Week 9: Excess formula

We start with an example:

Let $C_i = (X_0 X_i = 0) \subseteq \mathbb{P}^2$, $i = 1, 2$

$C_1 \cap C_2 = \ell \cup \{p\}$

but "should be" 4 pts

$\Rightarrow$ ℓ should "account for" 3 pts

We deform the conics:

$F_i := X_0 X_i + t Q_i$, Q_i = conic "deformation" of C_i

For $t \neq 0 \Rightarrow F_1 \cap F_2 = \{p, p_1(t), p_2(t), p_3(t)\}$

we want to find $\lim_{t \to 0} p_j(t) \in \ell$, $j = 1, 2, 3$

Note $X_2 F_1 - X_1 F_2 = t(X_2 Q_1 - X_1 Q_2)$

Its restriction to ℓ:

$$\underset{\neq 0}{t}\underbrace{\left(X_2 Q_1(0, X_1, X_2) - X_1 Q_2(0, X_1, X_2) \right)} = 0$$

has three roots $p_j = \lim_{t \to 0} p_j(t) \in \ell$

These are the points in ℓ that we are seeing in the Chow-theoretic intersection $[C_1] \cdot [C_2] = [p] + \sum_{j=1}^{3} [q_j]$

Of course, p_j depend on the choice of Q_i, but their classes in $CH^{\cdot}$ don't.

Idea: think $\ell = (x_0 = 0) \subset D = (x_0 x_1)$

$D \in |\mathcal{O}_{\mathbb{P}^2}(2)|$

ℓ & D should intersect at 1 pt.

$\rightsquigarrow$ replace D by <u>another</u> section of $\mathcal{O}_{\mathbb{P}^2}(2)$

(similar with $x_0 x_2$)

· <u>More generally</u>

$i : C \hookrightarrow X$ where $C \subseteq D \subseteq X$

$D = (s = 0)$, $s \in H^0(X, \mathcal{L})$

Say C = curve

We want to replace s by s' so that

$(s' = 0) \cap C$ intersect transversely

<u>Obs</u>: $\mathcal{L}|_D = N_{D/X}$

so $[C] \cdot [D] = \underbrace{i_* c_1(N_{D/X}|_C)}$

will appear in the excess formula!

Can think of s' as D_t = deformation of $D = D_0$

so $D_t \cap C = \{ p_1(t), \dots, p_k(t) \}$ $t \neq 0$

$D_t \longleftrightarrow$ section of $N_{D/X}$

<u>Deformation to the Normal Cone</u>

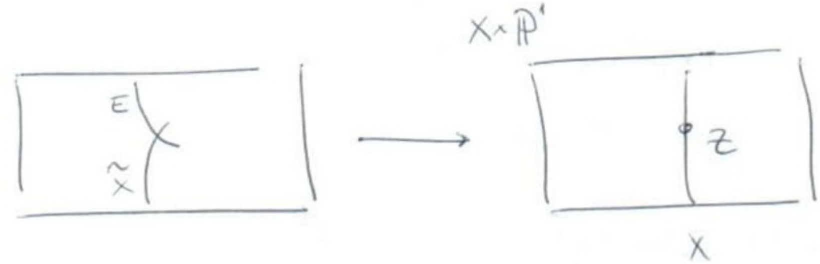
· Let $Z \subset X$ smooth subvariety

$M = Bl_{Z \times \{0\}} X \times \mathbb{P}^1$ $\qquad E = \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z) \hookleftarrow \mathbb{P}\mathcal{O}_Z = Z$

$\widetilde{X} = Bl_Z X$

$$\begin{array}{c} E \\ \cup \\ \mathbb{P}N_{Z/X} = \widetilde{X} \cap E \end{array}$$

M



Recall $CH^\cdot E = \dfrac{CH^\cdot Z[\xi]}{\xi^{r+1} + \xi^r c_1(N) + \ldots + \xi c_r(N)}$

Note in $CH^\cdot E$ $\quad 0 = \xi \cdot (\xi^r + \xi^{r-1} c_1(N) + \ldots + c_r(N))$

$$= [\mathbb{P}N_{Z/X}] \cdot [\mathbb{P}\mathcal{O}_Z]$$

where $\mathbb{P}N_{Z/X} \cap \mathbb{P}\mathcal{O}_Z = \phi$

$(r = \operatorname{rk} N_{Z/X} = \operatorname{codim}(Z \subset X))$

Excision:

generated by $CH.Z$ & ξ^j

$$\begin{tikzcd} CH.(\mathbb{P}N) \arrow[r] & CH.E \arrow[r] & CH.N \arrow[r] & 0 \\ & CH^\cdot Z \arrow[u] \arrow[ur] & & \end{tikzcd}$$

onto since $\xi^j \mapsto 0$

$\Longrightarrow \quad \pi^* : CH^\cdot Z \twoheadrightarrow CH^\cdot N$

Lemma

$\pi^* : CH^\cdot Z \xrightarrow{\sim} CH^\cdot N$

Proof:

