

For odd k the argument is modified due to clashes with $-I$ and the absence of period 2 elliptic points, but it is essentially analogous

Theorem 2.4.5. *Let k be an odd integer. Let Γ be a congruence subgroup of $Sp_2(\mathbb{Z})$. If Γ contains the negative identity matrix $-I$ then $\mathcal{M}_k(\Gamma) = S_k(\Gamma) = \{0\}$. If $-I \notin \Gamma$, let g be the genus of $X(\Gamma)$, ϵ_3 the number of elliptic points with period 3, ϵ_∞^{reg} the number of regular cusps, and ϵ_∞^{irr} the number of irregular cusps. Then*

$$\dim(\mathcal{M}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty^{reg} + \left(\frac{k-1}{2}\right) \epsilon_\infty^{irr} & \text{if } k \geq 3, \\ 0 & \text{if } k < 0, \end{cases}$$

and

$$\dim(S_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k-2}{2}\epsilon_\infty^{reg} + \left(\frac{k-1}{2}\right) \epsilon_\infty^{irr} & \text{if } k \geq 3, \\ 0 & \text{if } k < 0. \end{cases}$$

If $\epsilon_\infty^{reg} > 2g - 2$ then $\dim(\mathcal{M}_1(\Gamma)) = \epsilon_\infty^{reg}/2$ and $\dim(S_1(\Gamma)) = 0$. If $\epsilon_\infty^{reg} \leq 2g - 2$ then $\dim(\mathcal{M}_1(\Gamma)) \geq \epsilon_\infty^{reg}/2$ and $\dim(S_1(\Gamma)) = \dim(\mathcal{M}_1(\Gamma)) - \epsilon_\infty^{reg}/2$.

These dimension formulas are the basis for multiple calculations, and the machinery used to get to them sheds light on general methods to derive results about dimensions in moduli spaces.

Chapter 3

Kodaira dimension of A_g

In the same spirit as last chapter, we now review the process to calculate dimensions in the moduli of principally polarized abelian varieties using general modular forms. In particular, following [Tai1982] we will prove the following:

Theorem 3.0.1. *A_g is of general type for $g \geq 9$.*

We will now introduce the definitions of plurigenus, Kodaira dimension, and general type to better aim at the results we want to show.

Definition 3.0.1. *The **m-th plurigenus** of an n-dimensional algebraic variety X over a field k is*

$$P_m(X) := \dim_k H^0(X, \omega_X^{\otimes m})$$

where ω_X is the canonical line bundle of n-forms.

Definition 3.0.2. *The **Kodaira dimension** of X is $-\infty$ if the plurigenera are all zero, and the minimum k such that $P_m(X)/m^k$ is bounded otherwise.*

Remark 3.0.1. The Kodaira dimension can therefore be $-\infty$ or an integer between 0 and n.

Definition 3.0.3. *We say a variety is **of general type** when its Kodaira dimension is maximal.*

3.1 Extending forms

For the rest of this chapter, let $\mathfrak{A}_k = \mathfrak{A}_{g+1,k}$ now denote the space of modular forms of weight $k(g+1)$ and

$$\omega = \bigwedge_{1 \leq i \leq j \leq g} dZ_{ij}.$$

Given $f(Z) \in \mathfrak{A}_k$, its modularity makes $f(Z)\omega^{\otimes k}$ invariant under $Sp_{2g}(\mathbb{Z})$, so it is a canonical k-fold form on A_g^0 : the smooth locus of A_g, which corresponds to non-elliptic fixed points for $g < 3$, otherwise the non-singular points. We want to extend these forms to the cusps in a compactification. We now define

Definition 3.1.1. *A modular form $f(Z)$ vanishes at infinity of order m, if for all Fourier-Jacobi expansions of $f(Z)$:*

$$f(Z) = \sum \theta_S(\tau, w)e^{2\pi i tr(SZ)}, \quad Z = \begin{pmatrix} \tau & w \\ w^t & z \end{pmatrix},$$

where the block shape of τ is $n' \times n'$, w is $n' \times n$, and z is $n \times n$ satisfy

$$\min_{\theta_S \neq 0} \mu(S) = m \quad \text{where} \quad \mu(S) = \min_{0 \neq x \in \mathbb{Z}^n} x^t S x.$$

θ_S is the Fourier-Jacobi coefficent with matrix index S as defined in [11] running through $n \times n$ semi-integral, positive or semi-positive matrices

Remark 3.1.1. Every modular form has such a Fourier series thanks to the holomorphy condition for $g = 1$ and the Koecher principle for $g \geq 2$.

We'll reenact the toroidal compactification construction of A_g, $\overline{A_g}$ developed in [1]:
Let $F = H_{n'}$ be a boundary component of H_g,
$N(F)$: the normalizer of F (i.e. the elements that fix F under conjugation),
$U(F)$: the center of the unipotent radical of $N(F)$,
$U(F)_{\mathbb{Z}} = U(F) \cap Sp_{2g}(\mathbb{Z})$,
$D(F) = F \times \mathbb{C}^{n' \times n} \times U(F)_{\mathbb{C}}$,
$C(F)$: the cone of $n \times n$ positive definite symmetric matrices,
$\{\sigma_\alpha\}$: a $GL_n(\mathbb{Z})$-admissible decomposition of $C(F)$,
$(H_g/U(F)_{\mathbb{Z}})_{\sigma_\alpha}$: the interior of the closure of $H_g/U(F)_{\mathbb{Z}}$ in $(D(F)/U(F)_{\mathbb{Z}})_{\sigma_\alpha}$.

