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Counting rational points on Hirzebruch–Kleinschdmidt varieties over global fields

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To Dante and Dmitri

Abstract

We study the asymptotic growth of the number of rational points of bounded height on smooth projective split toric varieties with Picard rank 2 defined over a global field K, with respect to Arakelov height functions associated to big metrized line bundles. In the case of $\mathrm{char}(K) = 0$, we use Hermitian vector bundles to relate the height zeta function over a certain open subset of our studied varieties with the height zeta function of the base projective space, thereby studying its analytical behavior. In the case of $\mathrm{char}(K) > 0$, we use the description of the equations defining the variety to perform explicit calculations on the height zeta function over the corresponding open subset.

Contents

Introduction

Given an equation with integer coefficients (or a system of equations), an ancient problem is finding solutions in the ring of integers, which in some cases is equivalent to finding them in the field of rational numbers. Although this is a beautiful problem, it is generally very difficult to tackle. A significant portion of modern ideas for addressing this problem involves considering the algebraic varieties that the equation (or system) defines in some affine or projective space and using the available machinery in algebraic geometry.

There is a correspondence between the non-trivial solutions of the equations and the rational points of the variety. In this sense, given an abstract variety X defined over a field K (assumed not algebraically closed), many questions arise about its rational points, that is, the set of morphisms $\mathrm{Spec}(K) \to X$. These questions cover, for example, the existence and nature of its cardinality. In the case of finite cardinality, one might ask for the exact number or even dare to list all of them. In the case of infinite cardinality, questions about asymptotic low or high bounds and their distribution on the variety arise. It is this latter nature of investigating the rational points of varieties that we will adopt.

To study the distribution of rational points, it is common to work with the concept of height and investigate the asymptotic behavior of rational points of bounded height. Heights are a way to measure the arithmetic complexity of these points.

In the projective space $\mathbb{P}^n$, for $\mathbb{Q}$-rational points, there is the naive height given by:

$$H_{\mathbb{P}^n}(x) = \max\{|x_0|, |x_1|, \ldots, |x_n|\}$$

where $x = [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n(\mathbb{Q})$, with integer coordinates x_i and $\gcd(x_0, x_1, \ldots, x_n) = 1$. This height ensures that the set $\{x \in \mathbb{P}^n(\mathbb{Q}) : H_{\mathbb{P}^n}(x) \leq B\}$ of rational points with bounded height is finite.

For a global field K, the naive height is given by

$$H_{\mathbb{P}^n}(x) = \prod_{v \in \mathrm{Val}(K)} \max\{|x_0|_v, |x_1|_v, \ldots, |x_n|_v\}$$

which is well-defined for all $x \in \mathbb{P}^n(K)$ by the product formula.

Given a smooth projective algebraic variety X defined over a global field K and L an ample line bundle over X, with $L^{\otimes m}$ very ample that induces the closed embedding $\iota : X \hookrightarrow \mathbb{P}^n$, we define the height on the rational points of X by:

$$H_L(x) = H_{\mathbb{P}^n}(\iota(x))^{\frac{1}{m}}$$

for $x \in X(K)$. This definition leverages the naive height on projective space and translates it to the context of the variety X via the embedding induced by the power $L^{\otimes m}$ of the line bundle L that is very ample.

In this way, we are interested in studying the asymptotic behavior of the cardinality

$$N(X, L, B) := \#\{x \in X(K) : H_L(x) \leq B\}.$$

It is known that this number can be dominated by the number of rational points in certain subvarieties of X called accumulating subvarieties. Therefore, to have a proper analysis of the asymptotic behavior of rational points of bounded height, it is necessary to remove these subvarieties, and in the complementary open subset U, Manin conjectured that in the case where the anticanonical bundle is ample, that is, the variety X is Fano, we have

$$N(U, -K_X, B) = CB \log B^{t-1}(1 + o(1)), \quad B \to \infty.$$

Furthermore, t should be equal to $\operatorname{rk} \operatorname{Pic}(X)$.

In 1996, Batyrev and Tschinkel [3] provided a counterexample to the first version of Manin's Conjecture. Nowadays, the current expectation is as follows in the case of number fields (see e.g. [28, Formule empirique 5.1] or [1, Conjecture 6.3.1.5] for a detailed discussion and the precise definition of the relevant concepts).

Conjecture 1 (Manin–Peyre)**.** *Let X be an almost Fano variety*[1] *over a global field K, with dense set of rational points $X(K)$, finitely generated $\Lambda_{\mathrm{eff}}(X_{\overline{K}})$ and trivial Brauer group $\operatorname{Br}(X_{\overline{K}})$. Let $H = H_{-K_X}$ be the anticanonical height function, and assume that there is an open subset U of X that is the complement of the weakly accumulating subvarieties on X with respect to H. Then, there is a constant $C > 0$ such that*

$$N(U, H_{-K_X}, B) \sim CB(\log B)^{\operatorname{rk}\operatorname{Pic}(X)-1} \quad \text{as } B \to \infty. \tag{0.0.1}$$

Moreover, the leading constant is of the form

$$C = \alpha(X)\beta(X)\tau_H(X),$$

where

$$\alpha(X) := \frac{1}{(\operatorname{rk}\operatorname{Pic}(X) - 1)!}\int_{\Lambda_{\mathrm{eff}}(X)^{\vee}} e^{-\langle -K_X, \mathbf{y}\rangle}\mathrm{d}\mathbf{y},$$
$$\beta(X) := \#H^1(\mathrm{Gal}(\overline{K}/K), \operatorname{Pic}(X_{\overline{K}})),$$

and $\tau_H(X)$ is the Tamagawa number of X with respect to H (see Section 1.5.2).

There is also a conjecture, originating in the work of Batyrev and Manin [2], concerning the asymptotic growth of the number

$$N(U, H_L, B) = \#\{P \in U(K) : H_L(P) \leq B\},$$

[1] Following [28, Définition 3.1], an almost Fano variety is a smooth, projective, geometrically integral variety X defined over a field K, with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, torsion-free geometric Picard group $\operatorname{Pic}(X_{\overline{K}})$ and $-K_X$ big.

when considering height functions H_L associated to big metrized line bundles L on a variety X as above, and for appropriate open subsets $U \subset X$. More precisely, if we denote by $\tau \preceq \sigma$ whenever τ is a face of a cone σ, then one defines the following classical numerical invariants for a big line bundle L on X:

$$a(L) := \inf\{a \in \mathbb{R} : aL + K_X \in \Lambda_{\mathrm{eff}}(X)\},$$
$$b(L) := \max\{\operatorname{codim}(\tau) : a(L)L + K_X \in \tau \preceq \Lambda_{\mathrm{eff}}(X)\},$$

which measure the position of L inside the cone $\Lambda_{\mathrm{eff}}(X)$. With this notation, the more general version of the above conjecture states that there exists a constant $C > 0$ such that

$$N(U, H_L, B) \sim C B^{a(L)} (\log B)^{b(L)-1} \quad \text{as } B \to \infty. \tag{0.0.2}$$

We refer the reader to [4] for extensions, and a conjectural description of the leading constant C in terms of geometric, cohomological and adelic invariants associated to U and L.

The above conjectures have been proven by various authors, either in specific examples or in certain families of varieties (see for instance [36] for an account of such results).

Typically, asymptotic formulas of the form (0.0.1) and (0.0.2) are deduced, via a Tauberian theorem (see Section 1.6 for a precise statement), from analytic properties of the associated height zeta functions. More precisely, considering a height function H_L associated to a big metrized line bundle L, its height zeta function is defined as

$$\zeta_{U,L}(s) := \sum_{P \in U(K)} H_L(P)^{-s} \quad \text{for } s \in \mathbb{C} \text{ with } \Re(s) \gg 0.$$

If $\zeta_{U,L}(s)$ converges absolutely on $\Re(s) > a > 0$, it has an analytic continuation to $\Re(s) > a - \varepsilon$, $s \neq a$ for some $\varepsilon > 0$ and it has a pole of order $b \geq 1$ at $s = a$, then one obtains

$$N(U, H_L, B) \sim C B^{a} (\log B)^{b-1} \quad \text{as } B \to \infty, \tag{0.0.3}$$

with

$$C := \frac{1}{(b-1)!\,a} \lim_{s \to a} (s-a)^b \zeta_{U,L}(s)$$

In this thesis we focus on smooth projective split toric varieties with Picard rank 2, and consider Arakelov height functions associated to big metrized line bundles. In order to be more precise, we recall a geometric result due to Kleinschmidt [19] stating that all smooth projective toric varieties of Picard rank 2 are (up to isomorphism) of the form

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r)), \tag{0.0.4}$$

where $r \geq 1$, $t \geq 2$, and $0 \leq a_1 \leq \cdots \leq a_r$ are integers. We refer to these varieties as Hirzebruch–Kleinschmidt varieties.

As is well known, global fields are completely classified. These are number fields, i.e., finite extensions of $\mathbb{Q}$, and global function fields, i.e., finite extensions of the rational function field of a smooth, irreducible, geometrically integral curve defined over a finite field $\mathbb{F}_q$. We will deal with each of these cases separately.

Number Fields case

For computational convenience, we choose a different normalization than the given in (0.0.4) and put

$$X_d(a_1,\dots,a_r) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1}-a_r)) \simeq X,$$

with $d := r+t-1$ the dimension of $X_d(a_1,\dots,a_r)$.

Our results show that these varieties can be naturally decomposed into a finite disjoint union of subvarieties where explicit asympotic formulas for the number of rational points of bounded height can be given, in the spirit of Schanuel's work [34]. We achieve these results by using suitable algebraic models for such varieties and by performing explicit computations on the associated height zeta functions.

If $X = X_d(a_1,\dots,a_r)$, then we consider the projective subbundle $F := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1}-a_r))$ and note that $F \simeq X_{d-1}(a_1,\dots,a_{r-1})$ when $r>1$, while $F \simeq \mathbb{P}^{t-1}$ when $r=1$. Then, we define the "good open subset"

$$U_d(a_1,\dots,a_r) := X \setminus F.$$

We note that this open subset is larger than the dense toric orbit of X.

Our main result gives an asymptotic formula for $N(U_d(a_1,\dots,a_r), H_L, B)$ of the form (0.0.2), with an explicit constant $C = C_{L;K}$, for every big divisor class $L \in \mathrm{Pic}(X)$ that we equip with a "standard metrization". This general result is given in Section 2.10 (see Theorem 70). In the case of the anticanonical line bundle wich in a smooth toric variety is always big, our result take the following form

Theorem 2. *Let $X = X_d(a_1,\dots,a_r)$ be a Hirzebruch–Kleinschmidt variety of dimension $d = r+t-1$ over a number field K, let $H = H_{-K_X}$ denote the anticanonical height function on $X(K)$, and let $U = U_d(a_1,\dots,a_r)$ be the good open subset of X. Then, we have*

$$N(U,H,B) \sim CB\log(B) \quad \text{as } B \to \infty,$$

with

$$C := \frac{R_K^2 h_K^2 |\Delta_K|^{-\frac{(d+2)}{2}}}{w_K^2 (r+1)(t+(r+1)a_r - |a|)\xi_K(r+1)\xi_K(t)}, \tag{0.0.5}$$

where R_K, h_K, Δ_K are the regulator, class number and discriminant of K, respectively, $|a| := \sum_{i=1}^r a_i$ and

$$\xi_K(s) := \left(\frac{\Gamma(s/2)}{2\pi^{s/2}}\right)^{r_1}\left(\frac{\Gamma(s)}{(2\pi)^s}\right)^{r_2}\zeta_K(s),$$

with r_1 and r_2 the number of real and complex Archimedean places of K, respectively, and ζ_K the Dedekind zeta function of K.

Remark 3. In Lemma 26 we show that

$$\alpha(X) = \frac{1}{(r+1)(t+(r+1)a_r - |a|)}.$$

Also, since X is a split toric variety, we have $\beta(X) = 1$ (this is well-known and follows e.g. from [4, Remark 1.7 and Corollary 1.18] and [32, Lemma 2.21]). Hence, Theorem 2 together with the main theorem of [4] implies that the Tamagawa number of X with respect to the anticanonical height function H_{-K_X} is

$$\tau_H(X) = \frac{R_K^2 h_K^2 |\Delta_K|^{-\frac{(d+2)}{2}}}{w_K^2 \xi_K(r+1)\xi_K(t)}.$$

Since the good open subset U in Theorem 2 is obtained by removing from $X_d(a_1, \dots, a_r)$ the subbundle F, which is either another Hirzebruch–Kleinschmidt variety or a projective space, we can also describe the asymptotic behaviour of the number $N(F, H, B)$, provided the restricted divisor class $-K_X|_F$ is big in $\mathrm{Pic}(F)$ (it is easily seen that when this is not the case, one has $N(F, H, B) = \infty$ for every $B \geq 1$). In general, the restricted divisor $-K_X|_F$ does not coincide with the anticanonical divisor $-K_F$ of F (see Section 2.3), and this is the main reason why we are lead to study the asymptotic behaviour of $N(U, H_L, B)$ for height functions H_L associated to general big divisors. These ideas are illustrated in the following example.

Example 4. Given integers $0 \leq a_1 \leq a_2$ consider the Hirzebruch–Kleinschmidt threefold

$$X := X_3(a_1, a_2) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1 - a_2)),$$

with projection map $\pi : X \to \mathbb{P}^1$ and good open subset $U = U_3(a_1, a_2)$. Using results from Sections 2.2 and 2.3, we get that $-K_X = \mathcal{O}_X(3) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(2 + 2a_2 - a_1))$, and the restriction $-K_X|_F$ of the anticanonical divisor of X to the projective subbundle $F = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1 - a_2))$ corresponds, under the isomorphism $F \simeq X_2(a_1) =: X'$, to the line bundle $L := \mathcal{O}_{X'}(3) \otimes (\pi')^*(\mathcal{O}_{\mathbb{P}^1}(2 + 2a_1 - a_2))$, where $\pi' : X' \to \mathbb{P}^1$ is the corresponding projection map. If we write $X' = U' \sqcup F'$ with $U' = U_2(a_1)$ the good open subset of X' and F' its complement in X', then we have $F' \simeq \mathbb{P}^1$, and the restriction $L|_{F'}$ corresponds to the line bundle $M := \mathcal{O}_{\mathbb{P}^1}(2 - a_1 - a_2)$. Hence, we have a disjoint decomposition

$$X = U \sqcup U' \sqcup F' \simeq U_3(a_1, a_2) \sqcup U_2(a_1) \sqcup \mathbb{P}^1, \tag{0.0.6}$$

and denoting by $H = H_{-K_X}$ the anticanonical height function, we obtain

$$N(U, H, B) \sim CB \log(B) \quad \text{as } B \to \infty,$$

with an explicit constant $C > 0$ by Theorem 2. Now, by Lemma 33 in Section 2.3 the line bundle L is big if and only if $a_2 < 2a_1 + 2$, in which case Theorem 70 in Section 2.10 gives

$$N(U', H, B) = N(U_2(a_1), H_L, B) \sim C' B^{\frac{a_1+2}{2a_1+2-a_2}} \quad \text{as } B \to \infty,$$

with another explicit constant $C' > 0$. Finally, by Lemma 34 M is big if and only if $a_1 + a_2 < 2$, in which case Schanuel's estimate, in the form of Corollary 66 in Section 2.9, gives

$$N(F', H, B) = N(\mathbb{P}^1, H_M, B) \sim C'' B^{\frac{2}{2-a_1-a_2}} \quad \text{as } B \to \infty,$$

with yet another explicit constant $C'' > 0$. We can then distinguish several different cases in order to compare the contribution of each of the subsets U, U', F' in the decomposition (0.0.6) to the asymptotic growth of the number of rational points of bounded anticanonical height on X, as represented in the following table.

Case	Is L big?	Is M big?	Comparison
$(a_1, a_2) = (0, 0)$	Yes	Yes	$N(F', H, B) = o(N(U', H, B))$ $N(U', H, B) = o(N(U, H, B))$ $N(U, H, B) \sim CB\log(B)$
$(a_1, a_2) = (0, 1)$	Yes	Yes	$N(U, H, B) = o(N(U', H, B))$ $N(U', H, B) \sim C'B^2$ $N(F', H, B) \sim C''B^2$
$1 \leq a_1 < a_2 < 2a_1 + 2$	Yes	No	$N(F', H, B) = \infty$ $N(U, H, B) = o(N(U', H, B))$ $N(U', H, B) \sim C'B^{\frac{a_1+2}{2a_1+2-a_2}}$
$1 \leq a_1 = a_2$	Yes	No	$N(F', H, B) = \infty$ $N(U', H, B) = o(N(U, H, B))$ $N(U, H, B) \sim CB\log(B)$
$2a_1 + 2 \leq a_2$	No	No	$N(F', H, B) = \infty$ $N(U', H, B) = \infty$ $N(U, H, B) \sim CB\log(B)$

Since the values $r = t = 2$ are fixed, the constants C, C', C'' above depend only on the base field K and on the coefficients a_1, a_2. In order to give a concrete numerical example, let us for simplicity assume $K = \mathbb{Q}$ and choose $(a_1, a_2) = (0, 1)$. Using that $\xi_{\mathbb{Q}}(s) = (2\pi^{s/2})^{-1}\Gamma(s/2)\zeta(s)$, we get

$$C = \frac{\pi^2}{6\zeta(3)\zeta(2)} = 0.83190737\ldots,$$

while the values of the constants C' and C'' can be extracted from Example 77 and Corollary 66, and are given by

$$C' = \frac{3}{\pi}\left(1 + \frac{945\zeta(3)}{16\pi^3}\right) = 3.14147564\ldots, \qquad C'' = \frac{\pi}{2\zeta(2)} = \frac{3}{\pi} = 0.95492965\ldots.$$

In particular, this shows that U' "contributes more" than F' to the number of rational points of bounded anticanonical height in $X_3(0, 1)$.

