Teoría de Intersección (Semana 6): Clases de Chern (parte II)

Splitting lemma: Podemos asumir siempre $E = L_1 \oplus \dots \oplus L_r$.

Dada $X \subseteq \mathbb{P}^n$ proy, $F(X) := \{ \ell \subseteq X \text{ rectas} \} \subseteq G(1,n)$ es la var. de Fano de rectas de X.

$\dim(X) = 0 \Rightarrow \#$ rectas en $X = \# F_1(X)$

$\dim(X) > 0 \rightarrow$ Inf. rectas, param. por $m = \dim F(X)$ param. $\longleftrightarrow L \subseteq X = V(f)$.

$\Psi := \{ ([f],[L]) \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d)) \times G(1,n),\ f|_L = 0 \}$

$\dim H^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d} = N+1$, ie, $[f] \in \mathbb{P}^N \longleftrightarrow X_d^f = \{x \in \mathbb{P}^n,\ f(x)=0\}$

Aquí: $pr_1 : \Psi \longrightarrow \mathbb{P}^N$, $([f],[L]) \mapsto [f]$ con $pr_1^{-1}([f]) \simeq F(X_d^f)$.

y $pr_2 : \Psi \rightarrow G(1,n)$, $([f],[L]) \mapsto [L]$, $pr_2^{-1}([L]) = \{ X_d^f \text{ tq } L \subseteq X_d^f \}$

¿Cuándo f se anula a L?

Cambio de coord: $L = \{ x_2 = \dots = x_n = 0 \}$

$f = \sum a_I x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. $\Rightarrow f|_L = 0$

$f|_L = 0 \iff f \notin \langle x_0^d, x_0^{d-1}x_1, \dots, x_1^d \rangle \leftarrow \dim = d+1$

$\iff a_I = 0 \ \forall I = (\alpha_0, \alpha_1, 0, \dots, 0)$

Así, $f|_L = 0$ impone $d+1$ cond. lineales.

$\rightarrow \dim pr_2^{-1}([L]) = N - (d+1)$ de dim constante

$\Rightarrow \dim \Psi = N - (d+1) + \dim G(1,n) = 2n-2 + N - (d+1) = N + 2n - d - 3$.

$\dim \Psi = \dim \underbrace{pr_1^{-1}([f])}_{\text{general}} + N$

$\Rightarrow \boxed{\dim F(X_d^f) = 2n - d - 3}$ si X_d^f genérica

Conj (Debarre - de Jong): $d \leq n$ y X_d suave $\Rightarrow \dim F(X_d) = 2n-d-3$

Eg. $d = 3$, $n = 3 \Rightarrow \dim F(X_3) = 0$.

Recuerdo $E \rightarrow X$ $\operatorname{rg} E = r$ glob. gen, $\sigma_0, \dots, \sigma_{r-i} \in H^0(X,E)$ generales
$\Rightarrow [Z(\sigma_0 \wedge \dots \wedge \sigma_{r-i})] = c_i(E)$. y luego $[Z(\sigma_0)] = c_r(E)$.

Considere $E := \operatorname{Sym}^d(S^\vee) \longrightarrow G(1,n)$ con fibra $H^0(L, \mathcal{O}_L(d))$

$\Rightarrow [F(X_d)] = c_{d+1}(E) \in A^{d+1}(G(1,n))$.

$\hookrightarrow$ Aquí: Para f vos $\sigma_f : G(1,n) \rightarrow E$, $L \mapsto f|_L$ y así $Z(\sigma_f) = F(X_d^f)$

Sup. $S^\vee = L_1 \oplus L_2$ y escribimos $x = c_1(L)$, $y = c_1(L_2)$

$$\Rightarrow Sym^d(S^\vee) \cong \bigoplus_{i=0}^{d} L_1^{\otimes i} \otimes L_2^{\otimes d-i} \quad \text{y} \quad c_1(L_1^{\otimes i} \otimes L_2^{\otimes d-i}) = ix + (d-i)y$$

$$c(E) = \prod_{i=0}^{d} (1 + ix + (d-i)y) = 1 + (gr=1) + (gr=2) + \dots + (gr = d+1)$$

$$c_{d+1}(E) \in \langle \sigma_{n-1,n-1} \rangle$$

$$\boxed{\begin{array}{l} 2n - d - 3 = 0 \\ \text{ie, } (n-1)+(n-1) = d+1 \ \checkmark \end{array}}$$

$$c_{d+1}(E) = \prod_{i=0}^{d} (ix + (d-i)y)$$

simétrico en x,y y luego gen por $\underbrace{x+y}_{= c_1(S^\vee)}$, $xy = c_2(S^\vee)$

$$\left(c(S^\vee) = c(L_1)(c(L_2)) = (1+x)(1+y) = 1 + x + y + xy \right)$$

En $A^*(G(1,n))$, $\sigma_1 = x+y = c_1(S^\vee)$, $\sigma_{1^2} = \sigma_{1,1} = xy = c_2(S^\vee)$

Ejemplo: $X_3 \subseteq \mathbb{P}^3$ sup. cúbica $(d=3, n=3) \Rightarrow F(X_3) = \{L_1, \dots, L_m\}$

$$\Rightarrow [F(X_3)] = c_4(E) = \prod_{i=0}^{3} (ix + (3-i)y) = (3y)(x+2y)(2x+y)(3x)$$

$$= 9xy\,(2(x+y)^2 + xy) = 9\,c_2(S^\vee)\,(2\,c_1^2(S^\vee) + c_2(S^\vee))$$

cf Semana 1

$$= 9\,\sigma_{1,1}\,(2\sigma_1^2 + \sigma_{1,1}) = m\,\sigma_{2,2} \quad (\star)$$

Recuerdo (Pieri): $\sigma_1 \cdot \sigma_1 = \sigma_2 + \sigma_{1,1}$ y $\sigma_{1,1}^2 = \sigma_1^4 - 2\sigma_1^2\sigma_2 + \sigma_2^2$, $\sigma_1^4 = 2\sigma_{2,2}$

$$\star = 9\,\sigma_{1,1}\,(2(\sigma_2 + \sigma_{1,1}) + \sigma_{1,1}) = 9\,\sigma_{1,1}\,(2\sigma_2 + 3\sigma_{1,1}) = 18\,\sigma_{1,1}\sigma_2 + 27\,\sigma_{1,1}^2$$

$$\sigma_{1,1}^2 = \sigma_1^4 - 2\sigma_1^2\sigma_2 + \sigma_2^2 = 2\cancel{\sigma_{2,2}} - 2\cancel{\sigma_{2,2}} + \sigma_2^2 = \sigma_2^2 = \sigma_{2,2}$$

$\sigma_{1,1}\sigma_2 = 0$

No hay rectas en un plano y estan cont en un plano.

$$\Rightarrow \boxed{m = 27}$$