By the main theorem in [1], there's a compact analytic space $\overline{A}_g$ and open analytic morphisms $\pi_F : (H_g/U(F)_{\mathbb{Z}})_{\sigma_d} \to \overline{A}_g$ such that

1. A_g is an open dense subset of $\overline{A}_g$
2. every point of $\overline{A}_g$ is in the image of the maps π_F.

We also define $\overline{A_g^0}$ as the open subset of $\overline{A_g}$ such that π_F are unramified for all F. With this,

Theorem 3.1.1. *$f(Z)\omega^{\otimes k}$ defines a k-fold canonical differential form on $\overline{A_g^0}$ if $f(Z)$ vanishes at infinity of order $\geq k$.*

Proof: We just need to check at $H_g/U(F)_{\mathbb{Z}} \to (H_g/U(F)_{\mathbb{Z}})_{\sigma_d}$ if $f\omega^{\otimes k}$ extends. By thm 4.1.1 in [1] $f\omega^{\otimes k}$ extends if for all Fourier series of $f(Z)$, $\theta_S \neq 0 \implies tr(SX) \geq k$ for all integral (semi) positive matrices X.

We can prove the trace restriction thanks to a result by Barnes and Cohn [2]

Lemma 3.1.1. *For a fixed S, $tr(SX)$ attains its minimum at rank one X, i.e.,*

$$\min_{\substack{X \text{ integral} \\ \text{(semi)positive}}} tr(SX) = \min_{\substack{X = x^t x \\ 0 \neq x \in \mathbb{Z}^n}} tr(SX) = \mu(S).$$

□

We can reduce the theorem to just checking the coefficients with integer index at the highest dimensional cusp:

Theorem 3.1.2. *For* $Z = \begin{pmatrix} \tau & w \\ w^t & z \end{pmatrix}$ *with the top blocks having* $g-1$ *rows, and the left blocks having* $g-1$ *columns,* $f(Z)\omega^{\otimes k}$ *can be extended to* $\bar{A}_g^0$ *if in the Fourier-Jacobi expansion of* $f(Z)$ *over the highest dimensional cusp:*

$$f(Z) = \sum \theta_m(\tau, w) e^{2\pi i m z},$$

$\theta_m = 0$ *for* $m < k$.

Proof: For any $Z \in H_g$, $\lambda \in \mathbb{R}^+$ define $Z_\lambda \in H_g$ by

$$(Z_\lambda)_{jk} = Z_{jk} \quad (j,k) \neq (g,g)$$

$$(Z_\lambda)_{gg} = Z_{gg} + i\lambda.$$

Consider any Fourier-Jacobi expansion of $f(Z)$

$$f(Z) = \sum \theta_S(\tau', w') e^{2\pi i \mathrm{tr}(Sz')}, \quad Z = \begin{pmatrix} \tau' & w' \\ w'^t & z' \end{pmatrix},$$

where the τ' block is $n' \times n'$ and the z' block is $n \times n$. If $\min_{\theta_S \neq 0} \mu(S) = l$, then

$$\lim_{\lambda \to \infty} f(Z_\lambda) e^{2\pi i \lambda l} = \sum_{S_{nn} = l} \theta_S(\tau', w') e^{2\pi i \mathrm{tr}(Sz')}.$$

But if $l < k$, since $\theta_m = 0$ for $m < k$, we have

$$\lim_{\lambda \to \infty} f(Z_\lambda) e^{2\pi l \lambda l} = 0 \implies \sum_{S_{nn} = l} \theta_S(\tau', w') e^{2\pi i \mathrm{tr}(Sz')} = 0$$

$$\implies \theta_S(\tau', w') = 0 \quad \text{for all } S \text{ such that } S_{nn} = l$$

$$\implies \theta_S(\tau', w') = 0 \quad \text{for all } S \text{ such that } \mu(S) = l.$$

Since $\mu(S) = l$, $\exists U \in GL_n(\mathbb{Z})$ such that $S[U]_{n,n} = l$ and $\theta_S = \theta_{S[U]}$. This contradicts the definition of l, hence $l \geq k$ or f vanishes at infinity of order $\geq k$, $f\omega^{\otimes k}$ is extendable to $\overline{A_g^0}$ by last theorem. □

3.2 Dimension formulas

In this section we prove asymptotical relations for the dimension of $\mathfrak{A}_k$ and the space of Fourier coefficients.

Proposition 3.2.1. *For a fixed g and large k,* $\dim \mathfrak{A}_k \sim 2^{\frac{1}{2}(g-1)(g-2)} \prod_{j=1}^{g} \frac{(j-1)!}{2j!} B_j [(g+1)k]^{\frac{1}{2}g(g+1)}$ *where B_j's are Bernoulli numbers:*

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum (-1)^{j+1} B_j \frac{x^{2j}}{2j!}$$

Proof: Let $\Gamma(1) = Sp_{2g}(\mathbb{Z})/\{\pm I\}$, $\overline{\Gamma} = \Gamma(1)/\Gamma(l)$, $\mathfrak{A}_k(l), S_k(l)$: the spaces of $\Gamma(l)$-modular forms and cusp forms of weight $k(g+1)$ on H_g.

Note that $\bar{\Gamma}$ acts on $\mathfrak{A}_k(l)$ and $S_k(l)$ as follows:

$$\gamma^* f(Z) = f(Z)|(CZ+D)|^{-k(g+1)}$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$. Then $\mathfrak{A}_k = \mathfrak{A}_k(l)^{\bar{\Gamma}}$: the modular forms in $\mathfrak{A}_k(l)$ fixed by $\bar{\Gamma}$. Assuming $\Gamma(l)$ is neat, we can apply Mumford's extension of Hirzebruch's proportionality principle from [10] to compute

$$\dim S_k(l) \sim \dim \mathfrak{A}_k(l) \sim 2^{-N-g(g+1)} N^k k^N [\Gamma(1) : \Gamma(l)] V_g \pi^{-N}$$

where $N = g(g+1)/2$, and V_g : the Siegel symplectic volume$= 2^{g^2+1} \pi^N \prod_{j=1}^{g} \frac{(j-1)!}{2j!} B_j$.