In the general case, our strategy to study the number of rational points of bounded height on a Hirzebruch–Kleinschmidt varierty is as follows: Given $X = X_d(a_1, \ldots, a_r)$ we start by writing

$$X \simeq U_d(a_1, \ldots, a_r) \sqcup U_{d-1}(a_1, \ldots, a_{r-1}) \sqcup \cdots \sqcup U_t(a_1) \sqcup \mathbb{P}^{t-1}. \tag{0.0.7}$$

Then, starting with the anticanonical height function H_{-K_X} on $X(K)$, we give simple criteria to decide if the induced height functions H_i on $U_{t+i-1}(a_1, \ldots, a_i)(K)$ (for $1 \leq i \leq r$), and H_0 on $\mathbb{P}^{t-1}(K)$, are associated to big divisors in the corresponding Picard groups. Finally, using Theorem 70 in Section 2.10.1 together with Schanuel's estimate (Corollary 66 in Section 2.9), we give explicit asymptotic formulas for the numbers $N(U_{d+i-r}(a_1, \ldots, a_i), H_i, B)$ and $N(\mathbb{P}^{t-1}, H_0, B)$, obtaining the counting of rational points of bounded height on each piece of the decomposition (0.0.7). Note that this approach also works if we start with a height function H_L on $X(K)$ associated to $L \in \operatorname{Pic}(X)$ big.

Remark 5. Assume $a_1 = \ldots = a_j = 0$ and $0 < a_{j+1} \leq \ldots \leq a_r$, for some $j \in \{1, \ldots, r-1\}$. In this case, instead of (0.0.7), one can rather write

$$X \simeq U_d(a_1, \ldots, a_r) \sqcup U_{d-1}(a_1, \ldots, a_{r-1}) \sqcup \cdots \sqcup U_{t+j}(a_1, \ldots, a_{j+1}) \sqcup (\mathbb{P}^{t-1} \times \mathbb{P}^j), \quad (0.0.8)$$

and proceed as above by giving explicit asymptotic formulas for the numbers $N(U_{d+i-r}(a_1, \ldots, a_i), H_i, B)$ (for $j+1 \leq i \leq r$) and $N(\mathbb{P}^{t-1} \times \mathbb{P}^j, H_j, B)$. This variant seems more natural to us in this case since there is no gain in removing a closed subvariety of $X_{t+j-1}(a_1, \ldots, a_j) = \mathbb{P}^{t-1} \times \mathbb{P}^j$ when counting rational points of bounded height on this particular component in the decomposition (0.0.8).

We refer the reader to Section 2.10.4 where we apply the above strategy to the case of Hirzebruch surfaces $X = X_2(a)$ with $a > 0$ an integer. In particular, we recover a classical example going back to Serre [35], and revisited by Batyrev and Manin in [2] and by Peyre in [27] (see Remark 81).

Our proof of Theorem 2 (and of Theorem 70 for general big divisors) is based on the analytic properties of the associated height zeta functions, which we relate to height zeta functions of projective spaces and to the zeta function ξ_K of the base field K. Then, as usual, a direct application of a Tauberian theorem leads to the desired results. The analytic continuation and identification of the first pole of our height zeta functions is achieved via explicit computations, and by exploiting the concrete algebraic models of our Hirzebruch–Kleinschmidt varieties. As such, it would be interesting to investigate if the techniques used here can be applied to other families of algebraic varieties.

Remark 6. For the number fields case we came across with an unpublished manuscript of Maruyama [22], where an asymptotic formula for the number of rational points of bounded anticanonical height on Hirzebruch surfaces is proposed. Unfortunately, the proposed result is incorrect due to convergence issues of the relevant height zeta function, that occur when one does not remove the corresponding subbundle F of the Hirzebruch surface $X_2(a)$, as we have done here in greater generality. We remark, nevertheless, that the approach used in loc. cit. to study the analytic properties of height zeta functions of projective bundles has served us as an inspiration for this present work (see Section 2.9).

Function fields case

It is natural to ask for analogous statements like Conjecture 1 and (0.0.2) for global fields of positive characteristic. However, in the case that K is a global function field of positive characteristic, the relevant height functions have values typically contained in $q^{\mathbb{Z}}$ with q the cardinality of the constant subfield $\mathbb{F}_q \subset K$, and this implies that the associated height zeta functions are invariant under $s \mapsto s + \frac{2\pi i}{\log(q)}$. In particular, having a pole at a point $s = a > 0$ implies the existence of infinitely many poles on the line $\Re(s) = a$, which makes it impossible to apply a Tauberian theorem. Moreover, in this setting, it is hopeless to expect an asymptotic formula of the form (0.0.3) since $N(U, H_L, q^n) = N(U, H_L, q^{n+\frac{1}{2}})$ would imply $\sqrt{q} = 1$ (as explained in [8, Section 1.1]).

For this reason, for varieties defined over global fields of positive characteristic, the Manin Conjecture is presented in terms of the analytical properties of the respective height zeta function. In [29], Peyre succeeds to do this in the case of flag varieties. As a variant of Conjecture 1, it is then expected that the anticanonical height zeta function $\zeta_{U,-K_X}(s)$, for appropriate open subsets $U \subseteq X$, converges absolutely for $\Re(s) > 1$, it admits meromorphic continuation to $\Re(s) > 1 - \varepsilon$ for some $\varepsilon > 0$, and it has a pole at $s = 1$ of order $\operatorname{rk}\operatorname{Pic}(X)$ satisfying

$$\lim_{s \to 1}(s-1)^{\operatorname{rk}\operatorname{Pic}(X)}\zeta_{U,-K_X}(s) = \alpha(X)\beta(X)\tau_H(X), \tag{0.0.9}$$

where

$$\alpha(X) := \int_{\Lambda_{\text{eff}}(X)^\vee} e^{-\langle -K_X, \mathbf{y}\rangle} \mathrm{d}\mathbf{y},$$
$$\beta(X) := \# H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), \operatorname{Pic}(X_{K^{\mathrm{sep}}})),$$

and $\tau_H(X)$ is the Tamagawa number of X with respect to the anticanonical height function $H = H_{-K_X}$ defined by Peyre in [29, Section 2] in the case of global fields of positive characteristic. Similarly, as a variant of (0.0.2), one expects $\zeta_{U,L}(s)$ to converge absolutely for $\Re(s) > a(L)$ and have meromorphic continuation to $\Re(s) > a(L) - \varepsilon$ for some $\varepsilon > 0$ with a pole at $s = a(L)$ of order $b(L)$.

We show that similarly to the case of number fields, the varieties of Hirzebruch–Kleinschdmidt can be naturally decomposed into a finite disjoint union of subvarieties where precise analytic properties of the corresponding height zeta functions can be given. Hence, in this setting, we go beyond the scope of the classical expectations mentioned above. We achieve these results by using concrete algebraic models for such varieties and by performing explicit computations on the height zeta functions.

This ideas were motivated by the results of Bourqui in [6], [7] and [8], where the above expectation for the anticanonical height zeta function is verified for toric varieties over global fields of positive characteristic, with U the dense torus orbit. Bourqui's work was inspired by Batyrev and Tschinkel's proof of Conjecture 1 for toric varieties over number fields [4].

As usual, in order to avoid accumulation of rational points of bounded height, we find it necessary to restrict our attention to rational points in a dense open subset. To this purpose, we define the *good open subset* $U_d(a_1, \dots, a_r) \subseteq X_d(a_1, \dots, a_r)$ as the complement of the closed subvariety define by the equation $x_{tr} = 0$.

Our main results describe the analytic properties of the height zeta function $\zeta_{U,L}(s)$ for the good open subset $U := U_d(a_1, \dots, a_r) \subseteq X_d(a_1, \dots, a_r)$, for every big line bundle class $L \in \operatorname{Pic}(X)$ that we equip with a "standard metrization". For simplicity, we present here the statement for $L = -K_X$.

Theorem 7. *Let $X = X_d(a_1, \dots, a_r)$ be a Hirzebruch–Kleinschmidt variety of dimension $d = r+t-1$ over a global function field $K = \mathbb{F}_q(C)$, let $H = H_{-K_X}$ denote the anticanonical height function on $X(K)$, and let $U = U_d(a_1, \dots, a_r)$ be the good open subset of X. Then, the anticanonical height zeta function $\zeta_{U,-K_X}(s)$ converges absolutely for $\Re(s) > 1$, and it is a rational function on q^{-s}. In particular, it has meromorphic continuation to $\mathbb{C}$. Moreover, it has a pole of order two at $s = 1$ with*

$$\lim_{s \to 1}(s-1)^2\zeta_{U,-K_X}(s) = \frac{q^{(d+2)(1-g)}h_K^2}{\zeta_K(t)\zeta_K(r+1)((r+1)a_r - |a| + t)(r+1)(q-1)^2\log(q)^2}, \tag{0.0.10}$$

where h_K is the class number of K, g is the genus of C and ζ_K is the zeta function of K.

As mentioned before, we go beyond the classical expectations and study further the height zeta function of the complement of the good open subset inside $X_d(a_1, \dots, a_r)$. This is achieved by a natural decomposition of $X = X_d(a_1, \dots, a_r)$ of the form

$$X \simeq \begin{cases} X_{d-1}(a_1, \dots, a_{r-1}) \sqcup \mathbb{A}^r \sqcup \left(\bigsqcup_{2 \leq t' \leq t} U_{t'+r-1}(a_1, \dots, a_r) \right) & \text{if } r > 1, \\ \mathbb{P}^{t-1} \sqcup \mathbb{A}^1 \sqcup \left(\bigsqcup_{2 \leq t' \leq t} U_{t'}(a_1) \right) & \text{if } r = 1 \end{cases} \tag{0.0.11}$$

(see Section 3.5.1 for details), which allows us to work inductively on the dimension of X, and decompose $\zeta_{X, -K_X}(s)$ as a finite sum of height zeta functions of projective (and affine) spaces and height zeta functions of good open subsets of Hirzebruch–Kleinschmidt varieties of dimension $d = t + r - 1, t + r, \dots, t$. On the one hand, height zeta functions of projective (and affine) spaces are well understood (see Section 3.4). On the other hand, for each good open subset $U' = U_{d'}(a_1, \dots, a_r)$ in the decomposition (0.0.11), we get a contribution of a height zeta function of the form $\zeta_{U',L}(s)$ with $L \in \mathrm{Pic}(X_{d'}(a_1, \dots, a_r))$ generally distinct from the corresponding anticanonical class. Nevertheless, provided L is big, we can still describe the analytic properties of $\zeta_{U',L}(s)$ in detail, see Theorems 98 and 100 in Section 3.5.3. Note that the bigness condition on L is necessary, because when L is not big, $\zeta_{U',L}(s)$ has no finite abscissa of absolute convergence. The following example illustrates these ideas.

Example 8. Given an integer $a > 0$, let us consider the Hirzebruch–Kleinschmidt threefold $X := X_3(a)$ with projection map $\pi : X_3(a) \to \mathbb{P}^2$. The decomposition (0.0.11) in this case becomes

$$X \simeq \mathbb{P}^2 \sqcup \mathbb{A}^1 \sqcup U_3(a) \sqcup U_2(a). \tag{0.0.12}$$

Put $U := U_3(a)$ and $U' := U_2(a)$, $X' := X_2(a)$ and let $\pi' : X' \to \mathbb{P}^1$ denote the corresponding projection map. The anticanonical class on X is $-K_X = \mathcal{O}_X(2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^2}(3 - a))$. Using Theorem 7 we get that $\zeta_{U, -K_X}(s)$ converges absolutely in $\Re(s) > 1$ and has meromorphic continuation to $\mathbb{C}$ with a pole of order 2 at $s = 1$ with

$$\lim_{s \to 1} (s - 1)^2 \zeta_{U, -K_X}(s) = \frac{q^{5(1-g)} h_K^2}{\zeta_K(3)\zeta_K(2)(a + 3)2(q - 1)^2 \log(q)^2}.$$

By Lemma 87 the component $U_2(a)$ in (0.0.12) contributes with the height zeta function $\zeta_{U',L}(s)$ with $L := \mathcal{O}_{X'}(2) \otimes \pi'^*(\mathcal{O}_{\mathbb{P}^1}(3 - a)) \in \mathrm{Pic}(X')$ big but different from the anticanonical class $-K_{X'}$. Nevertheless, by Theorem 98 we know that $\zeta_{U',L}$ has meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$.

Now, by Lemma 87 the component $\mathbb{P}^2$ in (0.0.12) contributes with the height zeta function $\zeta_{\mathbb{P}^2}((3 - a)s)$ where $\zeta_{\mathbb{P}^2}(s)$ denotes the standard height zeta function of $\mathbb{P}^2$ defined in Section 3.4. We see that $\zeta_{\mathbb{P}^2}((3 - a)s)$ has no finite abscissa of absolute convergence when $a \geq 3$. Assuming $a < 3$, we can apply Theorem 84 to deduce that $\zeta_{\mathbb{P}^2}((3 - a)s)$ converges absolutely for $\Re(s) > \frac{3}{3-a}$ and has meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = \frac{3}{3-a} > 1$.

Finally, by Lemma 87 the component $\mathbb{A}^1$ in (0.0.12) contributes with the height zeta function $\zeta_{\mathbb{A}^1}(2s) = \zeta_{\mathbb{P}^1}(2s) - 1$, where $\zeta_{\mathbb{P}^1}(s)$ denotes the standard height zeta function of $\mathbb{P}^1$.

In particular, by Theorem 84 we know that $\zeta_{\mathbb{A}^1}(2s)$ converges absolutely in $\Re(s) > 1$ and has meromorphic continuation to $s \in \mathbb{C}$ with a simple pole at $s = 1$.

We conclude:

1. If $a = 1, 2$, then the number of rational points of bounded anticanonical height in $X_3(a)$ is "dominated" by the number of those points in the component $\mathbb{P}^2$ in (0.0.12).

2. If $a \geq 3$, then the component $\mathbb{P}^2$ in (0.0.12) has "too many" rational points of bounded height (in fact, $N(\mathbb{P}^2, -K_X, B) = \infty$ for all $B \geq 1$).

In both cases, we see that only after removing the closed subvariety $\mathbb{P}^2 \subset X_3(a)$ we obtain a height zeta function

$$\zeta_{X \setminus \mathbb{P}^2; -K_X}(s) := \zeta_{\mathbb{A}^1}(2s) + \zeta_{U; -K_X}(s) + \zeta_{U'; L}(s)$$

satisfying the analogue of Conjecture 1 over global function fields.

We can also include here the easy case $a = 0$, namely $X = X_3(0) \simeq \mathbb{P}^1 \times \mathbb{P}^2$. It follows from Theorem 100 that $\zeta_{X; -K_X}(s)$ converges absolutely on $\Re(s) > 1$, has meromorphic continuation to $\mathbb{C}$ and has a pole of order two at $s = 1$ with

$$\lim_{s \to 1}(s-1)^2 \zeta_{X; -K_X}(s) = \frac{q^{5(1-g)} h_K^2}{\zeta_K(3)\zeta_K(2)6(q-1)^2 \log(q)^2}.$$

In particular, in this easy case, there is no need to remove a closed subvariety of X in order to verify the analogue of Conjecture 1 over global function fields.

The techniques used were inspired by the work of Bourqui [6] on the anticanonical height zeta function on Hirzebruch surfaces. Combining Bourqui's ideas with some technical computations, we are able to express the height zeta function $\zeta_{U;L}(s)$, for a good open subset of a Hirzebruch–Kleinschmidt variety and for general big metrized line bundles L, as a rational function of degree 2 on the Dedekind zeta function $\zeta_K(s)$ of the base field K. In that regard, it would be interesting to investigate if this method can also be applied to other families of algebraic varieties defined over global function fields.

We now proceed to describe the structure of this thesis.