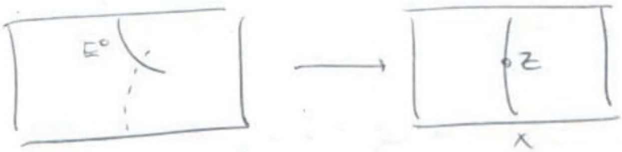
It is onto, so it suffices to find a left-inverse.

take $\quad \beta \longmapsto \pi_*(\bar{\beta} \cdot [\mathbb{P}\mathcal{O}_Z])$

where $\bar{\beta}$ is the closure in $\mathbb{P}(N \oplus \mathcal{O}_Z)$

It is an inverse by projection formula. □

Now let $M^\circ := M \setminus \tilde{X} \to \mathbb{P}^1$



We are "replacing" X by the normal cone $E^\circ = N_{Z/X}$
(of Z in X)

<u>Thm</u>

The map
$$\begin{array}{ccc} CH^\bullet X & \xrightarrow{\sigma} & CH^\bullet N_{Z/X} \\ [B] & \longmapsto & [C_{B\cap Z/B}] \end{array}$$
is well defined

where $C_{B\cap Z/B}$ = normal cone of $B \cap Z$ in B

($= N_{B\cap Z/B}$ if $B\cap Z$ & B are smooth).

<u>Proof</u>. Excision

$$\begin{tikzcd} CH_{\bullet+1}(N) \arrow[r, "i_*"] & CH_{\bullet+1}(M^\circ) \arrow[r] \arrow[d, "i^*"] & CH_{\bullet+1}(X \times \mathbb{A}^1) \arrow[r] & 0 \\ & CH_\bullet(N) & CH_\bullet(X) \arrow[u, "\cong"] \arrow[l, dashed] & \end{tikzcd}$$

well defined since $i^* \circ i_* = 0$

<u>Corollary</u>: Let $j : Z \hookrightarrow X$

Then
$$\begin{tikzcd} CH^\bullet X \arrow[r, "\sigma"] \arrow[rr, bend right, "j^*"'] & CH^\bullet N \arrow[r, "(\pi^*)^{-1}"] & CH^\bullet Z \end{tikzcd}$$
is well defined!

j^* is called the "Gysin pullback".

Remark:
Using j^* we can define intersection products with $[Z]$.

Recall: on $E = \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$

$$0 \to \mathcal{O}_E(-1) \to \pi^*(N \oplus \mathcal{O}_Z) \to Q \to 0 \quad \text{tautological}$$

$$Z = \mathbb{P}\mathcal{O}_Z \subseteq \mathbb{P}(N \oplus \mathcal{O}_Z)$$

$$\overset{\shortparallel}{(s=0)}, \quad s \in H^0(E, Q)$$

For $\beta \in CH_k(N)$

$$(\pi^*)^{-1}\beta = \pi_*(\bar{\beta} \cdot [\mathbb{P}\mathcal{O}_Z])$$

$$= \pi_*(\bar{\beta} \cdot c_r(Q))$$

$$= \pi_*(\bar{\beta} \cdot c(Q))_{k-r}$$

$$c(Q) = \frac{c(\pi^*(N \oplus \mathcal{O}_Z))}{c(S)}$$

$$= c(\pi^*(N \oplus \mathcal{O}_Z)) \cdot (1 + \xi + \xi^2 + \cdots)$$

$$\therefore \underset{\substack{\cap \\ CH_k(Z)}}{(\pi^*)^{-1}\beta} = [c(N) \cdot \underbrace{\pi_*(\bar{\beta} \cdot (1 + \xi + \xi^2 + \cdots))}_{\text{total Segre class}}]_{k-r}$$

Recall: For $F \to X$ a v.b. of rank r

$$s(F) := \pi_*(1 + \xi + \xi^2 + \cdots) \quad \text{"Segre class"}$$

where $\pi : \mathbb{P}(F \oplus \mathcal{O}_X) \to X$

Note $s_i(F) = \pi_*(\xi^{r+i})$

Remark

Segre class can also be defined on normal cone:

Recall $C_{B\cap Z/Z} = \underline{\mathrm{Spec}}_Z \bigoplus_{d\geq 0} I^d / I^{d+1}$

where I = ideal of $B\cap Z \subseteq Z$.