To study the dimension of S_k we now apply Hirzebruch's method of [8]. Note that

$$\dim S_k = \dim S_k^{\Gamma}(l) = \frac{1}{|\overline{\Gamma}|} \sum_{\gamma \in \Gamma} \operatorname{tr}(\gamma^* | S_k(l))$$

By the Atiyah-Bott fixed point theorem,

$$\operatorname{tr}(\gamma^* | S_k(l)) = (\text{a polynomial in } k \text{ of degree} \leq \dim \operatorname{Fix}(\gamma)).$$

where $\operatorname{Fix}(\gamma)$ is the space of points fixed by the action of γ. So, for $\gamma \neq I$, $tr(\gamma^* | S_k(l))$ doesn't contribute to the leading term of $\dim S_k$, therefore

$$\dim \mathfrak{A}_k \sim \dim S_k \sim \frac{1}{[\Gamma(1) : \Gamma(l)]} \dim S_k(l) \sim 2^{-N-g(g+1)} N^k k^N V_g \pi^{-N}.$$

□

Now we look closely at the Fourier coefficients of modular forms at cusps.

Proposition 3.2.2. *Suppose $f(Z) \in \mathfrak{A}_k$ and has the Fourier-Jacobi expansion (as in thm. 3.1.2)*

$$f(Z) = \sum \theta_m(\tau, w) e^{2\pi i m z}, \quad Z = \begin{pmatrix} \tau & w \\ w & z \end{pmatrix},$$

at the highest dimensional cusp then $\theta_m(\tau, w)$ satisfies the following:

1. $\theta_m(\tau, w + \tau n_1 + n_2) = \theta_m(\tau, w) e^{-2\pi i m (n_1^t \tau n_1 + 2 n_1^t w)}$, $n_1, n_2 \in \mathbb{Z}^{g-1}$ *i.e., for fixed τ, θ_m is a theta function of degree $2m$ in w.*

2. $\theta_m(\tau, -w) = \theta_m(\tau, w)$

3. If $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g-2}(\mathbb{Z})$, *then*

$$\theta_m(\gamma\tau, (C\tau + D)^{-t}w) = \theta_m(\tau, w)e^{2\pi imw(C\tau+D)^{-t}Cw}|(C\tau + D)|^{k(g+1)}$$

Proof: Consider the following matrices and their transformations on the Z blocks:

$$\gamma_1 = \begin{pmatrix} 1_{g-1} & 0 & 0 & n_2 \\ {}^t n_1 & 1 & {}^t n_2 & 0 \\ 0 & 0 & 1_{g-1} & -n_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad : \quad (\tau, w, z) \mapsto (\tau, w + \tau n_1 + n_2, z + n_1^t \tau n_1 + 2n_1^t w - n_2^t n_1)$$

$$\gamma_2 = \begin{pmatrix} I_{g-1} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & I_{g-1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad : \quad (\tau, w, z) \mapsto (\tau, -w, z)$$

$$\gamma_3 = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad : \quad (\tau, w, z) \mapsto (\gamma\tau, (C\tau + D)^{-t}w, z - w^t(C\tau + D)^{-1}Cw)$$

Apply the automorphic condition to these three elements and f to prove (1),(2), and (3) respectively. □

We now define

Definition 3.2.1. $\tilde{H}_m(l) = \tilde{H}_{g,m}(l)$ *is the space of holomorphic functions* $\theta(\tau, w)$ *on* $H_g \times \mathbb{C}^g$ *such that:*

$$(\theta_1) \quad \theta(\tau, w + \tau n_1 + n_2) = \theta(\tau, w)e^{-2m\pi i(n_1^t \tau n_1 + 2n_1^t w)}, \quad n_1, n_2 \in \mathbb{Z}^g$$
$$(\theta_2) \quad \theta(\tau', w') = \theta(\tau, w)e^{2mU'}|C\tau + D|^{k(g+1)}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})(l)$$

$$(\tau', w') = ((A\tau + B)(C\tau + D)^{-1}, (C\tau + D)^{-t}w)$$

$$U' = \pi i w^t(C\tau + D)^{-1}Cw.$$

By the recent proposition, we see that $\theta_m(\tau, w) \in \tilde{H}_{g-1,m}(1)^{\text{even}}$.

Let

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau, w) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}(n+a)^t\tau(n+a)+(n+a)^t(w+b)\right)}$$

with

$$(\tau, w) \in H_g \times \mathbb{C}^g, \quad a, b \in \frac{1}{2m}\mathbb{Z}^g.$$

(the **Riemann theta function**). It is a known fact that the Riemann theta functions $\Theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2m\tau, 2mw)$ span the Jacobi theta functions of degree $2m$ by varying a over the set of representatives of $\frac{1}{2m}\mathbb{Z}^g/\mathbb{Z}^g$, so let's look at

Proposition 3.2.3. *For* $(\tau, w) \in H_g \times \mathbb{C}^g, a, a^* \in \frac{1}{2m}\mathbb{Z}^g$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$, *we have*

$$\Theta \begin{bmatrix} a^* \\ 0 \end{bmatrix} (\tau^*, w^*) = e^{U^*}(C(2m)^{-1}\tau + D^{\frac{1}{2}} \sum_{a \mod \mathbb{Z}^g} u_{a^*a}\Theta \begin{bmatrix} a \\ 0 \end{bmatrix} (\tau, w)$$

where

$$(\tau^*, w^*) = (2m(A\tau + 2mB)(C\tau + 2mD)^{-1}, 2mt(C\tau + 2mD)^{-1}w)$$
$$U^* = \pi i t_w (C\tau + 2mD)^{-1}Cw$$

and u_{a^*a} *is a constant unitary matrix of degree* $(2m)^g$.

Proof: Follows from transformation laws of theta functions □

If we define $\vec{\Theta}(\tau, w) = \left(\Theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2m\tau, 2mw)\right)_{a\in\frac{1}{2m}\mathbb{Z}^g/\mathbb{Z}^g}$, the next result follows

Corollary 3.2.1. *If* $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$ *then*

$$\vec{\Theta}(\tau', w') = e^{2mU'}|C\tau + D|^{\frac{1}{2}}\rho(\gamma)\vec{\Theta}(\tau, w)$$

where τ', w', U' *are the same as in* (θ_2), $\rho(\gamma)$ *is a unitary representation of* $\gamma \in Sp_{2g}(\mathbb{Z})$.