In Chapter 1, we present a quick introduction to the necessary geometric concepts, including line bundles, the effective cone, the ample cone, toric varieties, and toric bundles as well as arithmetic aspects such as heights, height zeta functions, and a more detailed description of Peyre's constant.

In Chapter 2, we work on the case of number fields. To do this we revisit the theory of Hermitian vector bundles over arithmetic curves and the notion of Arakelov degree of a Hermitian line bundle. Additionally, we state the Poisson–Riemann–Roch formula, and demonstrate several estimates on the number of non-zero sections of a Hermitian vector bundle that are key in our proofs. Then we define a "standard (Arakelov) height function" on the set of rational points of the projective space $\mathbb{P}^n$, in terms of the Arakelov degree of tautological Hermitian line bundles. With these ideas in mind, we revisit Maruyama's proof of

Schanuel's estimate on the number of rational points of bounded standard height on projective spaces (see Corollary 66). Finally, we proceed to state and prove our main results on the asymptotic growth of the number of rational points in Hirzebruch–Kleinschmidt varieties, with respect to any big divisor class (Theorems 70 and 76), and provide more examples to illustrate the scope of our results.

In Chapter 3, we work on the case of function fields. First, we present several lemmas used in the proof of the Theorem 98. The proof is divided into four steps, with the central idea being to relate the height zeta function of an open subset of the variety to the Dedekind zeta function of the base field.

Chapter 1

Preliminaries

1.1 Some geometric concepts

Here we follow the exposition made by Tschinkel in [36].

Let X be a variety, which in this Thesis will be assumed to be irreducible, reduced and separated schemes of finite type over the base field K, we refer the reader to [15] for details. The Picard group of X is defined as

$$\mathrm{Pic}(X) = \{\text{Line bundles over } X\}/\text{isomorphism}$$

endowed with the tensor product. The identity of $\mathrm{Pic}(X)$ is the trivial line bundle. Also we have

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\,\mathrm{PDiv} = \mathrm{Div}(X)/(\mathbb{C}(X)^*/\mathbb{C}^*),$$

where $\mathrm{Div}(X)$ is the group of Cartier divisors and $\mathrm{PDiv}(X)$ is the subgroup of principal Cartier divisors.

A Cartier divisor $D = [(U_i, f_i)] \in \mathrm{Div}(X)$, induces a line bundle $\mathcal{O}_X(D)$ given by the transition functions $g_{ij} = f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)$.

Given $L \in \mathrm{Pic}(X)$ the induced rational application

$$\varphi_L : X \dashrightarrow |H^0(X, L)| = \mathbb{P}(H^0(X, L)^\vee) = \{\text{hyperplanes} \subset H^0(X, L)\}$$

is defined by

$$x \mapsto M_x := \{s \in H^0(X, L) \text{ such that } s(x) = 0_{L_x}\}.$$

We say that L is

- **very ample** if φ_L is a closed embedding;
- **ample** if $L^{\otimes m}$ is very ample for some $m \in \mathbb{Z}_{>0}$.

The Iitaka dimension of $L \in \mathrm{Pic}(X)$ is given by

$$\kappa(L) = \begin{cases} \max_{m \in \mathbb{Z}_{>0}} \dim(\overline{\varphi_{L^{\otimes m}}(X)}), & \text{if there exist } m_0 \in \mathbb{Z}_{>0} \text{ such that } H^0(X, L^{\otimes m_0}) \neq \{0\}, \\ -\infty, & \text{in other case.} \end{cases}$$

We have $\kappa(L) \in \{-\infty, 1, 2, \ldots, \dim(X)\}$ and we say that L is **big** when $\kappa(L) = \dim(X)$.

Let C be an irreducible smooth projective algebraic curve. A Weil divisor over C has the form $D = \sum_{i=1}^{k} n_i p_i$ with $n_i \in \mathbb{Z}$ and $p_i \in C$. We define the degree of D as

$$\deg(D) = \sum_{i=1}^{k} n_i .$$

For all $D \in \mathrm{PDiv}(C)$ we have $\deg(D) = 0$ and a surjective homomorphism $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$.

Let now X be an irreducible projective algebraic variety, $D \in \mathrm{Div}(X)$ and $C \subset X$ an irreducible smooth curve. We define the intersection number between D and C by $D \cdot C := \deg(\mathcal{O}_X(D)|_C) \in \mathbb{Z}$. If C is not smooth, then we take the normalization morphism $\nu : \tilde{C} \to C$ and we define $D \cdot C := \deg(\nu^*(\mathcal{O}_X(D)|_C))$.

We say that two Cartier divisors D_1, D_2 are **numerically equivalent** if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curve $C \subset X$. In such case we write $D_1 \equiv D_2$. The quotient group $\mathrm{NS}(X) := \mathrm{Div}(X)/\equiv$ is called the **Neron–Severi group** of X. By Severi's base theorem we know that $\mathrm{NS}(X)$ is finitely generated, so, $\mathrm{NS}(X) = \mathbb{Z}^r \oplus$ torsion. The integer $r = rk(\mathrm{NS}(X)) =: \rho(X)$ is called the **Picard number** of X.

Given a line bundle $L = \mathcal{O}_X(D)$, we say that L is **nef** if $D \cdot C \geq 0$ for all irreducible curve $C \subset X$.

We have an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

where $\mathrm{Pic}^0(X)$ are the divisors such that $D \cdot C = 0$ for all irreducible curves $C \subset X$.

Given a projective variety $X \subset \mathbb{P}^n$, via an explicit system of homogeneous equations, we can easily write down at least one divisor on X, a hyperplane section L in this embedding. Another divisor, the divisor of zeroes of a differential form of top degree on X, can also be computed from the equations. In general, for an irreducible smooth algebraic variety X, we define the canonical line bundle of X as

$$\omega_X := \det(\Omega^1_X) = \bigwedge^{\dim(X)} \Omega^1_X .$$

Furthermore, a canonical divisor is a divisor $K_X \in \mathrm{Div}(X)$ such that $\omega_X = \mathcal{O}_X(K_X)$. The dual $-K_X$ is the anticanonical line bundle.

For an irreducible smooth projective algebraic variety X the Kodaira dimension of X, denoted by $\kappa(X)$, is the Iitaka dimension of ω_X.

Elements in $\mathrm{Pic}(X)$ corresponding to projective embeddings generate the ample cone $\mathcal{C}_{\text{ample}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$; ample divisors arise as hyperplane sections of X in a projective embedding. The closure $\mathcal{C}_{\text{nef}}(X)$ of $\mathcal{C}_{\text{ample}}$ in $\mathrm{Pic}(X)_{\mathbb{R}}$ is called **the nef cone.** An **effective divisor** is a sum with nonnegative coefficients of irreducible subvarieties of codimension one. Their classes span the **effective cone** $\mathcal{C}_e(X)$; divisors arising as hyperplane sections of projective embeddings of some Zariski open subset of X form the interior of $\mathcal{C}_e(X)$.

In general, the computation of the ample and effective cones, and the position of K_X with respect to these cones, is a difficult problem. However, we have the following theorem.

Theorem 9. *Let X be a smooth projective variety with $-K_X \in \mathcal{C}_{\text{ample}}$. Then $\mathcal{C}_{\text{nef}}$ is a finitely generated rational cone. If $-K_X$ is big and nef, then $\mathcal{C}_{\text{eff}}$ is finitely generated.*

A very rough classification of smooth algebraic varieties is based on the position of the anticanonical class with respect to the cone of ample divisors. Numerically this is reflected in the value of the Kodaira dimension. We say that X is of *general type* if K_X is ample and $\kappa(X) = \dim(X)$. When $-K_X$ is ample and $\kappa(X) = -\infty$, we say that X is a *Fano variety* and we say that X is of *intermediate type* in the other cases *e.g.* when $\kappa(X) = 0$.

1.2 Toric varieties

We refer the reader to [11] for the general theory of toric varieties.

Let $N \simeq \mathbb{Z}^d$ be a rank d lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice. Let us denote by $\mathbb{T} = \mathrm{Spec}(K[M]) \simeq \mathbb{G}_m^d$ the corresponding split algebraic torus, where $K[M]$ is the K-algebra generated by M as a semigroup. We identify the lattice M with the group of characters of the torus $\mathbb{T}$ and N with the one-parameter subgroups of $\mathbb{T}$.

Let Σ be a fan in $N_{\mathbb{R}} := N \otimes \mathbb{R}$. This is, Σ is a finite collection of strongly convex, rational polyhedral cones $\sigma \subseteq N_{\mathbb{R}}$ containing all the faces of its elements, and such that for every $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 (hence it is also in Σ). We denote by $\Sigma(1)$ the set of rays (i.e., one-dimensional cones) in Σ. More generally, for $\sigma \in \Sigma$ we denote by $\sigma(1) = \sigma \cap \Sigma(1)$ the set of rays on σ and, by abuse of notation, we identify rays with their primitive generators, i.e., with the unique primitive element $u_\rho \in N$ that generates the ray $\rho \in \Sigma(1)$. Also, given vector $v_1, \ldots, v_r \in N_{\mathbb{R}}$ we denote by $\mathrm{cone}(v_1, \ldots, v_r)$ the cone that they generate.

Given a cone $\sigma \in \Sigma$, its dual $\sigma^\vee := \{m \in M : \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma\}$ is a cone in $M_{\mathbb{R}}$ and $U_\sigma = \mathrm{Spec}(K[\sigma^\vee \cap M])$ is the associated affine toric variety. The toric variety X_Σ associated to Σ is obtained by gluing the affine toric varieties $\{U_\sigma\}_{\sigma \in \Sigma}$ along $U_{\sigma_1} \cap U_{\sigma_2} \simeq U_{\sigma_1 \cap \sigma_2}$. It is a normal and separated variety that contains a maximal torus $U_{\{0\}} \simeq \mathbb{T}$ as an open subset and admits an effective regular action of the torus $\mathbb{T}$ extending the natural action of the torus over itself. The toric variety X_Σ is smooth if and only if Σ is regular, meaning that every cone in Σ is generated by vectors that are part of a basis of N.

On a toric variety X_Σ, each ray $\rho \in \Sigma(1)$ corresponds to a prime $\mathbb{T}$-invariant Weil divisor D_ρ, and the classes of D_ρ with $\rho \in \Sigma(1)$ generate the class group $\mathrm{Cl}(X_\Sigma)$. In particular, every Weil divisor D on X is linearly equivalent to $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ for some integers $a_\rho \in \mathbb{Z}$. Similarly, the classes of the $\mathbb{T}$-invariant Cartier divisor generate the Picard group $\mathrm{Pic}(X_\Sigma)$. If the fan Σ contains a cone of maximal dimension $d = \dim_{\mathbb{R}}(N_{\mathbb{R}})$, then $\mathrm{Pic}(X_\Sigma)$ is a free abelian group of rank $\#\Sigma(1) - d$.

The relevant toric varieties appearing in this paper are all smooth. We recall that on smooth varieties every Weil divisor is Cartier, and in particular $\mathrm{Cl}(X_\Sigma) \simeq \mathrm{Pic}(X_\Sigma)$.

1.3 Toric vector bundles

Let X be an algebraic variety. A variety V is a vector bundle of rank r over X if there is a morphism $\pi : V \to X$ and an open cover $\{U_i\}_{i \in I}$ of X such that:

1. For every $i \in I$, there exists an isomorphism
$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r,$$
such that $p_1 \circ \varphi_i = \pi|_{\pi^{-1}(U_i)}$, where p_1 denotes the projection onto the first coordinate.

2. For every pair $i, j \in I$ there exists $g_{ij} \in \mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j))$ such that the following diagram is commutative:

$$\begin{array}{ccc} & \overset{\varphi_i|_{\pi^{-1}(U_i \cap U_j)}}{\nearrow} & (U_i \cap U_j) \times \mathbb{A}^r \\ \pi^{-1}(U_i \cap U_j) & & \uparrow \mathrm{Id} \times g_{ij} \\ & \underset{\varphi_j|_{\pi^{-1}(U_i \cap U_j)}}{\searrow} & (U_i \cap U_j) \times \mathbb{A}^r \end{array}$$

The data $\{(U_i, \varphi_i)\}_{i \in I}$ satisfying (1) and (2) is called a trivialization for $\pi : V \to X$. The g_{ij} are called transition matrices, and $V_p := \pi^{-1}(p) \cong \mathbb{A}^r$ is called the fiber at $p \in X$.

Let $\pi : V \to X$ be a vector bundle of rank r and $\{(U_i, \phi_i)\}_{i \in I}$ be a trivialization with transition matrices g_{ij}. Then the functions $\mathrm{Id} \times g_{ij}$ induce isomorphisms

$$\mathrm{Id} \times \overline{g}_{ij} : (U_i \cap U_j) \times \mathbb{P}^{r-1} \to (U_i \cap U_j) \times \mathbb{P}^{r-1},$$

where $\overline{g}_{ij} \in \mathrm{PGL}_r(\mathcal{O}_X(U_i \cap U_j))$ is the projective map induced by g_{ij}. This gives gluing data for a variety $\mathbb{P}(V)$ and π induces a morphism

$$\overline{\pi} : \mathbb{P}(V) \to X,$$

with trivializations $\{(U_i, \overline{\varphi}_i)\}_{i \in I}$ for $\mathbb{P}(V)$ where

$$\overline{\varphi}_i : \overline{\pi}^{-1}(U_i) \to U_i \times \mathbb{P}^{r-1}.$$

The algebraic variety $\mathbb{P}(V)$ constructed in this way is called the projective bundle associated to V.

If $\mathcal{E}$ is a locally free sheaf over X of rank r, then $\mathcal{E}$ is the sheaf of sections of a vector bundle $\pi_{\mathcal{E}} : V_{\mathcal{E}} \to X$ of rank r. In this case, we define the projectivization of $\mathcal{E}$ to be

$$\mathbb{P}(\mathcal{E}) := \mathbb{P}(V_{\mathcal{E}}^{\vee}), \tag{1.3.1}$$

where $V_{\mathcal{E}}^{\vee}$ denotes the vector bundle dual to $V_{\mathcal{E}}$. The projective bundle $\mathbb{P}(\mathcal{E})$ has fiber over $p \in X$ given by $\mathbb{P}(\mathcal{E})_p = \mathbb{P}(W_p)$ where $W_p = (V_{\mathcal{E}}^{\vee})_p$.

Remark 10. Some authors define $\mathbb{P}(\mathcal{E})$ as $\mathbb{P}(V_{\mathcal{E}})$, see e.g. [15, Chapter II, Section 7].

Let X be a toric variety defined by a fan Σ in $N_{\mathbb{R}}$ as in Section 1.2. A toric vector bundle over X is a vector bundle $\pi : V \to X$ such that the action of $\mathbb{T}$ on X extends to an action on V in such a way that π is $\mathbb{T}$-equivariant and the action is linear on the fibers. The algebraic variety V is not toric in general, and Oda [25, §7.6] notes that the toric vector bundles which are toric varieties are precisely the decomposables ones, i.e., those of the form $V_{\mathcal{E}}$ with $\mathcal{E} = \mathcal{O}_X(D_0) \oplus \cdots \oplus \mathcal{O}_X(D_r)$ for some $\mathbb{T}$-invariant Cartier divisors $D_0, \ldots, D_r$ on X.

For decomposables toric vector bundles $V_{\mathcal{E}} \to X$ of rank $r+1$ we can construct the fan that defines the projective bundle $\mathbb{P}(\mathcal{E}) \to X$ as a toric variety following [11, §7.3]: Let $D_0, \ldots, D_r$ be $\mathbb{T}$-invariant Cartier divisors, and write $D_i = \sum_{\rho \in \Sigma(1)} a_{i\rho} D_\rho$ with $a_{i\rho} \in \mathbb{Z}$ for $i \in \{0, \ldots, r\}$. To construct the fan of $V_{\mathcal{E}}$ we work in the vector space $N_{\mathbb{R}} \times \mathbb{R}^{r+1}$. We will denote by $e_0, \ldots, e_r$ the canonical basis of $\mathbb{R}^{r+1} \subset N_{\mathbb{R}} \times \mathbb{R}^{r+1}$ and write the elements of $N_{\mathbb{R}} \times \mathbb{R}^{r+1}$ in the form $u + \lambda_D e_0 + \cdots + \lambda_r e_r$, with $u \in N_{\mathbb{R}}$ and $\lambda_D, \ldots, \lambda_r \in \mathbb{R}$. Then, given $\sigma \in \Sigma$ we define $\tilde{\sigma} \subset N_{\mathbb{R}} \times \mathbb{R}^{r+1}$ to be the Minkowski sum

$$\tilde{\sigma} := \operatorname{cone}(u_\rho - a_{0\rho} e_0 - \cdots - a_{r\rho} e_r : \rho \in \sigma(1)) + \operatorname{cone}(e_0, \ldots, e_r),$$

where $u_\rho \in N$ is the primitive generator of the ray $\rho \in \sigma(1)$. The set of cones $\tilde{\sigma}$ where $\sigma \in \Sigma$, together with their faces, defines a fan $\tilde{\Sigma} \subset N_{\mathbb{R}} \times \mathbb{R}^{r+1}$ for a toric variety $X_{\tilde{\Sigma}}$ with a vector bundle structure $X_{\tilde{\Sigma}} \to X$ whose sheaf of sections is isomorphic to $\mathcal{E}$. It follows that $X_{\tilde{\Sigma}} \cong V_{\mathcal{E}}$.