The Segre class of $C_{B\cap Z/Z}$ is defined using

$$\mathbb{P}(C_{B\cap Z/Z} \oplus \mathcal{O}_Z) \longrightarrow Z$$

Now let $[B] \in CH^\cdot X$

$$\sigma([B]) = [C_{B\cap Z/B}] =: \beta$$

By the previous computation:

$$\bar{\beta} = [\mathbb{P}(C_{B\cap Z/B} \oplus \mathcal{O}_Z)] \in CH^\cdot \mathbb{P}(N \oplus \mathcal{O}_Z)$$

$$\& \quad j^*([B]) = [c(N_{Z/X}) \cdot g_* s(C_{B\cap Z/B})]_{k-r}$$

$k = \dim B$

or: $\boxed{[B]\cdot[Z] = [g^* c(N_{Z/X}) \cdot s(C_{B\cap Z/B})]_{k-r}}$ $\in CH^\cdot(B\cap Z)$

where $g : B\cap Z \hookrightarrow Z$

This is called the Excess Formula

Remark

1) This is a formula for the "Chow-theoretic" intersection of B & Z, and it is supported in $B\cap Z$.

2) Here we have assumed that Z is smooth, B not necessarily.

If all three of $Z, B, B\cap Z$ are smooth, we can write the Excess Formula as

$$\boxed{[B]\cdot[Z] = \left[\frac{c(N_{Z/X}|_{B\cap Z})\cdot c(N_{B/X}|_{B\cap Z})}{c(N_{B\cap Z/X})}\right]_{k-r}}.$$

using the normal sequence on $B\cap Z \subseteq Z$

Remark

This is enough to define intersection product for arbitrary cycles $A, B \subseteq X$, not necessarily smooth (without using the moving lemma):

use $\Delta : X \longrightarrow X\times X$

& take the Gysin pullback $\Delta^*(A\times B)$.

The Excess Formula can be written a bit more generally as:

Thm (Excess Formula)

Suppose $S \subset X$

$T \subset X$, T is local complete intersection

$$\Longrightarrow \boxed{[S]\cdot[T] = \sum_D (i_D)_* \gamma_D}$$

where:

D = connected components of $S\cap T$

$i_D : D \hookrightarrow X$ the inclusion

& $\boxed{\gamma_D = \left[s(C_{D/S})\cdot c(N_{T/X}|_D)\right]_d} \in CH_d D$

$d = \dim X - \operatorname{codim} S - \operatorname{codim} T$

If S is also l.c.i. then

$$\boxed{\gamma_D = \left[s(C_{D/X}) \cdot c(N_{S/X}|_D) \cdot c(N_{T/X}|_D) \right]_d}$$

Example

$S_1, S_2, S_3 \subseteq \mathbb{P}^3$ surfaces of degree d_1, d_2, d_3

with $S_1 \cap S_2 \cap S_3 = L \cup \Gamma$ $\qquad L =$ line

$\Gamma =$ finite number of pts

We want to know $\deg \Gamma$ ($=$ # isolated pts in $S_1 \cap S_2 \cap S_3$)

Say S_1 is smooth & let

$$\left. \begin{array}{l} S_1 \cap S_2 = L + D \\ S_1 \cap S_3 = L + F \end{array} \right\} \in Z_1(S_1)$$

with D, F divisors in S_1

$\Rightarrow$ $S_1 \cap S_2 \cap S_3 = L \sqcup \Gamma$, where $\Gamma = D \cap F$

$H :=$ hyperplane section in S_1

$D \sim d_2 H - L$

$F \sim d_3 H - L$

$$[D] \cdot [F] = d_2 d_3 H^2 - (d_2 + d_3) H L + L^2$$

$$= d_1 d_2 d_3 - (d_2 + d_3) + L^2$$

By excess formula (or by adjunction):

$$L^2 = i_* c_1(N_{L/S_1}) = 2 - d_1$$

$$\Rightarrow \quad |\Gamma| = d_1 d_2 d_3 - (d_1 + d_2 + d_3) + 2$$

$$\left(d_1 + d_2 + d_3 - 2 = \text{"excess"} \right)$$