These results play an important role to prove another piece of the puzzle

Proposition 3.2.4. *For large* k, $\dim \tilde{H}_m(1) \sim (2m)^g \dim \mathfrak{A}_k$

Proof: Assuming that $\Gamma(l)$ is neat, we compute $\dim \tilde{H}_m(1)$ as follows: let $\theta_m(\tau, w) \in \tilde{H}_m(l)$, then we can write θ_m as

$$\theta_m(\tau, w) = \sum f_{\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]}(\tau)\Theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2m\tau, 2mw) = \vec{F}(\tau)^t\vec{\Theta}(\tau, w)$$

where $\vec{F}(\tau) = (f_{\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]}(\tau))$ is a holomorphic vector valued function in τ. By (θ_2) and the previous corollary we have

$$\vec{F}(\gamma\tau) = \underline{\rho(\gamma)^{-t}}\vec{F}(\tau)|C\tau + D|^{k(g+1)\underline{-\frac{1}{2}}}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(l). \tag{3.1}$$

Now consider the following:

1. The action of $\gamma \in \Gamma(l)$ on $H_g \times \mathbb{C}^{(2m)^g}$ given by $(\tau, w) \mapsto (\gamma\tau, \rho(\gamma)^{-t}w),\ \gamma \in \Gamma(l)$ defines a vector bundle on $A_g(l)$ extendible to a vector bundle W_l on $X = \overline{A_g(l)}$ of rank $(2m)^g$ by construction.

2. The action of $\Gamma(l)$ on $H_g \times \mathbb{C}$ given by $(\tau, w) \mapsto (\gamma\tau, |C\tau + D|^{\frac{1}{2}}w)$ defines a line bundle M on X

3. Let L be the line bundle on X corresponding to the cusp forms of weight $k(g+1)$.

4. Define $V_l = W_l \otimes L \otimes M^{-1}$

So, we see that $\tilde{H}_m(l) = \Gamma(X, V_l)$

Indeed, note that relation 3.1 is "almost automorphic" in the sense that if the highlighted factors disappeared this would be indeed a vector automorphic condition and $\vec{F}$ would be a vector cusp form. So, the sections of the tensor product V_l represent cusp forms (L) with a $\rho(\gamma)^{-t}$ factor (W_l) and the inverse of the square root of a $|C\tau + D|$ factor (M^{-1}). Considering this, and the Riemann-Roch theorem, we have:

$$\begin{aligned}\sum(-1)^i \dim H^i(X, V_l) &= \{ch(V_l) \cdot Td(X)\}[X] \\ &= (2m)^g\{ch(L \otimes M^{-1}) \cdot Td(X)\}[X] \\ &= (2m)^g \sum(-1)^i \dim H^i(X, L \otimes M^{-1}).\end{aligned}$$

Extracting the first and last members of this equality, we note that, for big enough k, $V_l \otimes K_X^{-1}$ and $L \otimes M^{-1} \otimes K_X^{-1}$ are quasi-positive in the sense of Grauert and Riemenschneider in [7], and by their vanishing theorem, $H^i(X, V_l)$ and $H^i(L \otimes M^{-1}) = 0$ for $i > 0$. So we're left with

$$\dim \tilde{H}_m(l) = \dim H^0(X, V_l) = \dim H^0(X, L \otimes M^{-1})(2m)^g \sim (2m)^g \dim \mathfrak{A}_k(l).$$

To find $\dim \tilde{H}_m(1)$ recall $\overline{\Gamma}$ and consider its action on $\tilde{H}_m(l)$ by

$$\theta(\tau, w) \mapsto \theta(\tau', w')e^{-2mU'}|C\tau + D|^{-k(g+1)}$$

with τ', w', U', γ are as before.

Replicating the argument in prop. 3.2.1

$$\dim \tilde{H}_m(1) = \dim \tilde{H}_m(l)^{\Gamma} = \frac{1}{|\overline{\Gamma}|}\sum_{\gamma\in\Gamma} \mathrm{tr}(\gamma^*|\tilde{H}_m(l))$$

and $\mathrm{tr}(\gamma^*|\tilde{H}_m(l))$ is bounded by $(2m)^g$ times a polynomial of degree $\leq \dim X^\gamma$, showing ultimately that

$$\dim \tilde{H}_m(1) = \frac{1}{|\overline{\Gamma}|}\dim \tilde{H}_m(l) \sim \frac{1}{|\overline{\Gamma}|}(2m)^g \dim \mathfrak{A}_k(l) \sim (2m)^g \dim \mathfrak{A}_k.$$

□

Corollary 3.2.2. $\dim \tilde{H}_{g-1,m}(l) \sim (2m)^{g-1} \dim \mathfrak{A}_{g-1,k}$.

Now see that, since by prop. 3.2.2 the Fourier coefficients are in $\tilde{H}_{g-1}(1)^{even}$ and $\sum_{m=1}^{k}(2m)^{g-1} \sim 2^{g-1} \cdot \frac{k^g}{g}$, by the previous corollary and prop. 3.2.1, we have

$$\sum_{m \le k} \dim \tilde{H}_{g-1,m}(1)^{\text{even}} \sim \frac{1}{2}2^{g-1} \cdot \frac{k^g}{g} \cdot 2^{\frac{1}{2}(g-2)(g-3)} \prod_{j=1}^{g-1} \frac{(j-1)!}{2j!} B_j[k(g+1)]^{\frac{1}{2}(g-1)}$$

The ratio $\dim \mathfrak{A}_k / \sum_{m \le k} \dim \tilde{H}_{g-1,m}(1)^{\text{even}}$ is k-asymptotically equal to $r_g := \frac{g!}{2g!} B_g (g+1)^g$, and by numerical calculation we get table 3.2

Table 3.1: Numerical calculation for r_g

g	r_g
2	0.0250000
3	0.0126984
4	0.0124008
5	0.0194805
6	0.0447609
7	0.1414486
8	0.5883277
9	3.1157102
10	20.470239
11	163.38885
12	1557.2931