Now consider $\mathbb{P}(\mathcal{E}) \to X$, which is a projective bundle with fibers isomorphic to $\mathbb{P}^r$. To construct the fan of $\mathbb{P}(\mathcal{E})$ we need to consider the dual sheaf $\mathcal{E}^\vee = \mathcal{O}_X(-D_0) \oplus \cdots \oplus \mathcal{O}_X(-D_r)$ and the associated vector bundle $V_{\mathcal{E}^\vee} = V_{\mathcal{E}}^\vee$. By the above construction, $V_{\mathcal{E}^\vee}$ is built from the cones

$$\operatorname{cone}(u_\rho + a_{0\rho} e_0 + \cdots + a_{r\rho} e_r : \rho \in \sigma(1)) + \operatorname{cone}(e_0, \ldots, e_r),$$

and their faces when σ ranges over all the cones $\sigma \in \Sigma$. The fan of $\mathbb{P}(\mathcal{E})$ is obtained as follows: for each $\sigma \in \Sigma$ and $i \in \{0, \ldots, r\}$ we put $F_i = \operatorname{cone}(e_0, \ldots, \hat{e}_i, \ldots, e_r)$, where $\hat{e}_i$ means that we omit the vector e_i, and define

$$\tilde{\sigma}_i := \operatorname{cone}(u_\rho + a_{0\rho} e_0 + \cdots + a_{r\rho} e_r : \rho \in \sigma(1)) + F_i \subset N_{\mathbb{R}} \times \mathbb{R}^{r+1}.$$

Let σ_i be the image of $\tilde{\sigma}_i$ under the canonical projection $N_{\mathbb{R}} \times \mathbb{R}^{r+1} \to N_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$, where $\overline{N}_{\mathbb{R}} = \mathbb{R}^{r+1}/\mathbb{R}(e_0 + e_1 + \cdots + e_r)$. Then it follows from [11, Proposition 7.3.3] that:

Proposition 11. *The cones $\{\sigma_i\}_{\sigma \in \Sigma,\, i \in \{0,\ldots,r\}}$ and their faces form a fan $\Sigma_{\mathcal{E}}$ in $N_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$ whose associated toric variety $X_{\Sigma_{\mathcal{E}}}$ is isomorphic to $\mathbb{P}(\mathcal{E})$.*

In practice, we will replace $\overline{N}_{\mathbb{R}} = \mathbb{R}^{r+1}/\mathbb{R}(e_0 + e_1 + \cdots + e_r)$ by $\mathbb{R}^r$ with basis $e_1, \ldots, e_r$ and define $e_0 := -e_1 - \cdots - e_r$. Hence, we put

$$F_i = \operatorname{cone}(e_0, \ldots, \hat{e}_i, \ldots, e_r) \subset \mathbb{R}^r,$$

and for a cone $\sigma \in \Sigma$ we get

$$\sigma_i = \operatorname{cone}(u_\rho + (a_{1\rho} - a_{0\rho}) e_1 + \cdots + (a_{r\rho} - a_{0\rho}) e_r : \rho \in \sigma(1)) + F_i \subset N_{\mathbb{R}} \times \mathbb{R}^r.$$

The cones σ_i and their faces define a fan $\Sigma_{\mathcal{E}}$ in $N_{\mathbb{R}} \times \mathbb{R}^r$ for $\mathbb{P}(\mathcal{E})$.

1.4 Hirzebruch–Kleinschmidt varieties

Given integers $r \geq 1, t \geq 2, 0 \leq a_1 \leq \dots \leq a_r$, consider the vector bundle

$$\mathcal{E} := \mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r).$$

Here, as usual, we put $\mathcal{O}_{\mathbb{P}^{t-1}}(a_i) := \mathcal{O}_{\mathbb{P}^{t-1}}(a_i H_0)$ where $H_0 \subset \mathbb{P}^{t-1}$ is a hyperplane, which we can choose as $H_0 = \{x_0 = 0\}$ in homogeneous coordinantes $[x_0 : \dots : x_t]$.

We can use Proposition 11 to describe the fan $\Sigma_{\mathcal{E}}$ of the smooth toric variety $\mathbb{P}(\mathcal{E})$. For this, it is enough to describe the maximal cones in $\Sigma_{\mathcal{E}}$. Consider $N \oplus \overline{N} = \mathbb{Z}^{t-1} \oplus \mathbb{Z}^r$ with canonial bases $u_1, \dots, u_{t-1}$ and $e_1, \dots, e_r$ for N and $\overline{N}$, respectively. As before, we set $u_0 := -u_1 - \cdots - u_{t-1}$ and $e_0 := -e_1 - \cdots - e_r$. Note that $\{u_0, \dots u_{t-1}\}$ is the set of primitive generators of the rays of a fan for the toric variety $\mathbb{P}^{t-1}$ with u_0 corresponding to the divisor H_0.

Put $a_0 := 0$ and $D_i := a_i H_0$ for $i \in \{0, \dots, r\}$. As in Section 1.3, we write $D_i = \sum_{\rho \in \Sigma(1)} a_{i\rho} D_\rho$. For $\rho \in \Sigma(1)$ with $u_\rho = u_i$ we define

$$v_i := u_\rho + (a_{1\rho} - a_{0\rho})e_1 + \dots + (a_{r\rho} - a_{0\rho})e_r = \begin{cases} u_0 + a_1 e_1 + \cdots + a_r e_r & \text{if } i = 0, \\ u_i & \text{if } i \in \{1, \dots, t-1\}. \end{cases}$$

Since the maximal cones of $\mathbb{P}^{t-1}$ are $\{\mathrm{cone}(u_0, \dots, \hat{u}_i, \dots, u_{t-1})\}_{i \in \{0, \dots, t-1\}}$ we see that the maximal cones of $\Sigma_{\mathcal{E}}$ are

$$\mathrm{cone}(v_0, \dots, \hat{v}_j, \dots, v_{t-1}) + \mathrm{cone}(e_0, \dots, \hat{e}_i, \dots, e_r),$$

for $j \in \{0, \dots, t-1\}$ and $i \in \{0, \dots, r\}$. Therefore, the primitive generators of the rays of $\Sigma_{\mathcal{E}}$ are $v_0, \dots, v_{t-1}, e_0, e_1, \dots, e_r$ (compare with [19, p. 256]).

Note that $\mathbb{P}(\mathcal{E})$ has dimension

$$d := \dim(\mathbb{P}(\mathcal{E})) = r + t - 1.$$

Since $\Sigma_{\mathcal{E}}$ contains cones of maximal dimension d and $\#\Sigma_{\mathcal{E}} = d + 2$, we conclude that $\mathrm{Pic}(\mathbb{P}(\mathcal{E})) \simeq \mathbb{Z}^2$. Conversely, Kleinschmidt [19] proved the following classification result of smooth projective[1] varieties of Picard rank 2.

Theorem 12 (Kleinschmidt)**.** *Let X be a smooth projective toric variety with $\mathrm{Pic}(X) \simeq \mathbb{Z}^2$. Then there exists integers $r \geq 1, t \geq 2, 0 \leq a_1 \leq \cdots \leq a_r$ with $r + t - 1 = d = \dim(X)$ such that*

$$X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r)).$$

Moreover, if we write $I_k = \{1, \dots, k\} \subset \mathbb{Z}$, then X is isomorphic to the subvariety of $\mathbb{P}^{rt-1} \times \mathbb{P}^{t-1}$ given in homogeneous coordinates $([x_0 : (x_{ij})_{i \in I_t, j \in I_r}], [y_1 : \dots : y_t])$ by the equations

$$x_{mj} y_n^{a_j} = x_{nj} y_m^{a_j}, \quad \text{for all } j \in I_r \text{ and all } m, n \in I_t \text{ with } m \neq n. \tag{1.4.1}$$

[1] Actually, the classification in [19] does not assume the projectivity hypothesis.

1.5 Manin's Conjecture

The purpose of this section is to introduce concepts related to Conjecture 1 such as heights, height zeta functions, and a more detailed description of Peyre's constant.

Heights

As we say in the Introduction, for the projective space $\mathbb{P}^n$, for $\mathbb{Q}$-rational points, there is the naive height given by:

$$H_{\mathbb{P}^n}(x) = \max\{|x_0|, |x_1|, \ldots, |x_n|\}$$

where $x = [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n(\mathbb{Q})$ with $\gcd(x_0, x_1, \ldots, x_n) = 1$. This height ensures that the set $\{x \in \mathbb{P}^n(\mathbb{Q}) : H_{\mathbb{P}^n}(x) \leq B\}$ of rational points with bounded height is finite.

For a global field K, the naive height is given by

$$H_{\mathbb{P}^n}(x) = \prod_{v \in \mathrm{Val}(K)} \max\{|x_0|_v, |x_1|_v, \ldots, |x_n|_v\}$$

which is well-defined for all $x \in \mathbb{P}^n(K)$ by the product formula.

For this height, the following finiteness result due to Northcott holds

Proposition 13 (Nothcott)**.** *Given $B \in \mathbb{R}$, the set*

$$\{x \in \mathbb{P}^n(K) : H_{\mathbb{P}^n}(x) \leq B\},$$

is finite.

Metrization of Line Bundles

In this section, we discuss a theory of height functions based on the notion of an adelically metrized line bundle, following very close [36]. We recall some notation. Let K be a number field. As before, we denote the set of places of K by Val(K) and write $v|\infty$ if v is archimedean and $v \nmid \infty$ if v is nonarchimedean. For a place v, let K_v be the completion of K at v, and by $\mathfrak{o}_v$, when $v \nmid \infty$, the ring of v-adic integers. Let q_v be the cardinality of the residue field κ_v of K_v for nonarchimedean valuations. The local absolute value $|\cdot|_v$ on K_v is the multiplier of the Haar measure, i.e., $\mathrm{d}(ax_v) = |a|_v \mathrm{d}x_v$ for some Haar measure $\mathrm{d}x_v$ on K_v. We denote by

$$\mathbb{A}_K = \prod_v{}' K_v$$

the adele ring of K.

Definition 14. Let X be an algebraic variety over K and L a line bundle on X. A v-adic metric on L is a family $(\|\cdot\|_x)_{x \in X(K_v)}$ of v-adic Banach norms on the fibers L_x such that for all Zariski open subsets $X^\circ \subset X$ and every section $f \in H^0(X^\circ, L)$ the map

$$X^\circ(K_v) \to \mathbb{R}, \quad x \mapsto \|f\|_x,$$

is continuous in the v-adic topology on $X^\circ(K_v)$.

Example 15. Assume that L is generated by global sections. Choose a basis $(f_j)_{j \in \{0,\dots,n\}}$ of $H^0(X, L)$ over K. If f is a section such that $f(x) \neq 0$, then we can define

$$\|f\|_x := \max_{0 \le j \le n} \left(\left| \frac{f_j(x)}{f(x)} \right|_v \right)^{-1},$$

otherwise $\|0\|_x := 0$. This defines a v-adic metric on L. Of course, this metric depends on the choice of the basis.

Definition 16. Assume that L is generated by global sections. An adelic metric on L is a collection of v-adic metrics, for $v \in \mathrm{Val}(K)$, such that for all but finitely many $v \in \mathrm{Val}(K)$ the v-adic metric on L is defined by means of some *fixed* basis $(f_j)_{j \in \{0,\dots,n\}}$ of $H^0(X, L)$.

We shall write $\|\cdot\|_{\mathbb{A}_K} := (\|\cdot\|_v)$ for an adelic metric on L and call a pair $\mathcal{L} = (L, \|\cdot\|_{\mathbb{A}_K})$ an adelically metrized line bundle. Metrizations extend naturally to tensor products and duals of metrized line bundles, which allows us to define adelic metrizations on arbitrary line bundles L on a projective variety X. First, write L as $L = L_1 \otimes L_2^{-1}$ with very ample L_1, L_2 that we assume are already adelically metrized. An adelic metrization of L is any metrization which for all but finitely many v is induced from metrizations on L_1 and L_2.

Definition 17. Let $\mathcal{L} = (L, \|\cdot\|_{\mathbb{A}_K})$ be an adelically metrized line bundle on X and f an K-rational section of L. Let $X^\circ \subset X$ be the maximal Zariski open subset of X where f is defined and does not vanish. For all $x = (x_v)_v \in X^\circ(\mathbb{A})$ we define the *local height function*

$$\mathrm{H}_{\mathcal{L},f,v}(x_v) := \|f\|_{x_v}^{-1},$$

and the *global height function*

$$\mathrm{H}_{\mathcal{L}}(x) := \prod_{v \in \mathrm{Val}(K)} \mathrm{H}_{\mathcal{L},f,v}(x_v).$$

By the product formula, the restriction of the global height to $X^\circ(K)$ does not depend on the choice of f.

1.5.1 Height zeta functions

Some analytic approaches to the conjectures of Manin et al. make use of height zeta functions. In this section we present these functions.
Let X be an algebraic variety over a global field K, $\mathcal{L} = (L, \|\cdot\|_{\mathbb{A}_K})$ an adelically metrized

ample line bundle on X, H_{L} a height function associated to L, X a subvariety of X, and $a_{X}(L)$ the abscissa of convergence of the height zeta function

$$\mathrm{Z}(X, L, s) := \sum_{x \in X(K)} \mathrm{H}_{L}(x)^{-s}.$$

Proposition 18. *The value of $a_{X}(L)$ depends only on the class of L in $\mathrm{NS}(X)$.*

So, we may write $a_{X}(L) := a_{X}(L)$.

1.5.2 Peyre's constants

In this section, we present in more detail the constants mentioned in Conjecture 1. We we follow [26]. Let $\mathfrak{p}$ a nonarchimedean place of a number field K and $K_{\mathfrak{p}}$ the local field corresponding to $|\ |_{\mathfrak{p}}$. If $\mathfrak{p}$ is above the rational prime p, then we can choose $|x|_{\mathfrak{p}} = |N_{K_{\mathfrak{p}}/\mathbb{Q}_p}(x)|_p$.

Let X be a smooth projective algebraic variety over K and $(\|\ \|_v)_{v \in \mathrm{Val}(K)}$ an adelic metric over ω_X^{-1}. Let H be the corresponding height over $X(K)$, and denote by $\overline{X}$ the base change of X over $\overline{K}$ (an algebraic closure of K). The adelic metrization of the anticanonical line bundle yields for any place v of K a measure $\omega_{H,v}$ on the locally compact space $X(K_v)$, given by the local formula

$$\omega_{H,v} = \left\| \frac{\partial}{\partial x_{1,v}} \wedge \cdots \wedge \frac{\partial}{\partial x_{n,v}} \right\|_v \mathrm{d}x_{1,v} \ldots \mathrm{d}x_{n,v},$$

where $x_{1,v}, \ldots, x_{n,v}$ are local v-adic analytic coordinates, $\frac{\partial}{\partial x_{1,v}} \wedge \cdots \wedge \frac{\partial}{\partial x_{n,v}}$ is seen as section of ω_X^{-1} and the Haar measures $\mathrm{d}x_{j,v}$ are normalized as follows:

- If v is a finite place, then $\int_{\mathcal{O}_v} \mathrm{d}x_{j,v} = 1$;
- If v is real, $\mathrm{d}x_{j,v}$ is the standard Lebesgue measure;
- If v is complex, then $\mathrm{d}x_{j,v} = -i dz d\overline{z}$.

Let S a finite set of bad places containing the archimedean ones. Increasing S if necessary, we can assume that X lifts to a projective smooth scheme $\mathcal{X}$ over $\mathcal{O}_S := \{x \in \mathcal{O}_K : v(x) \geq 0$ for all $v \notin S\}$, the ring of S-integers.
For any $\mathfrak{p} \in \mathrm{Val}(K) \setminus S$ let $\mathrm{N}(\mathfrak{p})$ denote its ideal norm. The local term of the L-function correspond to the Picard group is defined by

$$L_{\mathfrak{p}}(s, \mathrm{Pic}(\overline{X})) = \frac{1}{\det(1 - \mathrm{N}(\mathfrak{p})^{-s}\mathrm{Frob}_{\mathfrak{p}} | \mathrm{Pic}(X_{\overline{K_{\mathfrak{p}}}}) \otimes \mathbb{Q})}.$$

Example 19. If $\mathrm{Pic}(X_{\overline{\mathbb{Q}}}) = \mathbb{Z}$, then $\det(1 - \mathrm{N}(\mathfrak{p})^{-s}\mathrm{Frob}_\mathfrak{p} | \mathrm{Pic}(X_{\overline{\mathbb{Q}}})^{I_\mathfrak{p}}) = 1 - p^{-s}$ (see [18]), where $\mathrm{Pic}(X_{\overline{\mathbb{Q}}})^{I_\mathfrak{p}}$ is the fixed module under the inertia group $I_\mathfrak{p}$.