Since $B_g = \frac{2(2g)!}{(2\pi)^{2g}} \zeta(2g)$, with the usual Riemann zeta, we can check

$$\frac{B_{g+1}}{B_g} = \frac{(2g+2)(2g+1)}{(2\pi)^2} \cdot \frac{\zeta(2g+2)}{\zeta(2g)} \text{ and } \frac{r_{g+1}}{r_g} = \left(\frac{g+2}{g+1}\right)^g \cdot \frac{g+2}{4g+2} \cdot \frac{B_{g+1}}{B_g}$$

from this is direct that r_g is increasing and $r_g \to \infty$ when $g \to \infty$ growing faster than a^g for any a. Since we know how to extend forms to cusps by thm. 3.1.2, we have:

Theorem 3.2.1. *For $g \ge 9$, $\dim \Gamma(\overline{A_g^0}, (\Omega^N)^{\otimes k}) \sim ck^N$, where $N = \frac{1}{2}g(g+1)$, and c is a positive constant.*

We can now deduce:

Theorem 3.2.2. *For any positive integer a, there exists $f(Z) \in \mathfrak{A}_k$ with order of vanishing ak at infinity if g is sufficiently large.*

Proof: By prop. 3.2.1 and cor. 3.2.2

$$\frac{\dim \mathfrak{A}_k}{\sum_{m \le ak} \dim H_{g-1,m}(1)^{\text{even}}} \sim r_g \left(\frac{k}{ak}\right)^g = \frac{r_g}{a^g}$$

but for g sufficiently large $r_g \ge a^g$. □

3.3 Extension over quotient singularities

In the following Γ is a finite group acting linearly on $\mathbb{C}^N$, $X = \mathbb{C}^N/\Gamma$. If $\gamma \in \Gamma$, $\langle\gamma\rangle$ = the cyclic group generated by γ, $X_\gamma = \mathbb{C}^N/\langle\gamma\rangle$. Let $\tilde{X}$, $\tilde{X}_\gamma$ be the non-singular models of X and X_γ.

This first result proves that we can reduce to the cyclic subgroups:

Proposition 3.3.1. *Given η a pluricanonical form on $\mathbb{C}^N$ invariant under Γ, then η extends to $\tilde{X}$ if and only if η extends to $\tilde{X}_\gamma$ for every $\gamma \in \Gamma$.*

Proof: By negation, if η does not extend to $\tilde{X}$, then η has poles at certain exceptional divisor $F \subseteq \tilde{X} \setminus X$. Let's normalize $\tilde{X}$ with respect to the rational function field of $\mathbb{C}^N$ and call that Y. Let D' a component of the preimage of D in Y. The subgroup for which its action is the identity when restricted to D' has to be cyclic, therefore it's generated by a single element, say, γ. Let U a γ-stable open set. Then, $U \cap D' \neq \emptyset$ and the map $U/\langle\gamma\rangle \to \tilde{X}$ is unramified. Then η is holomorphic in U but has poles in $U/\langle\gamma\rangle$. So η does not extend to the resolution of X_γ. Reversing this argument proves the converse.

□

Now we know that we can restrict to an arbitrary $\Gamma = \langle\gamma\rangle$. Assume γ acts on $\mathbb{C}^N$ by

$$\gamma(Z_1, \ldots, Z_N) = (e^{2\pi i S_1} Z_1, \ldots, e^{2\pi i S_N} Z_N), \quad S_i \in \mathbb{Q}, \quad 0 \leq S_i < 1$$

and let $X = \mathbb{C}^N/\langle\gamma\rangle$, $\tilde{X}$ be a resolution of X. X and $\tilde{X}$ can be described by torus embedding as follows:

Let $\sigma = \left\{\sum_{i=1}^N \lambda_i e_i \mid \lambda_i \geq 0\right\}$, $e_i \in \mathbb{Z}^N$, assume that $e_1, e_2, \ldots, e_N$ generate a lattice L such that

$$\mathbb{Z}^N/L \approx \text{the cyclic group generated by } S_1 e_1 + \ldots + S_N e_N.$$

Then in terms of torus embedding of [9]

$$X_\sigma \approx \mathbb{C}^N/\langle\gamma\rangle = X.$$

Let $X_{\{\sigma_i\}}$: the toric resolution of X_σ by decomposing σ into the unit simplices, i.e., each face of σ_α is generated by a part of a basis of $\mathbb{Z}^N$.

Proposition 3.3.2. *Given η a γ-invariant pluricanonical form on $\mathbb{C}^N$, then η extends to $X_{\{\sigma_\alpha\}}$ if*

$$(*)\{\mu S_1\} + \{\mu S_2\} + \ldots + \{\mu S_N\} \geq 1$$

for $0 < \mu < m$ where $\{\cdot\}$ denotes the fractional part function and m is the order of γ; the least common denominator of the S_i.

Proof: Let's write $\eta = f(Z_1, \ldots, Z_N)(dZ_1 \wedge \ldots \wedge dZ_N)^{\otimes k}$ and $\{Z_1^*, \ldots, Z_N^*\}$ the local coordinate system defined by σ_α, then

$$\begin{aligned}\eta &= g(Z_1, \ldots, Z_N)\frac{(dZ_1 \wedge \ldots \wedge dZ_N)^{\otimes k}}{(Z_1, \ldots, Z_N)^k} \\ &= g(Z_1^*, \ldots, Z_N^*)\frac{(dZ_1^* \wedge \ldots \wedge dZ_N^*)^{\otimes k}}{(Z_1^*, \ldots, Z_N^*)^k}\end{aligned}$$

where $\mathrm{ord}_{Z_i} g \geq k$. η extends to X_{σ_α} if $\mathrm{ord}_{Z_j^*} g \geq k$ for all j.

When Z_j^* corresponds to a vertex $v = \sum \lambda_i e_i \in \mathbb{Z}^n$ of σ_α, with $\lambda_i \in \mathbb{Q}_0^+$, then

$$\begin{aligned}\mathrm{ord}_{Z_j^*} g(Z_1^*, \ldots, Z_N^*) &= \lambda_1 \mathrm{ord}_{Z_1} g + \ldots + \lambda_N \mathrm{ord}_{Z_N} g \\ &\geq (\lambda_1 + \ldots + \lambda_N)k.\end{aligned}$$

so we can extend η to the toric resolution $X_{\{\sigma_\alpha\}}$ if for each α, at each vertex v as above:

$$\lambda_1 + \ldots + \lambda_N \geq 1$$

but since $v \in \mathbb{Z}^N$ and $\mathbb{Z}^N/L$ is generated by $S_1 e_1 + \ldots + S_N e_N$, we have

$$v = \mu_1 e_1 + \ldots + \mu_N e_N + \mu(S_1 e_1 + \ldots + S_N e_N)$$

where μ_i, μ are non negative integers. So $\lambda_j = \mu_j + \mu S_j$ and $\sum \lambda_j \geq 1$ for $\mu \geq m$ or $\mu = 0$ on the other hand,

$$\sum \lambda_j \geq \{\mu S_1\} + \ldots + \{\mu S_N\}.$$

Therefore if $(*)$ is true, then we win. □

Let $\gamma \in \Gamma$ which acts on $\mathbb{C}^N$ with a fixed point x and an induced tangent space action given by $e^{2\pi i S_j}$ with $0 \leq S_j < 1 \in \mathbb{Q}$. We write the pairing $\{\gamma, x\} = \sum S_j$. Combining the last two propositions it is proven that:

Theorem 3.3.1. *Given η a Γ-invariant pluricanonical form on $\mathbb{C}^N$, if Γ is finite and acts linearly on $\mathbb{C}^N$, then η extends to a non-singular model of $\mathbb{C}^N/\Gamma$ if for every non-identity $\gamma \in \Gamma, x \in Fix(\gamma)$ the pairing $\{\gamma, x\} \geq 1$.*

3.4 Extensions over elliptic points

We've studied the elliptic points of A_1, now we'll prove a useful lemma that characterizes elliptic points of A_g with our north being finding a way to extend forms over these.

Lemma 3.4.1. *For $Z \in H_g, \gamma \in Sp_{2g}(\mathbb{Z})$ such that $\gamma Z = Z$, in the local coordinate system (x_{ij}) around Z, γ is given by*

$$x_{ij} \to \eta^{t_i + t_j} x_{ij}$$

with η an mth root of unity, $m = ord(\gamma)$, and $t_i, t_j \in \mathbb{Z}$.

Proof: γ fixing Z means that there is an $\alpha_1 \in Sp_{2g}(\mathbb{Z})$ such that $\alpha_1 \gamma \alpha_1^{-1} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with $A + iB \in U_g$. So, there's an $U \in U_g$ such that

$$\overline{U}^t (A + iB) U = \Lambda = \begin{pmatrix} \eta^{t_1} & & \\ & \ddots & \\ & & \eta^{t_g} \end{pmatrix}, \; t_j \in \mathbb{Z}.$$

If $U = V + iW$ with V, W real, let

$$\alpha_2 = \begin{pmatrix} V & W \\ -W & V \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} I & -iI \\ 0 & I \end{pmatrix}$$

then $\alpha = \alpha_3\alpha_2\alpha_1$ sends Z to 0 in $\mathbb{C}^{g(g+1)/2}$ and $\alpha\gamma\alpha^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$ with its action around 0 given by

$$(x_{ij}) \mapsto \Lambda(x_{ij})\Lambda = (\eta^{t_1+t_i}x_{ij}).$$

□

Proposition 3.4.1. *Given $Z \in H_g$ and a non identity $\gamma \in Sp_{2g}(\mathbb{Z})$ that fixes Z then $\{\gamma, Z\} \geq 1$ for $g \geq 5$.*

Proof: Consider the three following lemmas, with ϕ Euler's totient function.

Lemma 3.4.2. *For a positive integer m not $1, 2, 3, 4,$ or 6, $r = \phi(m)/2$, $t_1, \ldots, t_r$ positive integers such that*

$$0 < t_i < m, \quad (t_i, m) = 1 \quad \forall i$$
$$t_i \not\equiv \pm t_j \pmod{m} \quad \textit{for} \quad i \neq j$$

then

$$\sum \left\{ \frac{t_i + t_j}{m} \right\} \geq 1.$$

Proof: WLG, we fix i. Since t_i and t_j are incongruent mod m, we can bound

$$\sum \left\{ \frac{t_i + t_j}{m} \right\} \geq \frac{1}{4} r(r+1)^2 \cdot \frac{1}{m}.$$

By hand we can check that the RHS of the inequality is ≥ 1 for all m except $2, 3, 4, 6, 8, 10, 14$. But for $m = 8, 10, 14$ the LHS is still ≥ 1. □

Lemma 3.4.3. *Given γ fixing Z, γ is conjugate to $\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$ and*

$$\lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_g \end{pmatrix}, \quad \lambda_j = e^{2\pi i t_j / m}.$$

If $(t_j, m) = 1$ for some j and $m \neq 1, 2, 3, 4, 6$, then $\{\gamma, Z\} \geq 1$.

Proof: Since the characteristic polynomial of γ has rational coefficients, all the conjugates of λ_j over $\mathbb{Q}$ appear either in Λ or Λ^{-1}, so by the previous lemma

$$\{\gamma, Z\} \geq \sum \frac{t_i + t_j}{m} \geq 1.$$

□

Lemma 3.4.4. *Given γ that fixes Z conjugate to* $\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$ *and all the λ_j appearing in Λ have orders any $1, 2, 3, 4,$ or 6, then $\{\gamma, Z\} \geq 1$ for $g \geq 5$.*

Proof: If $\text{ord}\lambda_j = 6$ for some j, and $\lambda_i\lambda_j \neq 1$ for $i \neq j$, then $\{\gamma, \mathbb{Z}\} \geq (2 + g - 1)/6 \geq 1$ iff $g \geq 5$. If $\lambda_j\lambda_i = 1$ for some $i \neq j$, then $\{\gamma, \mathbb{Z}\} \geq 2/6 + 4/6 \geq 1$. For orders $2, 3, 4$ $\{\gamma, \mathbb{Z}\} \geq 1$ for $g \geq 3$.

□

□

3.5 Extensions over cuspidal singular points

In this final section we finish the treatment of the problematic points to extend forms in $\overline{A}_g^0$ to the desingularizations over the singular points in $\overline{A}_g - A_g$.