The corresponding global L-function is given by

$$L_S(s, \mathrm{Pic}(\overline{X})) = \prod_{\mathfrak{p} \in \mathrm{Val}(K) \setminus S} L_\mathfrak{p}(s, \mathrm{Pic}(\overline{X})).$$

It converges for $\Re(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$ with a pole of order $t = rk\,\mathrm{Pic}(X)$ at 1.
The local convergence factors are defined as $\lambda_v = L_v(1, \mathrm{Pic}(\overline{X}))$ if $v \in \mathrm{Val}(K) \setminus S$ and $L_v = 1$ otherwise. In addition, the local density in $\mathfrak{p} \in \mathrm{Val}(K) \setminus S$ is defined as

$$d_\mathfrak{p}(X) = \frac{\# X(K_\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{\dim X}}.$$

Lemma 20. *For almost everything $v \in \mathrm{Val}(K) \setminus S$, we have*

$$\omega_{H,v}(X(K_v)) = d_\mathfrak{p}(X).$$

Definition 21. The Tamagawa measure corresponding to H is defined by

$$\omega_H = \lim_{s \to 1}(s - 1)^t L_S(s, \mathrm{Pic}(\overline{X})) \sqrt{\Delta_K}^{-\dim(X)} \prod_{v \in \mathrm{Val}(K)} \lambda_v^{-1}\omega_{H,v}.$$

where Δ_K is the discriminant of the field K. The Tamagawa number is defined by

$$\tau_H(X) = \int_{\overline{X(K)}} \omega_H,$$

where $\overline{X(K)}$ is the closure of $X(K)$ in $X(\mathbb{A}_K)$.

Definition 22. The cohomological constant is given by

$$\beta(X) = \# H^1(K, \mathrm{Pic}(\overline{X})).$$

Let $\mathrm{NS}(X)^\vee$ be the dual lattice of $\mathrm{NS}(X)$. It determines the normalization of the Lebesgue measure $\mathbf{d}\mathbf{y}$ on $\mathrm{NS}(X)^\vee \otimes \mathbb{R}$ (with covolume one).

Example 23. If $\mathrm{Pic}(X_{\overline{\mathbb{Q}}}) \cong \mathbb{Z}$ then $\beta(X) = 1$.

As mentioned in the introduction in Remark 3, if X is a Hirzebruch–Kleinschmidt variety, we have that $\beta(X) = 1$.

Definition 24. The *Peyre's α-constant* of an almost Fano[2] variety X is defined as

$$\alpha(X) = \frac{1}{(\operatorname{rk}\operatorname{Pic}(X) - 1)!}\int_{\Lambda_{\mathrm{eff}}(X)^{\vee}} e^{-\langle -K_X, \mathbf{y}\rangle}\mathrm{d}\mathbf{y},$$

where $\langle\,,\,\rangle$ denotes the natural pairing $\operatorname{Pic}(X)_{\mathbb{R}} \times \operatorname{Pic}(X)^{\vee}_{\mathbb{R}} \to \mathbb{R}$ and $\mathrm{d}\mathbf{y}$ denotes the Lebesgue measure on $\operatorname{Pic}(X)^{\vee}_{\mathbb{R}}$ normalized to give covolume 1 to the lattice $\operatorname{Pic}(X)^{\vee}$ (see e.g. [31, Definition 2.5]). Also define the constant (see [29, *Proposition* 3.1.2]

$$\alpha^{*}(X) = \int_{\Lambda_{\mathrm{eff}}(X)^{\vee}} e^{-\langle -K_X, \mathbf{y}\rangle}\mathrm{d}\mathbf{y}.$$

Example 25. If $\operatorname{Pic}(X) \cong \mathbb{Z}$ and $[L]$ is the ample generator such that $[-K_X] = \delta[L]$ then $\alpha(X) = \frac{1}{\delta}$.

A straightforward computation using Propositions 30 and 83 give us the following result.

Lemma 26. *Let $X = X_d(a_1, \ldots, a_r)$ be a Hirzebruch–Kleinschmidt variety. Then, its α-constant is given by*

$$\begin{aligned}\alpha(X) &= \int_0^{\infty}\int_0^{\infty} e^{-(r+1)y_1 - ((r+1)a_r + t - |a|)y_2}\mathrm{d}y_1\,\mathrm{d}y_2 = \int_0^{\infty}\int_{a_r y_2}^{\infty} e^{-(r+1)y_1 - (t-|a|)y_2}\mathrm{d}y_1\,\mathrm{d}y_2\\ &= \frac{1}{(r+1)\left((r+1)a_r + t - |a|\right)} = \alpha^{*}(X).\end{aligned}$$

1.6 A Tauberian theorem

As mentioned in the Introduction, the Theorem 70 is obtained from the analytic properties of certain height zeta functions, by using a Tauberian theorem. We will use the following formulation, which follows from [12, Théorème III].

Theorem 27. *Let X be a countable set, $H : X \to \mathbb{R}^{+}$ a function and suppose that*

$$Z(s) = \sum_{x\in X} H(x)^{-s}$$

is absolutely convergent for $\Re(s) > a > 0$ and

$$Z(s) = \frac{g(s)}{(s-a)^b},$$

where b is a positive integer and $g(s)$ is a holomorphic function in the half-plane $\Re(s) > a - \varepsilon$, for some $\varepsilon > 0$, with $g(a) \neq 0$. Then, for every $B > 0$ the cardinality

$$N(X, H, B) = \#\{x \in X : H(x) \leq B\}$$

is finite, and

$$N(X, H, B) \sim \frac{g(a)}{(b-1)!\,a} B^{a}(\log B)^{b-1} \quad \text{as } B \to \infty.$$

[2]Every smooth projective toric variety is almost Fano.

Chapter 2

Number fields case

2.1 Basic notation

Throughout this chapter we let K denote a number field of degree n_K over $\mathbb{Q}$. Associated to K we have the following objects:

- The ring of integers $\mathcal{O}_K$ and the associated arithmetic curve $S := \mathrm{Spec}(\mathcal{O}_K)$.

- The number w_K of roots of unity in K.

- The discriminant Δ_K, regulator R_K and class number h_K.

- The set of discrete valuations $\mathrm{Val}_f(K)$, which is in bijection with the set of non-zero prime ideals $\mathfrak{p} \subset \mathcal{O}_K$.

- The set Σ_K of field embeddings $K \hookrightarrow \mathbb{C}$, and r_1, r_2 the number of real and complex Archimedean places, respectively. In particular, $\#\Sigma_K = n_K = r_1 + 2r_2$.

- Given $v \in \mathrm{Val}_f(K)$ we denote by K_v the corresponding completion, and for $x \in K_v$ we put $|x|_v := |\mathrm{Nr}_{K_v|\mathbb{Q}_p}(x)|_p$ where p is the unique prime associated to the restriction of v to $\mathbb{Q}$, and $|\cdot|_p$ denotes the standard p-adic norm (namely, with $|p|_p = p^{-1}$). Similarly, given $\sigma \in \Sigma_K$ we denote by K_σ the completion of K with respect to the norm $|x|_\sigma := |\sigma(x)|$ where $|\cdot|$ stands for the usual Euclidean norm on $\mathbb{C}$ (namely, $|x| = \sqrt{x\overline{x}}$). With this normalization, the product formula

$$\prod_{v \in \mathrm{Val}_f(K)} |x|_v \prod_{\sigma \in \Sigma_K} |x|_\sigma = 1$$

  holds for every $x \in K, x \neq 0$.

- $\eta : \mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbb{Z})$ is the morphism of schemes induced by the inclusion $\mathbb{Z} \hookrightarrow \mathcal{O}_K$.

For a vector $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}_0^r$, we write $|\mathbf{a}| := \sum_{i=1}^r a_i$.

We now recall the definition of Hirzebruch–Kleinschmidt variety mentioned in the Introduction, which in the case of number fields will be normalized in another way.

Definition 28. Given integers $r \geq 1$, $t \geq 2$ and $0 \leq a_1 \leq \dots \leq a_r$, the *Hirzebruch–Kleinschmidt variety* $X_d(a_1, \dots, a_r)$ is defined as

$$X_d(a_1, \dots, a_r) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1 - a_r) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r)),$$

where $d = \dim(X_d(a_1, \dots, a_r)) = r + t - 1$. We denote by $\pi : X_d(a_1, \dots, a_r) \to \mathbb{P}^{t-1}$ the associated projective bundle.

Remark 29. Recall that for any line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ and every locally free sheaf $\mathcal{E}$ on an algebraic variety X there is a canonical isomorphism of projective bundles $\mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$ (see e.g. [11, Lemma 7.0.8(b)]). In particular,

$$X_d(a_1, \dots, a_r) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r)).$$

We choose the description in Definition 28 because it allows for a simpler characterization of the cone of effective divisors (see Proposition 30 below).

Note that the minimal generators of the rays in the fan of $X_d(a_1, \dots, a_r)$ are the vectors

$$w_i := \begin{cases} u_0 - a_r e_1 + (a_1 - a_r)e_2 + \dots + (a_{r-1} - a_r)e_r & \text{if } i = 0, \\ u_i & \text{if } i \in \{1, \dots, t-1\}, \end{cases}$$

together with the primitive elements $e_0, \dots, e_r$. From now on we denote by D_i the divisor on $X_d(a_1, \dots, a_r)$ corresponding to the minimal generator w_i, for $i \in \{0, \dots, t-1\}$, and by E_j the corresponding divisor corresponding to e_j, for $j \in \{0, \dots, r\}$. It is easy to see that $\mathcal{O}_X(D_i) \cong \pi^* \mathcal{O}_{\mathbb{P}^{t-1}}(1)$ for every $i \in \{1, \dots, t-1\}$ and $\mathcal{O}_X(E_0) \cong \mathcal{O}_X(1)$ (e.g., by using local trivializations).

2.2 Effective divisors

The following description of the cone of effective divisors on $X_d(a_1, \dots, a_r)$ is a natural extension to higher dimensions of the description for Hirzebruch surfaces (see e.g. [15, Chapter V, Corollary 2.18]).

Proposition 30. *Let $X = X_d(a_1, \dots, a_r)$ be a Hirzebruch–Kleinschmidt variety and let us denote by f the class of $\pi^* \mathcal{O}_{\mathbb{P}^{t-1}}(1)$ and by h the class of $\mathcal{O}_X(1)$, both in $\mathrm{Pic}(X)$. Then:*

1. *$\mathrm{Pic}(X) \cong \mathbb{Z}h \oplus \mathbb{Z}f$.*

2. *The anticanonical divisor class of X is given by*

$$-K_X = (r + 1)h + ((r + 1)a_r + t - |\mathbf{a}|) f,$$

where $|\mathbf{a}| = \sum_{i=1}^r a_i$.

3. *The cone of effective divisors of X is given by*

$${}_{\mathrm{eff}}(X) = \{\lambda h + \mu f : \lambda \geq 0, \mu \geq 0\} \subset \mathrm{Pic}(X)_{\mathbb{R}}$$

where $\mathrm{Pic}(X)_{\mathbb{R}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proof. We follow [17, Section 7]. With the above notation we have that the vectors $w_1, \ldots, w_{t-1}$, $e_1, \ldots, e_r$ form a basis for $\mathbb{Z}^{t-1} \oplus \mathbb{Z}^r$. For $1 \leq i \leq t-1$, $1 \leq j \leq r$ we denote by w_i^*, e_j^* the corresponding dual basis elements. We then compute the divisors of the characters $\chi^{w_i^*}$ for $i \in \{1, \ldots, t-1\}$, which are

$$\mathrm{div}\left(\chi^{w_i^*}\right) = \sum_{k=0}^{t-1} \langle w_i^*, w_k \rangle D_k + \sum_{k=0}^{r} \langle w_i^*, e_k \rangle E_k = -D_0 + D_i,$$

and similarly the divisors of the characters $\chi^{e_j^*}$ for $j \in \{1, \ldots, r\}$ are

$$\mathrm{div}\left(\chi^{e_j^*}\right) = (a_{j-1} - a_r)D_0 - E_0 + E_j$$

(recalling that $a_0 := 0$). Therefore, in $\mathrm{Pic}(X_d(a_1, \ldots, a_r))$ we have the relations

$$D_i = D_0 \text{ and } E_j = E_0 + (a_r - a_{j-1})D_0 \text{ for } 1 \leq i \leq t-1, 1 \leq j \leq r. \tag{2.2.1}$$

In particular, $\mathrm{Pic}(X_d(a_1, \ldots, a_r)) = \mathbb{Z} \cdot E_0 \oplus \mathbb{Z} \cdot D_0 = \mathbb{Z}h \oplus \mathbb{Z}f$. This proves item (1).

It follows from [11, Theorem 8.2.3] that the anticanonical divisor class of $X = X_d(a_1, \ldots, a_r)$ is given by the class

$$\sum_{i=0}^{t-1} D_i + \sum_{j=0}^{r} E_j = (r+1)E_0 + ((r+1)a_r + t - |a|)\, D_0.$$

This proves item (2).

Finally, by [11, Lemma 15.1.8] the effective cone equals the cone generated by the classes of the divisors D_i and E_j. Hence, item (3) is a consequence of (2.2.1). This completes the proof of the proposition.

□

It follows from [20, Theorem 2.2.26] that a divisor class in $\mathrm{Pic}(X_d(a_1, \ldots, a_r))$ is big if and only if it lies in the interior of the effective cone ${}_{\mathrm{eff}}(X)$. Hence, we get the following corollary from Proposition 30.

Corollary 31. *Let $L = \lambda h + \mu f$ with $\lambda, \mu \in \mathbb{Z}$, where $\{h, f\}$ is the basis of $\mathrm{Pic}(X_d(a_1, \ldots, a_r))$ given in Proposition 30. Then, L is big if and only if $\lambda > 0$ and $\mu > 0$.*

In particular, Proposition 30 implies that the anticanonical divisor class in a Hirzebruch–Kleinschmidt variety is big. This is true for any smooth projective toric variety.

2.3 Restriction of big line bundles

Given integers $r \geq 1$, $t \geq 2$ and $0 \leq a_1 \leq \cdots \leq a_r$ we defined the Hirzebruch–Kleinschmidt variety $X = X_d(a_1, \dots, a_r)$ of dimension $d = r + t - 1$ as the projective bundle $\mathbb{P}(E)$ where

$$E := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1 - a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r)).$$

Definition 32. Put

$$Y := \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1 - a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r)$$

and define, as in the Introduction, the projective subbundle $F := \mathbb{P}(Y) \subset X_d(a_1, \dots, a_r)$.

Note that, when $r \geq 2$ we have (see Remark 29)

$$F \cong \mathbb{P}(Y \otimes \mathcal{O}_{\mathbb{P}^{t-1}}(a_r - a_{r-1})) = X_{d-1}(a_1, \dots, a_{r-1}), \tag{2.3.1}$$

hence F is a Hirzebruch–Kleinschmidt variety of dimension $d-1$, while in the case $r = 1$ we have that $F \cong \mathbb{P}^{t-1}$ is a projective space.

Denote by $\iota : F \to X$ the inclusion map. Given a class $L \in \mathrm{Pic}(X)$, we denote by $L|_F := \iota^* L$ its restriction to F.

In this section we prove the following results concerning the restriction to F of line bundles on X.

Lemma 33. *Assume $r \geq 2$, let $X = X_d(a_1, \dots, a_r)$, $X' = X_d(a_1, \dots, a_{r-1})$, and let $\{h, f\}$, $\{h', f'\}$ be the bases of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X')$, respectively, given in Proposition 30. If $L = \lambda h + \mu f \in \mathrm{Pic}(X)$, then $L|_F \in \mathrm{Pic}(F)$ corresponds under the canonical isomorphism (2.3.1) to the class*

$$\lambda h' + (\mu - \lambda(a_r - a_{r-1}))f' \in \mathrm{Pic}(X').$$

In particular, $L|_F$ is a big line bundle class in $\mathrm{Pic}(F)$ if and only if $\lambda > 0$ and $\mu > \lambda(a_r - a_{r-1})$.

Proof. On the one hand, given a closed immersion $\varphi : X \hookrightarrow \mathbb{P}^N$ (for some $N > 0$) we have

$$h|_F = \iota^* h = \iota^*(\mathcal{O}_X(1)) = \iota^*(\varphi^*(\mathcal{O}_{\mathbb{P}^N}(1))) = (\varphi \circ \iota)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_F(1).$$

Now, under the isomorphism (2.3.1), the class $\mathcal{O}_F(1) \in \mathrm{Pic}(F)$ corresponds to $\mathcal{O}_{X'}(1) \otimes (\pi')^* \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r)$, where $\pi' : X' \to \mathbb{P}^{t-1}$ is the projection map of X' (see [15, Chapter 2, Lemma 7.9]). This shows that $h|_F$ corresponds to $h' + (a_{r-1} - a_r)f'$. On the other hand, since (2.3.1) is an isomorphism of projective bundles over $\mathbb{P}^{t-1}$, we have that $f|_F = (\pi|_F)^*(\mathcal{O}_{\mathbb{P}^{t-1}}(1))$ corresponds to $(\pi')^*(\mathcal{O}_{\mathbb{P}^{t-1}}(1)) = f'$. This implies that $L|_F = \lambda h|_F + \mu f|_F$ corresponds to

$$\lambda(h' + (a_{r-1} - a_r)f') + \mu f' = \lambda h' + (\mu - \lambda(a_r - a_{r-1}))f'.$$

Finally, the last statement follows from Corollary 31. This proves the lemma. □

Lemma 34. *Assume $r = 1$, i.e. $X = X_d(a)$ with $a \geq 0$ an integer. If $L = \lambda h + \mu f \in \mathrm{Pic}(X)$, then $L|_F$ corresponds under the isomorphism $F \simeq \mathbb{P}^{t-1}$ to the class of the line bundle $\mathcal{O}_{\mathbb{P}^{t-1}}(\mu - a\lambda) \in \mathrm{Pic}(\mathbb{P}^{t-1})$. In particular, $L|_F$ is big if and only if $\mu > a\lambda$.*

Proof. The proof is similar to the case $r \geq 2$, but now we use that $h|_F = \pi^*(\mathcal{O}_{\mathbb{P}^{t-1}}(-a))$ corresponds to $\mathcal{O}_{\mathbb{P}^{t-1}}(-a)$, while $f|_F$ corresponds to $\mathcal{O}_{\mathbb{P}^{t-1}}(1)$. . □

Remark 35. When applied to the anticanonical class $-K_X = (r+1)h + ((r+1)a_r + t - |a|)$ (see Proposition 30(3)), Lemmas 33 and 34 show that $-K_X$ remains big when restricted to each component in the decomposition (0.0.7) if and only if $t > |a|$, and this is exactly the case when $X_d(a_1, \dots, a_r)$ is Fano according to [19, Theorem 2(2)].

2.4 Hermitian vector bundles over arithmetic curves

In this section we follow closely the presentation in [5]. Let us recall that $S = \mathrm{Spec}(\mathcal{O}_K)$.

Definition 36. A *Hermitian vector bundle* $\overline{E}$ over S is a pair (E, h) where E is a finitely generated projective $\mathcal{O}_K$-module and $h = \{h_\sigma\}_{\sigma \in \Sigma_K}$ is a family of positive definite Hermitian forms over the family of complex vector spaces $\{E \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}\}_{\sigma \in \Sigma_K}$, which are invariant under conjugation, i.e., $\|e \otimes \overline{\lambda}\|_{\overline{\sigma}} = \|e \otimes \lambda\|_\sigma$, for all $e \in E, \lambda \in \mathbb{C}$, where $\|\cdot\|_\sigma := \sqrt{h_\sigma(\cdot,\cdot)}$ is the usual Hermitian norm associated to h_σ. An element of E is called a *rational section*.

The rank of $\overline{E} = (E, h)$ is defined as the rank of E as $\mathcal{O}_K$-module, i.e., as the dimension of the complex vector spaces $E \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$. A morphism between two Hermitian vector bundles $\overline{E}_1 = (E_1, h_1)$ and $\overline{E}_2 = (E_2, h_2)$ is a $\mathcal{O}_K$-homomorphism $\varphi : E_1 \to E_2$ such that $\|\varphi(e \otimes \lambda)\|_{2,\sigma} \leq \|e \otimes \lambda\|_{1,\sigma}$ for all $e \in E, \lambda \in \mathbb{C}$ and $\sigma \in \Sigma_K$. An isomorphism of Hermitian vector bundles is a bijective morphism inducing an isometry $E_1 \otimes_{\mathcal{O}_K,\sigma} \mathbb{C} \to E_2 \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$ for every σ.

We denote by $\widehat{\mathrm{Pic}}(S)$ the set of *Hermitian line bundles* (i.e., Hermitian vector bundles of rank 1) over S up to isomorphism. Note that a Hermitian line bundle is uniquely determined by the underlying projective $\mathcal{O}_K$-module and the values $\|1\|_\sigma$ with $\sigma \in \Sigma_K$.

Example 37. Let $r \in \mathbb{N}_{\geq 1}$. The *trivial Hermitian vector bundle* $\overline{\mathcal{O}_K^r} := (\mathcal{O}_K^r, h)$ over S is defined by considering for each $\sigma \in \Sigma_K$ the Hermitian form

$$h_\sigma(a \otimes \lambda_1, b \otimes \lambda_2) := \lambda_1 \overline{\lambda_2} \langle \sigma(a), \overline{\sigma(b)} \rangle_{\mathbb{C}^r},$$

where for $a = (a_i) \in \mathcal{O}_K^r$ we put $\sigma(a) := (\sigma(a_i)) \in \mathbb{C}^r$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}^r}$ denotes the standard bilinear form in $\mathbb{C}^r$.

2.5 Operations with Hermitian vector bundles.

It is possible to extend the usual constructions of linear algebra to Hermitian vector bundles. Here we present the ones that will be needed in this chapter.

Let $\overline{E}_1 = (E_1, h_1)$ and $\overline{E}_2 = (E_1, h_2)$ be Hermitian vector bundles over S.

- **Direct sum.** We define $\overline{E}_1 \oplus \overline{E}_2$ as the pair (E,h) where $E := E_1 \oplus E_2$, and over
$$(E_1 \oplus E_2) \otimes_{O_K,\sigma} \mathbb{C} \simeq (E_1 \otimes_{O_K,\sigma} \mathbb{C}) \oplus (E_2 \otimes_{O_K,\sigma} \mathbb{C})$$
we define
$$h_\sigma(((e_1 \otimes \lambda_1),(d_1 \otimes \mu_1)),((e_2 \otimes \lambda_2),(d_2 \otimes \mu_2))) := h_{1,\sigma}(e_1 \otimes \lambda_1, e_2 \otimes \lambda_2) + h_{2,\sigma}(d_1 \otimes \mu_1, d_2 \otimes \mu_2).$$
We have $\mathrm{rk}(\overline{E}_1 \oplus \overline{E}_2) = \mathrm{rk}(\overline{E}_1) + \mathrm{rk}(\overline{E}_2)$.

- **Tensor product**. Define $\overline{E}_1 \otimes \overline{E}_2$ as the pair (E,h), where $E := E_1 \otimes_{O_K} E_2$, and over
$$(E_1 \otimes_{O_K} E_2) \otimes_{O_K,\sigma} \mathbb{C} \simeq (E_1 \otimes_{O_K,\sigma} \mathbb{C}) \otimes_{\mathbb{C}} (E_2 \otimes_{O_K,\sigma} \mathbb{C})$$
we define
$$h_\sigma((e_1 \otimes \lambda_1 \otimes e_2 \otimes \lambda_2),(d_1 \otimes \mu_1 \otimes d_2 \otimes \mu_2)) := h_{1,\sigma}(e_1 \otimes \lambda_1, d_1 \otimes \mu_1) h_{2,\sigma}(e_2 \otimes \lambda_2, d_2 \otimes \mu_2).$$
Then, $\mathrm{rk}(\overline{E}_1 \otimes \overline{E}_2) = \mathrm{rk}(\overline{E}_1)\,\mathrm{rk}(\overline{E}_2)$.

- **Dual**. Given a Hermitian vector space (V,h), we can identify V with its dual $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ by means of the application $v \mapsto H_v$, where H_v is the functional defined by $H_v(u) = h(u,v)$. With this identification, $V^\vee$ inherits a Hermitian structure given by $h_{V^\vee}(H_u,H_v) := \overline{h(u,v)}$. We thus define $\overline{E}_1^\vee$ as the pair (E,h) where $E = E_1^\vee = \mathrm{Hom}_{O_K}(E,O_K)$ and h is the family of Hermitian forms defined in
$$\begin{aligned}\mathrm{Hom}_{O_K}(E,O_K) \otimes_{O_K,\sigma} \mathbb{C} &\simeq \mathrm{Hom}_{\mathbb{C}}(E \otimes_{O_K,\sigma} \mathbb{C}, O_K \otimes_{O_K,\sigma} \mathbb{C}) \\ &\simeq \mathrm{Hom}_{\mathbb{C}}(E \otimes_{O_K,\sigma} \mathbb{C}, \mathbb{C}) \\ &= (E \otimes_{O_K,\sigma} \mathbb{C})^\vee\end{aligned}$$
as we explained before for Hermitian vector spaces. In particular, $\mathrm{rk}(\overline{E}_1^\vee) = \mathrm{rk}(\overline{E}_1)$.

- **Alternating products**. Given $m \in \mathbb{N}_{\geq 1}$, define $\bigwedge^m \overline{E}_1$ as the pair (E,h) where $E = \bigwedge^m E_1$ and h is the family of Hermitian forms defined by
$$h_\sigma(e_1 \wedge \cdots \wedge e_m, d_1 \wedge \cdots \wedge d_m) := \det(h_{1,\sigma}(e_i,d_j)).$$
We have $\mathrm{rk}\left(\bigwedge^m \overline{E}_1\right) = \binom{\mathrm{rk}(E_1)}{m}$. In particular, the *determinant* $\det\left(\overline{E}_1\right) := \bigwedge^{\mathrm{rk}(E_1)} \overline{E}_1$ is a Hermitian line bundle.

- **Direct image.** Recall that $\eta : \mathrm{Spec}(O_K) \to \mathrm{Spec}(\mathbb{Z})$ is the morphism induced by the inclusion $\mathbb{Z} \hookrightarrow O_K$. Given a Hermitian vector bundle $\overline{E_1} = (E_1,h_1)$ over S, we can define a Hermitian vector bundle $\eta_* \overline{E} = (E,h)$ over $\mathrm{Spec}(\mathbb{Z})$ in the following way: Consider $E = E_1$ but as a free $\mathbb{Z}$-module of rank $[K:\mathbb{Q}] \cdot \mathrm{rk}\, E$, and note that
$$E \otimes_{\mathbb{Z}} \mathbb{C} = E_1 \otimes_{\mathbb{Z}} \mathbb{C} \simeq E_1 \otimes_{O_K} (O_K \otimes_{\mathbb{Z}} \mathbb{C}) \simeq \bigoplus_\sigma (E_1 \otimes_{O_K,\sigma} \mathbb{C}).$$
Then, given $a = (a_\sigma)_\sigma, b = (b_\sigma)_\sigma \in \bigoplus_\sigma (E_1 \otimes_{O_K,\sigma} \mathbb{C})$, we define $h(a,b) := \sum_\sigma h_{1,\sigma}(a_\sigma, b_\sigma)$.

Remark 38. The set $\widehat{\mathrm{Pic}}(S)$ has a group structure induced by the tensor product. The inverse element is induced by the dual and the identity element is the class of the trivial Hermitian line bundle $\overline{\mathcal{O}_K}$ defined in Example 37.

Example 39. For an integer $n \geq 1$ consider the trivial Hermitian vector bundle $\overline{\mathcal{O}_K}^{n+1}$ (see Example 37) and the projective space of lines $\mathbb{P}^n(K) = \mathbb{P}(K^{n+1})$. Each line $\ell \subseteq K^{n+1}$ defines a finitely generated projective $\mathcal{O}_K$-module $\mathcal{O}_K^{n+1} \cap \ell$ whose corresponding complexifications are metrized using the restriction of the ambient Hermitian forms. Then, for each point $P = \ell \in \mathbb{P}^n(K)$ we get a Hermitian line bundle denoted by $\overline{\mathcal{O}_{\mathbb{P}^n}(-1)_P}$. Its dual is denoted by $\overline{\mathcal{O}_{\mathbb{P}^n}(1)_P}$. The metric on $\mathcal{O}_{\mathbb{P}^n}(1)_P \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$ constructed in this way is the Fubini–Study metric (see e.g. [21, Example 1.2.45]). As usual, by taking duals and tensor powers, we can define for every $a \in \mathbb{Z}$ the Hermitian line bundle $\overline{\mathcal{O}_{\mathbb{P}^n}(a)_P}$.

Remark 40. Note the analogy between the construction in the example above and the classical construction of the tautological line bundle of $\mathbb{P}^n$. In particular, we can interpret $\overline{\mathcal{O}_{\mathbb{P}^n}(-1)_P}$ as $\mathcal{O}_{\mathbb{P}^n}(-1)_P \cap \overline{\mathcal{O}_K}^{n+1}$ where $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological geometric line bundle over the projective space $\mathbb{P}^n$.

2.6 Arakelov degree

In this section, we will review some of the important properties of the Arakelov degree of Hermitian vector bundles.

Definition 41. Let $\overline{L} = (L, h)$ be a Hermitian line bundle over S and $s \in L \setminus \{0\}$ a non-trivial rational section. The *Arakelov degree of the line bundle* $\overline{L}$ is defined as

$$\begin{aligned}\widehat{\deg}(\overline{L}) :&= \log|L/\mathcal{O}_K s| - \sum_{\sigma \in \Sigma_K} \log\|s\|_\sigma \\ &= \sum_{\mathfrak{p} \subseteq \mathcal{O}_K} v_\mathfrak{p}(s) \log N(\mathfrak{p}) - \sum_{\sigma \in \Sigma_K} \log\|s\|_\sigma ,\end{aligned}$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of $\mathcal{O}_K$, $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ is the norm of the ideal $\mathfrak{p}$ and $v_\mathfrak{p}(s)$ denotes the $\mathfrak{p}$-adic valuation of s seen as a section of the invertible sheaf over S associated to L. More concretely, if we consider the localization $L_\mathfrak{p} := L \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{p}}$, then $L_\mathfrak{p}$ is a free $\mathcal{O}_{K,\mathfrak{p}}$-module of rank one and therefore there exists an isomorphism (a trivialization) $i_\mathfrak{p} : L_\mathfrak{p} \to \mathcal{O}_{K,\mathfrak{p}}$. Then, $v_\mathfrak{p}(s) = v_\mathfrak{p}(i_\mathfrak{p}(s \otimes 1))$ via this identification. It follows from the product formula that the definition above is independent of the choice of the non-trivial section s.

The *Arakelov degree of a Hermitian vector bundle* $\overline{E} = (E, h)$ over S is defined as

$$\widehat{\deg}(\overline{E}) := \widehat{\deg}(\det(\overline{E})),$$

and its *norm* is defined as $\mathrm{N}(\overline{E}) := e^{\widehat{\deg}(\overline{E})} \in \mathbb{R}_{>0}$.

Example 42. For the trivial Hermitian vector bundle $\overline{O_K{}^r} = (O_K{}^r, h)$, we have $\widehat{\deg}(\overline{O_K{}^r}) = 0$. Indeed, by definition $\det(\overline{O_K{}^r}) = \overline{O_K}$. Thus, choosing $s = 1$ we see immediately that $\widehat{\deg}(\overline{O_K}) = 0$.

Example 43. Let $\overline{O_{\mathbb{P}^n}(1)_P}$ defined as in Example 39, and let $P = [x_0 : \ldots : x_n] \in \mathbb{P}^n(K)$. Then, it follows from [23, Proposition 9.10] that

$$\widehat{\deg}(\overline{O_{\mathbb{P}^n}(1)_P}) = \sum_{\mathfrak{p} \subset O_K} \log \max_i \{|x_i|_{\mathfrak{p}}\} + \sum_{\sigma \in \Sigma_K} \log \sqrt{\sum_i |x_i|_\sigma^2}.$$

The following example can be found in [5, Section 1.2.2].

Example 44. Consider the *canonical module* defined as

$$\omega_{O_K} := \mathrm{Hom}_{\mathbb{Z}}(O_K, \mathbb{Z}),$$

which is a projective O_K-module by defining $a \cdot f$ via $(a \cdot f)(b) := f(ab)$ for $a, b \in O_K$, $f \in \omega_{O_K}$. The Hermitian bundle $\overline{\omega_{O_K}} = (\omega_{O_K}, h)$ is defined by imposing $\|\mathrm{tr}_{K/\mathbb{Q}}\|_\sigma = 1$ for all $\sigma \in \Sigma_K$, where $\mathrm{tr}_{K/\mathbb{Q}} : K \to \mathbb{Q}$ is the usual trace map. We call this Hermitian bundle over S the *canonical Hermitian bundle*. It has Arakelov degree $\widehat{\deg}(\overline{\omega_{O_K}}) = \log|\Delta_K|$.