Consider a boundary component $F = H_{n'}$ of H_g, $0 \leq n' < g$, and $n = g - n'$. If $\gamma \in N(F)_{\mathbb{R}}$ the normalizer of F, then we can write

$$\gamma = \begin{pmatrix} A & 0 & B & B_{12} \\ A_{21} & U & B_{21} & B_{22} \\ C & 0 & D & D_{12} \\ 0 & 0 & 0 & U^{-t} \end{pmatrix} \tag{3.2}$$

where $U \in GL_n(\mathbb{R})$, $\quad \gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n'}(\mathbb{R})$.

Recall the defined elements involved in the toroidal compactification $U(F)$, $T(F)$, $C(F)$, $D(F)$, $(H_f/U(F)_{\mathbb{Z}})_{\sigma_\alpha}$, and π_F.

We remember

$$D(F) = F \times V(F) \times U(F)_{\mathbb{C}} = \left\{ \begin{pmatrix} \tau & W \\ W^t & Z \end{pmatrix} : \tau \in F, W \in V(F) = \mathbb{C}^{n' \times n}, Z \in U(F)_{\mathbb{C}} \right\}$$

$(H_g/U(F)_{\mathbb{Z}})_{\{\sigma_\alpha\}}$: the interior of the closure of $H_g/U(F)_{\mathbb{Z}}$ in $(D(F)/U(F)_{\mathbb{Z}})_{\{\sigma_\alpha\}}$

$(H_g/U(F)_{\mathbb{Z}})_{\{\sigma_\alpha\}} \to \overline{A}_g$ and $\overline{A}_g^0$: the subset of $\overline{A}_g$ where π_F are unramified.

With these constructions, we can write $y_0 \in \overline{A}_g - A_g$ as

$$y_0 = \pi_F(\tau_0, W_0, \overline{Z}_0 + \sigma\infty) \tag{3.3}$$

where $\tau_0 \in F$, $W_0 \in V(F)$, $Z_0 \in U(F)_{\mathbb{C}}$, and $\overline{Z}_0$: Z_0's image in $T(F)$. $\overline{Z}_0 + \sigma\infty$ is the ideal point in $T(F)_\sigma$ obtained by moving the imaginary part of $\overline{Z}_0$ towards infinity in the direction of σ, a face of $\{\sigma_\alpha\}$.

If $y_0 \notin \overline{A}_g^0$, then there's a $\gamma \in N(F)_{\mathbb{Z}}$ not congruent to the identity mod $U(F)_{\mathbb{Z}}$ such that

$$\gamma(\tau_0, W_0, \overline{Z}_0 + \sigma\infty) = (\tau_0, W_0, \overline{Z}_0 + \sigma\infty). \tag{3.4}$$

If we write γ as in eq. 3.2, then γ acts on H_g by

$$\begin{aligned} \tau &\to \gamma'\tau \quad \gamma' \in Sp_{2n'}(\mathbb{Z}) \\ W &\to A(\tau)W + a(\tau) \\ Z &\to Z[U] + b(\tau, W) \quad U \in GL_n(\mathbb{Z}) \end{aligned}$$

where $A(\tau), a(\tau), b(\tau, W)$ are analitic in τ, W and can be expressed explicitly in terms of A, B, C, D.

We now rewrite eq. 3.4 as

$$\begin{aligned} \tau_0 &= \gamma'\tau_0 \\ W_0 &= A(\tau_0)W_0 + a(\tau_0) \\ Z_0 &= Z_0[U] + b(\tau_0, W_0) + c, \quad \text{where } c \in L(\sigma) \otimes \mathbb{C} \text{ and} \\ L(\sigma) &\text{ is the linear span of } \sigma. \quad U(\sigma) = \sigma \end{aligned}$$

By further decomposition of $C(F)$ we can assume that $U/\sigma = I$. Let $E_\sigma = (D(F)/U(F)_{\mathbb{Z}})_\sigma$. E_σ is γ-invariant and a fiber space over F and $F \times V(F)$. The eigenvalues of γ on the tangent space of E_σ at $(\tau_0, W_0, \overline{Z}_0 + \sigma\infty)$ can be obtained by calculating the eigenvalues on F, $V(F)$, and $T(F)_\sigma$ separately.

Assume the eigenvalues of γ are $\{\lambda_1, \ldots, \lambda_{n'}, \lambda_1^{-1}, \ldots, \lambda_{n'}^{-1}\}$, U has eigenvalues $\{\mu_1, \ldots, \mu_n\}$ and let Λ and M be the diagonal matrices with λ_i and μ_i as entries respectively.

Lemma 3.5.1.

1. *The eigenvalues of γ on the tangent space to F at τ_0 are $\lambda_i\lambda_j$, $1 \le i \le j \le n'$.*

2. *The eigenvalues of γ on the tangent space to $V(F)$ at W_0 are $\lambda_i\mu_j$, $1 \le i \le n'$, $1 \le j \le n$.*

Proof: The lemma follows from diagonalizing γ' and U and noting that the action of γ around (τ_0, W_0) is the same as the local action of γ_1 at $(0, W_1)$ where

$$\gamma_1 = \begin{pmatrix} \Lambda & 0 & 0 & * \\ * & M & * & * \\ 0 & 0 & \Lambda & * \\ 0 & 0 & 0 & M \end{pmatrix}, \quad \gamma_1(\tau, W) = (\Lambda\tau\Lambda, \Lambda W M + a)$$

with $W_1 \in V(F)$ and $\Lambda W_1 M + a = W_1$ □

Let T be $T(F)$, T_σ the torus embedding associated to σ. From [9] we know that in T_σ there's a unique closed orbit O_σ and a subtorus T'_σ such that $T/T'_\sigma \approx O_\sigma$:

O_σ is defined by $X^r = 0,\ r \ge 0$ on $\sigma,\ r > 0$ on $\mathrm{Int}(\sigma)$,

T'_σ is defined by $\mathrm{Spec}\ \mathbb{C}\,[X^r]_{r \in L(\sigma) \cap U(F)/\mathbb{Z}}$.