We refer the reader to [5, Section 1.3.1] for the following properties of the Arakelov degree.

Proposition 45. *Let $\overline{E}, \overline{F}$ be Hermitian vector bundles over S. Then:*

1. $\widehat{\deg}(\overline{E} \otimes \overline{F}) = \mathrm{rk}\, F \cdot \widehat{\deg}(\overline{E}) + \mathrm{rk}\, E \cdot \widehat{\deg}(\overline{F})$.

2. $\widehat{\deg}(\overline{E} \oplus \overline{F}) = \widehat{\deg}(\overline{E}) + \widehat{\deg}(\overline{F})$.

3. $\widehat{\deg}(\overline{E}^\vee) = -\widehat{\deg}(\overline{E})$.

2.7 Arakelov divisors over $\mathrm{Spec}(O_K)$

In this section we recall the language of Arakelov divisors and their relationship with Hermitian line bundles.

Definition 46. An *Arakelov divisor* over S is a formal finite sum

$$D = \sum_{\mathfrak{p} \subset O_K} x_{\mathfrak{p}} \mathfrak{p} + \sum_{\sigma \in \Sigma_K} x_\sigma \sigma, \tag{2.7.1}$$

with $x_{\mathfrak{p}} \in \mathbb{Z}$, and $x_\sigma \in \mathbb{R}$ satisfying $x_{\bar{\sigma}} = x_\sigma$ for all $\sigma \in \Sigma_K$.

Following [24, Chapter I, §5] we define

$$K_{\mathbb{R}}^+ := \Big\{(x_\sigma) \in \prod_{\sigma\in\Sigma_K}\mathbb{R} : x_{\bar\sigma} = x_\sigma\Big\}.$$

We then have an isomorphism of groups

$$\mathrm{Div}(K) \simeq \Big(\bigoplus_{\mathfrak{p}\subset\mathcal{O}_K}\mathbb{Z}\Big) \oplus K_{\mathbb{R}}^+,\quad \sum_{\mathfrak{p}\subset\mathcal{O}_K} x_{\mathfrak{p}}\mathfrak{p} + \sum_{\sigma\in\Sigma_K} x_\sigma\sigma \mapsto \Big((x_{\mathfrak{p}})_{\mathfrak{p}\subset\mathcal{O}_K}, (x_\sigma)_{\sigma\in\Sigma_K}\Big).$$

On $K_{\mathbb{R}}^+$ we consider the *canonical inner product*

$$\langle (x_\sigma), (y_\sigma)\rangle_{K_{\mathbb{R}}^+} := \sum_{\sigma\in\Sigma_K} n_\sigma x_\sigma y_\sigma,$$

where $n_\sigma = 1$ or 2 depending on whether σ is real or complex. This induces a canonical measure on $K_{\mathbb{R}}^+$ giving volume 1 to any cube generated by an orthonormal basis.

We endow $\mathrm{Div}(K)$ with the product topology of the discrete topology on $\mathbb{Z}$ and the Euclidean topology on $K_{\mathbb{R}}^+$, and with the product measure of the counting measure on $\mathbb{Z}$ and the canonical measure on $K_{\mathbb{R}}^+$.

Remark 47. In [13] Arakelov divisors are defined as formal sums as in (2.7.1), but with σ running over the Archimedean places of K. If we denote by $v_\sigma = v_{\bar\sigma}$ the Archimedean place associated to a pair of conjugated complex embeddings $\sigma, \bar\sigma \in \Sigma_K$, then the map $x_\sigma\sigma + x_{\bar\sigma}\bar\sigma \mapsto 2x_v$ induces an equivalence between the two notions of Arakelov divisor, which is compatible with the constructions presented in this section. In particular, the canonical measure on $K_{\mathbb{R}}^+ \simeq \mathbb{R}^{r_1+r_2}$ corresponds to the usual Lebesgue measure on $\mathbb{R}^{r_1+r_2}$.

Definition 48. The *degree* of the Arakelov divisor D is the real number

$$\deg(D) := \sum_{\mathfrak{p}\subset\mathcal{O}_K}\log(\mathrm{N}(\mathfrak{p}))x_{\mathfrak{p}} + \sum_{\sigma\in\Sigma_K} x_\sigma,$$

and we define the *norm* of D as $\mathrm{N}(D) := e^{\deg(D)}$.

Notation 49. Given $f \in K^*$, its associated *principal Arakelov divisor* is defined as

$$\mathrm{div}(f) := \sum_{\mathfrak{p}\subset\mathcal{O}_K} x_{\mathfrak{p}}\mathfrak{p} + \sum_{\sigma\in\Sigma_K} x_\sigma\sigma,$$

with $x_{\mathfrak{p}} = \mathrm{ord}_{\mathfrak{p}}(f)$ and $x_\sigma = -\log|\sigma(f)|$. The quotient group of $\mathrm{Div}(K)$ by its subgroup of principal Arakelov divisors is denoted by $\mathrm{Pic}(K)$ and is called the *Picard–Arakelov group*.

To each Arakelov divisor $D = \sum_{\mathfrak{p}} x_{\mathfrak{p}}\mathfrak{p} + \sum_\sigma x_\sigma\sigma$, we can associated a fractional ideal of K by means of $D \mapsto I_D = \prod_{\mathfrak{p}}\mathfrak{p}^{-x_{\mathfrak{p}}}$. Then, we have a surjective homomorphism

$$\mathrm{Div}(K) \to J(K),$$

where $J(K)$ is the group of fractional ideals of K. In particular, if the Archimedean part of D is zero, then $\mathrm{N}(D) = \mathrm{N}(I_D)^{-1}$. Moreover, we have the following result (see e.g. [24, Chapter III, Proposition 1.11 and Theorem 1.12]).

Lemma 50. *If we denote by $\mathrm{Pic}^0(K)$ the subgroup of degree zero Arakelov divisor classes in $\mathrm{Pic}(K)$ and by $\mathrm{Cl}(K)$ the ideal class group of the number field K, then we have an exact sequence*

$$0 \to H/\Lambda \to \mathrm{Pic}^0(K) \to \mathrm{Cl}(K) \to 0,$$

where $H := \{(x_\sigma) \in K_{\mathbb{R}}^+ : \sum_\sigma x_\sigma = 0\}$ and $\Lambda := \mathrm{Log}(O_K^*)$ where $\mathrm{Log}(a) = (\log|\sigma(a)|)_\sigma$ for $a \in K^*$. In particular, $\mathrm{Pic}^0(K)$ is compact.

We endow $\mathrm{Pic}(X)$ with the quotient measure of the product measure on $\mathrm{Div}(X)$. On $\mathrm{Div}^0(X)$ we consider the unique measure satisfying

$$\int_{\mathrm{Div}(X)} f(D)\,\mathrm{d}D = \int_{\mathbb{R}}\int_{\mathrm{Div}^0(X)} f(D_0 + xU)\,\mathrm{d}D_0\,\mathrm{d}x \tag{2.7.2}$$

for all $f \in L^1(\mathrm{Div}(X))$, where U is the divisor[1]

$$U := \frac{1}{\sqrt{r_1 + r_2}} \sum_{\sigma \in \Sigma_K} \frac{1}{n_\sigma}\sigma,$$

and endow $\mathrm{Pic}^0(K)$ with the corresponding quotient measure. The above lemma implies that the volume of $\mathrm{Pic}^0(K)$ equals

$$\mathrm{vol}(\mathrm{Pic}^0(K)) = h_K \,\mathrm{vol}(H/\Lambda) = h_K R_K \sqrt{r_1 + r_2}, \tag{2.7.3}$$

where R_K and h_K are the regulator and the class number of K, respectively (see [24, Chapter III, Proposition 7.5]).

In Sections 2.9 and 2.10 we will make use of the following formula.

Lemma 51. *Given a positive function $f \in L^1(\mathbb{R})$, we have*

$$\int_{\mathrm{Pic}(K)} f(\deg(D))\,\mathrm{d}D = h_K R_K \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

Proof. Property (2.7.2) implies

$$\int_{\mathrm{Pic}(K)} f(\deg(D))\,\mathrm{d}D = \int_{\mathbb{R}}\int_{\mathrm{Pic}^0(K)} f(x\deg(U))\,\mathrm{d}D\,\mathrm{d}x = \frac{\mathrm{vol}(\mathrm{Pic}^0(K))}{\deg(U)} \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

Then, the result follows from (2.7.3) together with $\deg(U) = \sqrt{r_1 + r_2}$. This proves the lemma. □

It is worth mentioning that given a Hermitian line bundle $\overline{L} = (L, h)$ over S, we can associate to it an Arakelov divisor in the following way: Let $s \in L$ be a non-trivial rational section and define

$$\mathrm{div}(s) := \sum_{\mathfrak{p} \subset O_K} v_{\mathfrak{p}}(s)\mathfrak{p} + \sum_{\sigma \in \Sigma_K} (-\log|s|_\sigma)\sigma.$$

[1] U corresponds to a vector in $K_{\mathbb{R}}^+$ that is orthogonal to H and has norm 1 with respect to the canonical inner product.

The class $D_{\overline{L}}$ of $\mathrm{div}(s)$ in $\mathrm{Pic}(K)$ is independent of the choice of the section s. Moreover, the degree $\widehat{\deg}(\overline{L})$ of the Hermitian line bundle $\overline{L}$ over S is equal to the degree $\deg(D_{\overline{L}})$ of the Arakelov divisor class $D_{\overline{L}}$.

Conversely, following [5, p. 32], given an Arakelov divisor $D = \sum_{\mathfrak{p}} x_{\mathfrak{p}}\mathfrak{p} + \sum_{\sigma} x_\sigma \sigma$, we can construct a Hermitian line bundle $\overline{O(D)} = (O_K(D), h = \{h_\sigma\})$ by defining $O_K(D) := I_D = \prod_{\mathfrak{p}} \mathfrak{p}^{-x_{\mathfrak{p}}}$, and for each embedding $\sigma \in \Sigma_K$ imposing that $\|1\|_\sigma = e^{-x_\sigma}$.

Notation 52. For an Arakelov divisor D and a rational section $f \in I_D$ we write

$$\|f\|_D = \|f\|_{\overline{O(D)}} = \sqrt{\sum_{\sigma\in\Sigma_K} \|f\|_\sigma^2} = \sqrt{\sum_{\sigma\in\Sigma_K} |\sigma(f)|^2 e^{-2x_\sigma}}.$$

Note that given Hermitian line bundles $\overline{L_1} = (L_1, h_1), \overline{L_2} = (L_2, h_2)$ with trivializations $i_{\mathfrak{p}} : L_{1,\mathfrak{p}} \to O_{K,\mathfrak{p}}$ and $j_{\mathfrak{p}} : L_{2,\mathfrak{p}} \to O_{K,\mathfrak{p}}$, the corresponding trivializations for $\overline{L_1} \otimes \overline{L_2}$, $k_{\mathfrak{p}} : (L_1 \otimes_{O_K} L_2)_{\mathfrak{p}} \to O_{K,\mathfrak{p}}$ are given by $k_{\mathfrak{p}}(s \otimes t) = i_{\mathfrak{p}}(s) j_{\mathfrak{p}}(t)$. Thus, $v_{\mathfrak{p}}(s \otimes t) = v_{\mathfrak{p}}(s) + v_{\mathfrak{p}}(t)$. This shows that the map $\widehat{\mathrm{Pic}}(S) \to \mathrm{Pic}(K)$ given by $\overline{L} \mapsto D_{\overline{L}}$ is a group homomorphism.

From the above discussion, we conclude the following.

Proposition 53. *The map $\widehat{\mathrm{Pic}}(S) \to \mathrm{Pic}(K)$, $\overline{L} \mapsto D_{\overline{L}}$ is a group isomorphism.*

By abuse of notation, we will employ this isomorphism to treat Arakelov divisor classes as Hermitian line bundles (and vice versa) when the context does not lead to confusion. For example, for an Arakelov divisor class $D \in \mathrm{Pic}(K)$ and $\overline{L} \in \widehat{\mathrm{Pic}}(S)$, we write $D \otimes \overline{L}$ to refer to the element $\overline{O(D)} \otimes \overline{L} \in \widehat{\mathrm{Pic}}(S)$.

2.8 The Poisson–Riemann–Roch formula

Definition 54. Let $\overline{E} = (E, h)$ be a Hermitian vector bundle over $\mathrm{Spec}\,\mathbb{Z}$, and define

$$h^0(\overline{E}) := \log \sum_{v\in E} e^{-\pi\|v\|_{\overline{E}}^2},$$

where $\|\cdot\|_{\overline{E}}$ denotes the norm on $E \otimes_{\mathbb{Z}} \mathbb{C}$ associated to h. More generally, for a Hermitian vector bundle $\overline{E} = (E, h)$ over S we put

$$h^0(\overline{E}) := h^0(\eta_* \overline{E}).$$

We also define the *number of non-trivial sections* of $\overline{E}$ by

$$\varphi(\overline{E}) := e^{h^0(\overline{E})} - 1.$$

It is worth mentioning that the previous definition coincides with the one given in [13, Section 3] for Arakelov divisors. More precisely, the authors consider an Arakelov divisor $D = \sum_{\mathfrak{p}\subset O_K} x_{\mathfrak{p}}\mathfrak{p} + \sum_{\sigma} x_\sigma \sigma$, and define $h^0(D) = \log k^0(D)$ where

$$k^0(D) := \sum_{f\in I_D} e^{-\pi\|f\|_D^2}.$$

Then, a simple computation shows that $h^0(\overline{\mathcal{O}(D)}) = h^0(D)$.

Lemma 55. *For $\overline{E}$ and $\overline{F}$ Hermitian vector bundles over S, we have that*

1. $h^0(\overline{E} \oplus \overline{F}) = h^0(\overline{E}) + h^0(\overline{F})$, *and*
2. $\varphi(\overline{E} \oplus \overline{F}) = \varphi(\overline{E}) + \varphi(\overline{F}) + \varphi(\overline{E})\varphi(\overline{F})$.

Proof. Since $\eta_*(\overline{E} \oplus \overline{F}) = \eta_*(\overline{E}) \oplus \eta_*(\overline{F})$, it is enough to prove item (1) for Hermitian vector bundles over $\mathrm{Spec}(\mathbb{Z})$. In that case, we have

$$\sum_{(x,y)\in E\oplus F} e^{-\pi\|(x,y)\|^2_{\overline{E}\oplus\overline{F}}} = \sum_{(x,y)\in E\oplus F} e^{-\pi\left(\|x\|^2_{\overline{E}}+\|y\|^2_{\overline{F}}\right)}$$
$$= \left(\sum_{x\in E} e^{-\pi\|x\|^2_{\overline{E}}}\right)\left(\sum_{y\in F} e^{-\pi\|y\|^2_{\overline{F}}}\right).$$

Taking logarithms we conclude (1). Item (2) is a direct consequence of (1). This proves the lemma. ☐

The following formula follows from Lemma 55(2) by induction.

Corollary 56. *Let $\sigma_1, \ldots, \sigma_n$ be the elementary symmetric polynomials in n variables and let $\overline{E}_1, \ldots, \overline{E}_n$ be Hermitian vector bundles over S. Then*

$$\varphi(\overline{E}_1 \oplus \ldots \oplus \overline{E}_n) = \sum_{i=1}^{n} \sigma_i(\varphi(\overline{E}_1), \ldots, \varphi(\overline{E}_n)).$$

We can now state the Poisson–Riemann–Roch formula for Hermitian vector bundles over the arithmetic curve S. See [5, Section 2.2.2] for details.

Theorem 57. *Let $\overline{E}$ be a Hermitian vector bundle over S. Then*

$$h^0(\overline{E}) - h^0(\overline{\omega_{\mathcal{O}_K}} \otimes \overline{E}^{\vee}) = \widehat{\deg}(\overline{E}) - \frac{1}{2}(\log|\Delta_K|)\,\mathrm{rk}(\overline{E}).$$

Equivalently, we have $\varphi(\overline{E}) = \left(\varphi(\overline{E}^{\vee} \otimes \overline{\omega_{\mathcal{O}_K}}) + 1\right)\mathrm{N}(\overline{E})|\Delta_K|^{-\frac{\mathrm{rk}(\overline{E})}{2}} - 1.$

In [13, Section 5, Corollary 1] the authors prove the following bound for the number of non-trivial sections of Arakelov divisors with bounded degree.

Proposition 58. *Let $C \in \mathbb{R}$ and let D be an Arakelov divisor over S with $\deg(D) \leq C$. Then*

$$\varphi(D) := \varphi(\overline{\mathcal{O}(D)}) \leq \beta e^{-\pi n_K e^{-\frac{2}{n_K}\deg(D)}},$$

for some $\beta > 0$ depending on C and K, where $n_K = [K : \mathbb{Q}]$.