Since we assumed $U/\sigma = I$, U acts trivially on T'_σ, and U fixes e_σ : the identity of O_σ. Now γ acts on T by $Z \to Z[U] + b \mod U(F)_{\mathbb{Z}}$, this action extends to T_σ, O_σ.

Lemma 3.5.2. *The eigenvalues of γ on the tangent space to T_σ at $\overline{Z}_0 + \sigma\infty$ are $\mu_{ij}, 1 \le i \le j \le n$ where $\mu_{ij} = \mu_j\mu_i$ if $\mu_i\mu_j \ne 1$.*

Proof: Since γ fixes $\overline{Z}_0 + \sigma\infty$, we have

$$Z_0 \equiv Z_0[U] + b + c \bmod U(F)_{\mathbb{Z}}$$

with $c \in L(\sigma) \otimes \mathbb{C}$. Let t_{Z_0} be the translation by Z_0, it is easy to see $t_{Z_0} \circ U = \gamma \circ t_{Z_0}$ on O_σ, since for any $Z \in U(F)_{\mathbb{C}}$

$$\gamma_0 \circ t_{Z_0}(Z) - t_{Z_0} \circ U(Z) \equiv c \mod U(F)_{\mathbb{Z}}.$$

In the O_σ-directions, the eigenvalues μ_{ij} are the same as the eigenvalues of U on the tangent space to e_σ. The latter eigenvalues contain all the $\mu_i\mu_j$ when $\mu_i\mu_j \neq 1$.

□

Remark 3.5.1. This result illuminates why $\overline{A}_g^0$ are the $\overline{A}_g$ singular points for $g \geq 3$, since the ramification divisor of π_F occurs when exactly one of $\lambda_i\lambda_j$, $\lambda_i\mu_j$, $\mu_i\mu_j \neq 1$, which can happen only if $g \leq 2$.

Proposition 3.5.1. *For $g \geq 5$, η a pluricanonical holomorphic form in $\overline{A}_g^0$, η extends to the desingularizations of the singular points in $\overline{A}_g \setminus A_g$.*

Proof: If $y_0 \in \overline{A_g} \setminus A_g$, $y_0 \notin \overline{A_g^0}$ then we have, as before

$$x_0 = (\tau_0, W_0, \overline{Z}_0 + \sigma\infty) \in E_\sigma$$
$$\gamma x_0 = x_0, \quad \gamma \in N(F)_{\mathbb{Z}}$$
$$y_0 = \pi_F(x_0).$$

By the two previous lemmas, the eigenvalues of γ on the tangent space at x_0 are $\lambda_i\lambda_j$, $\lambda_i\mu_k$ and μ_{jk} ($\mu_{jk} = \mu_i\mu_k$ if $\mu_i\mu_k \neq 1$). This is similar to the situation in the interior fixed points, by the same arguments as in the last 2 lemmas of the previous section, we have $\{\gamma, x_0\} \geq 1$. Hence η extends to the desingularizations over y_0.

□

Finally, using proposition 3.5.1 to extend forms to the desingularizations on singular cusp points, using proposition 3.4.1 to assure the condition for extending forms on quotient singularities through theorem 3.3.1, and using theorem 3.2.1 to understand the asymptotic behaviour of the dimension of the space of global forms, we've proven that:

Theorem 3.5.1. *For $g \geq 9$, $\dim \Gamma(\Omega^N(\tilde{A}_g)^{\otimes k}) \sim ck^N$, with $N = g(g+1)/2$, and $c > 0$. So A_g is of general type.*

Bibliography

[1] Avner Ash, David Mumford, Michael Rapoport, and Yung-sheng Tai. *Smooth Compactifications of Locally Symmetric Varieties*. Cambridge University Press, Jan. 2010. ISBN: 9780511674693. DOI: 10.1017/cbo9780511674693.

[2] Eric S. Barnes and Michael J. Cohn. "On the Inner Product of Positive Quadratic Forms". In: *Journal of the London Mathematical Society* s2-12.1 (Dec. 1975), pp. 32–36. ISSN: 0024-6107. DOI: 10.1112/jlms/s2-12.1.32.

[3] Ignacio Barros, Pietro Beri, Emma Brakkee, and Laure Flapan. *Kodaira dimension of moduli spaces of hyperkähler varieties*. 2022. DOI: 10.48550/ARXIV.2212.12586.

[4] David A Cox. *Primes of the form* $x^2 + ny^2$. en. 2nd ed. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Nashville, TN: John Wiley & Sons, Apr. 2013.

[5] Olivier Debarre. *Tores et variétés abéliennes complexes*. fr. EDP Sciences, 1999.

[6] Fred Diamond and Jerry Shurman. *A First Course in Modular Forms*. en. Graduate texts in mathematics. Springer Science+Business Media, Jan. 2005.

[7] Hans Grauert and Oswald Riemenschneider. "Verschwindungssätze Für Analytische Kohomologiegruppen Auf Komplexen Räumen". In: *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1970, pp. 97–109. ISBN: 9783540363446. DOI: 10.1007/bfb0060317.

[8] Friedrich Hirzebruch. "Elliptische Differentialoperatoren auf Mannigfaltigkeiten". In: *Festschrift zur Gedächtnisfeier für Karl Weierstraß 1815–1965*. VS Verlag für Sozialwissenschaften, 1966, pp. 583–608. ISBN: 9783663162810. DOI: 10.1007/978-3-663-16281-0_23.

[9] Günther Kempf, Finn F. Knudsen, David Mumford, and Bernard Saint-Donat. *Toroidal Embeddings 1*. en. 1973rd ed. Lecture notes in mathematics. Berlin, Germany: Springer, Sept. 1973.

[10] David Mumford. "Hirzebruch's Proportionality Theorem in the Non-Compact Case." In: *Inventiones mathematicae* 42 (1977), pp. 239–272. URL: http://eudml.org/doc/142502.

[11] C. Ziegler. "Jacobi forms of higher degree". In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 59.1 (Dec. 1989), pp. 191–224. ISSN: 1865-8784. DOI: 10.1007/bf02942329.