Remark 59. In the proof of [13, Section 5, Corollary 1], the authors assumed that $\deg(D) \geq \frac{1}{2}\log(|\Delta_K|)$. Their proof can be adapted to Arakelov divisors with $\deg(D) \geq C$ by considering

$$u = \frac{1}{n_K}(C - \deg(D)) \text{ and } D' = D + \sum u\sigma,$$

instead of the u and D' used in their proof of [13, Proposition 2].

In order to deal with Hirzebruch–Kleinschmidt varieties, we will need the following bound.

Proposition 60. *Let $\overline{E}$ be a split Hermitian vector bundle over S, i.e., $\overline{E} = \overline{L}_1 \oplus \cdots \oplus \overline{L}_r$ where each $\overline{L}_i$ is a Hermitian line bundle, and let $\overline{L} \in \widehat{\mathrm{Pic}}(S)$ such that $\widehat{\deg}(\overline{L}) \geq C$ for some $C \in \mathbb{R}$. Then, there exist $\beta, \gamma > 0$ depending on C, K and $\overline{E}$, such that*

$$\varphi(\overline{E} \otimes \overline{L}) \leq \beta e^{-\gamma e^{\frac{2}{n_K}\widehat{\deg}(\overline{L})}}$$

where $n_K = [K : \mathbb{Q}]$.

Proof. We have

$$\varphi(\overline{E} \otimes \overline{L}) = \varphi((\overline{L}_1 \oplus \cdots \oplus \overline{L}_r) \otimes \overline{L}).$$

As $\varphi((\overline{L}_1 \oplus \cdots \oplus \overline{L}_r) \otimes \overline{L})$ depends polynomially on $\varphi(\overline{L}_i \otimes \overline{L})$ by Corollary 56, it is enough to prove the bound in the particular case $\overline{E} = \overline{L}_i$. Since $\widehat{\deg}(\overline{L}_i \otimes \overline{L}) = \widehat{\deg}(\overline{L}_i) + \widehat{\deg}(\overline{L}) \geq C + \widehat{\deg}(\overline{L}_i)$, it follows from Proposition 58 that there exists $\beta_i > 0$, depending on C, K and $\overline{L}_i$, such that

$$\varphi(\overline{L}_i \otimes \overline{L}) \leq \beta_i e^{-\pi n_K e^{\frac{2}{n_K}\widehat{\deg}(\overline{L}_i \otimes \overline{L})}} = \beta e^{-\gamma_i e^{\frac{2}{n_K}\widehat{\deg}(\overline{L})}}$$

were $\gamma_i := \pi n_K e^{\frac{2}{n_K}\widehat{\deg}(\overline{L}_i)}$. This proves the desired result. □

In the particular case when $\overline{L}$ is a Hermitian line bundle, the following uniform bound can be obtained (see [5, Proposition 2.7.3]).

Proposition 61. *Let $\theta \in \mathbb{R}_{\geq 0}$ and $\overline{L}$ be a Hermitian line bundle over S such that $\widehat{\deg}(\overline{L}) \leq \theta$. Then $h^0(\overline{L}) \leq 1 + \theta$. In particular, $\varphi(\overline{L}) \leq e^{1+\theta}$.*

2.9 Height zeta function of the projective space

In this section, we introduce a zeta function of the field K defined in [13]. This zeta function will allow us to study the analytic properties of the height zeta function of the projective space via a suitable integral representation.

We first define the *effectivity* $e(D)$ of an Arakelov divisor $D = \sum_{\mathfrak{p}} x_{\mathfrak{p}}\mathfrak{p} + \sum_{\sigma} x_\sigma \sigma$ in $\mathrm{Div}(K)$ as

$$e(D) := \begin{cases} e^{-\pi \|1\|_D^2} = e^{-\pi \sum_\sigma e^{-2x_\sigma}} & \text{if } x_{\mathfrak{p}} \geq 0 \text{ for all } \mathfrak{p} \subset \mathcal{O}_K, \\ 0 & \text{otherwise.} \end{cases}$$

In [13, Section 4], the authors define the zeta function associated to K as

$$\xi_K(s) := \int_{\mathrm{Div}(K)} \mathrm{N}(D)^{-s} e(D)\, \mathrm{d}D, \quad s \in \mathbb{C}, \Re(s) > 1,$$

and they prove that[2] $\xi_K(s) = 2^{-r_1}\left(\pi^{-s/2}\Gamma(s/2)\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_K(s)$, where

$$\zeta_K(s) := \sum_{\{0\} \neq J \subseteq \mathcal{O}_K} N(J)^{-s}$$

is the Dedekind zeta function of the field K. In particular, $\xi_K(s)$ has meromorphic continuation to $s \in \mathbb{C}$. Moreover, the authors show that

$$\xi_K(s) = \frac{1}{w_K}\int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\varphi(D)\, \mathrm{d}D. \tag{2.9.1}$$

Let $n \geq 1$ be an integer. Given $P = [x_0 : \ldots : x_n] \in \mathbb{P}^n(K)$, we define the *standard height* of P as

$$H_{\mathbb{P}^n}(P) := \prod_{\mathfrak{p} \subseteq \mathcal{O}_K} \max_i \{|x_i|_{\mathfrak{p}}\} \prod_{\sigma \in \Sigma_K} \sqrt{\sum_i |x_i|_\sigma^2}.$$

It follows from Example 43 that

$$H_{\mathbb{P}^n}(P) = \mathrm{N}(\overline{\mathcal{O}_{\mathbb{P}^n}(1)_P}) \text{ for every } P \in \mathbb{P}^n(K).$$

The associated *height zeta function* is

$$\mathrm{Z}_{\mathbb{P}^n}(s) := \sum_{P \in \mathbb{P}^n(K)} H_{\mathbb{P}^n}(P)^{-s} = \sum_{P \in \mathbb{P}^n(K)} \mathrm{N}(\overline{\mathcal{O}_{\mathbb{P}^n}(1)_P})^{-s},$$

defined for $s \in \mathbb{C}$ with $\Re(s) > n+1$ (the series converges absolutely and uniformly on compact subsets of this domain). We will study this function by means of Arakelov geometry.

As in [22, Section 3.2], we will work with a K-vector space V of dimension $n+1$ that contains a complete $\mathcal{O}_K$-lattice E which is the underlying finitely generated projective $\mathcal{O}_K$-module of a Hermitian vector bundle $\overline{E} = (E, h)$. Analogous to the construction carried out in Example 39, given a point $P \in \mathbb{P}(V)$, we define its height by

$$H_{\mathbb{P}(V)}(P) := \mathrm{N}(\overline{\mathcal{O}_{\mathbb{P}(V)}(1)_P}).$$

We also denote by $\mathrm{Z}_{\mathbb{P}(V)}(s)$ the corresponding height zeta function. In particular, considering $V = K^{\oplus(n+1)}$ we have $H_{\mathbb{P}(K^{\oplus(n+1)})}(P) = H_{\mathbb{P}^n}(P)$.

[2] There is a misprint in the first power of 2 appearing in the third line of the computation leading to the formula for $\xi_K(s)$ in [13, p. 388]. In the computation of the integral over t for σ real, the factor 2 should appear in the denominator.

Recall that if $D \in \mathrm{Pic}(K)$, we denote the Hermitian line bundle $\overline{\mathcal{O}(D)}$ simply by D. As explained before, the key idea is to express the height zeta function $Z_{\mathbb{P}^n}(s)$ as a suitable integral. To do so, we note that (2.9.1) implies that

$$\begin{aligned} w_K \xi_K(s) H_{\mathbb{P}(V)}(P)^{-s} &= \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \varphi(D) H_{\mathbb{P}(V)}(P)^{-s} \, \mathrm{d}D \\ &= \int_{\mathrm{Pic}(K)} (\mathrm{N}(D) N(\overline{\mathcal{O}_{\mathbb{P}(V)}(1)_P}))^{-s} \varphi(D) \, \mathrm{d}D \\ &= \int_{\mathrm{Pic}(K)} \mathrm{N}(D \otimes \overline{\mathcal{O}_{\mathbb{P}(V)}(1)_P})^{-s} \varphi(D) \, \mathrm{d}D \\ &= \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{\mathcal{O}_{\mathbb{P}(V)}(-1)_P}) \, \mathrm{d}D. \end{aligned}$$

If we fix $D \in \mathrm{Pic}(K)$ and we let P run through $\mathbb{P}(V)$, then $D \otimes \overline{\mathcal{O}_{\mathbb{P}(V)}(-1)_P}$ runs through all subline bundles of $\overline{\mathcal{O}(D)} \otimes \overline{E}$. Therefore, the above formula implies that

$$\begin{aligned} w_K \xi_K(s) Z_{\mathbb{P}(V)}(s) &= \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \sum_{P \in \mathbb{P}(V)} \varphi(D \otimes \overline{\mathcal{O}_{\mathbb{P}(V)}(-1)_P}) \, \mathrm{d}D \\ &= \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E}) \, \mathrm{d}D. \end{aligned} \tag{2.9.2}$$

Notation 62. We denote by

$$\mathrm{Pic}(K) := \left\{ D \in \mathrm{Pic}(K) : \mathrm{N}(D) \le \sqrt{|\Delta_K|} \right\}$$

the set of Arakelov divisor classes with norm bounded above by $\sqrt{|\Delta_K|}$.

Proposition 63. *Let V be a K-vector space of dimension $n+1$ containing a complete $\mathcal{O}_K$-lattice E which is the underlying finitely generated projective $\mathcal{O}_K$-module of a Hermitian vector bundle $\overline{E} = (E, h)$. Then:*

1. *The integral*
$$\int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E}) \, \mathrm{d}D$$
converges absolutely and uniformly for s in compact subsets of $\mathbb{C}$.

2. *For $\Re(s) > n+1$ we have*

$$\begin{aligned} w_K \xi_K(s) Z_{\mathbb{P}(V)}(s) &= \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E}) \, \mathrm{d}D \\ &\quad + \mathrm{N}(\overline{E}) |\Delta_K|^{\frac{(n+1)}{2} - s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s-(n+1)} \varphi(D \otimes \overline{E}^{\vee}) \, \mathrm{d}D \\ &\quad + R_K h_K |\Delta_K|^{\frac{s}{2}} \left(\frac{\mathrm{N}(\overline{E})}{s-(n+1)} - \frac{1}{s} \right). \end{aligned}$$

Proof. Item (1) follows from Proposition 60. Indeed, since Arakelov divisor classes $D \in \mathrm{Pic}(K)$ satisfy $\widehat{\deg}(D) \geq -\frac{1}{2}\log(|\Delta_K|)$, there are constants $\beta, \gamma > 0$ depending on K and $\overline{E}$ such that

$$\int_{\mathrm{Pic}(K)} \left|\mathrm{N}(D)^{-s}\right| \varphi(D \otimes \overline{E})\, \mathrm{d}D \leq \int_{\mathrm{Pic}(K)} \left|\mathrm{N}(D)^{-s}\right| \beta e^{-e^{\frac{2}{n_K}\widehat{\deg}(D)}}\, \mathrm{d}D.$$

Using Lemma 51 we get

$$\int_{\mathrm{Pic}(K)} \left|\mathrm{N}(D)^{-s}\right| \varphi(D \otimes \overline{E})\, \mathrm{d}D \leq R_K h_K \beta \int_{-\frac{1}{2}\log(|\Delta_K|)}^{\infty} e^{-\Re(s)x - e^{\frac{2}{n_K}x}}\, \mathrm{d}x,$$

and this last integral converges uniformly for s in compact subsets of $\mathbf{C}$. This proves item (1).

Now, if we define $\mathrm{Pic}(K)_+ := \{D \in \mathrm{Pic}(K) : \mathrm{N}(D) \leq \sqrt{|\Delta_K|}\}$, then by (2.9.2) we have

$$w_K \xi_K(s) Z_{\mathrm{P}(V)}(s) = \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E})\, \mathrm{d}D + \int_{\mathrm{Pic}(K)_+} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E})\, \mathrm{d}D.$$

In order to compute the integral over $\mathrm{Pic}(K)_+$, we consider the change of variables $D \mapsto \omega_{O_K} \otimes D^{\vee}$ to get

$$\begin{aligned}
\int_{\mathrm{Pic}(K)_+} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E})\, \mathrm{d}D &= \int_{\mathrm{Pic}(K)} \mathrm{N}(\omega_{O_K} \otimes D^{\vee})^{-s} \varphi(\omega_{O_K} \otimes D^{\vee} \otimes \overline{E})\, \mathrm{d}D \\
&= \int_{\mathrm{Pic}(K)} |\Delta_K|^{-s} \mathrm{N}(D)^{s} \varphi(\omega_{O_K} \otimes D^{\vee} \otimes \overline{E})\, \mathrm{d}D.
\end{aligned}$$

By Theorem 57 we have

$$\begin{aligned}
\varphi(D^{\vee} \otimes \omega_{O_K} \otimes \overline{E}) &= e^{h^0(\omega_{O_K} \otimes (D \otimes \overline{E}^{\vee})^{\vee})} - 1 \\
&= e^{h^0(D \otimes \overline{E}^{\vee}) - \deg(D \otimes \overline{E}^{\vee}) + \log|\Delta_K| \frac{\mathrm{rk}(D \otimes \overline{E}^{\vee})}{2}} - 1 \\
&= \left(\varphi(D \otimes \overline{E}^{\vee}) + 1\right) \mathrm{N}(D)^{-(n+1)} \mathrm{N}(\overline{E}) |\Delta_K|^{\frac{n+1}{2}} - 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathrm{Pic}(K)_+} \mathrm{N}(D)^{-s} \varphi(D \otimes \overline{E})\, \mathrm{d}D \\
&\quad = \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s} |\Delta_K|^{-s} \left(\left(\varphi(D \otimes \overline{E}^{\vee}) + 1\right) \mathrm{N}(D)^{-(n+1)} \mathrm{N}(\overline{E}) |\Delta_K|^{\frac{n+1}{2}} - 1\right) \mathrm{d}D \\
&\quad = \mathrm{N}(\overline{E}) |\Delta_K|^{\frac{n+1}{2} - s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s-(n+1)} \varphi(D \otimes \overline{E}^{\vee})\, \mathrm{d}D \\
&\qquad + \mathrm{N}(\overline{E}) |\Delta_K|^{\frac{n+1}{2} - s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s-(n+1)}\, \mathrm{d}D \\
&\qquad - |\Delta_K|^{-s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s}\, \mathrm{d}D.
\end{aligned}$$

Now, for $\Re(s) > n+1$ we have (using Lemma 51)

$$\begin{aligned}\int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s-(n+1)}\,\mathrm{d}D &= \int_{\mathrm{Pic}(K)} e^{(s-(n+1))\deg(D)}\,\mathrm{d}D \\ &= R_K h_K \int_{-\infty}^{\frac{1}{2}\log(|\Delta_K|)} e^{(s-(n+1))x}\,\mathrm{d}x \\ &= R_K h_K \frac{|\Delta_K|^{\frac{s-(n+1)}{2}}}{s-(n+1)}.\end{aligned}$$

Analogously, for $\Re(s) > 0$, we have

$$\int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s}\,\mathrm{d}D = R_K h_K \frac{|\Delta_K|^{\frac{s}{2}}}{s}.$$

This proves item (2) and completes the proof of the proposition. □

Remark 64. Proposition 63 also holds in the case $n = 0$, $V = K$ and $E = \overline{\mathcal{O}_K}$, in which case $\mathrm{Z}_{\mathbb{P}(V)} = 1$. In particular:

$$\begin{aligned} w_K \xi_K(s) = \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\varphi(D)\,\mathrm{d}D + |\Delta_K|^{\frac{1-s}{2}} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{s-1}\varphi(D)\,\mathrm{d}D \\ + R_K h_K |\Delta_K|^{-\frac{s}{2}} \left(\frac{1}{s-1} - \frac{1}{s}\right), \end{aligned} \tag{2.9.3}$$

and this gives the meromorphic continuation of $\xi_K(s)$ to $\mathbb{C}$ (as in [13, Section 4]).

As consequence of Proposition 63 we have the following result (see [22, Theorem 3.2]).

Theorem 65 (Maruyama)**.** *Let V be a K-vector space of dimension $n+1$ containing a complete $\mathcal{O}_K$-lattice E which is the underlying finitely generated projective $\mathcal{O}_K$-module of a Hermitian vector bundle $\overline{E} = (E, h)$. Then, the function $\mathrm{Z}_{\mathbb{P}(V)}(s)$ has meromorphic continuation to the whole complex plane, which is holomorphic for $\Re(s) > 1$, $s \neq n+1$, and with a simple pole at $s = n+1$. Moreover, we have*

$$\mathrm{Res}_{s=n+1} \mathrm{Z}_{\mathbb{P}(V)}(s) = \frac{R_K h_K \mathrm{N}(\overline{E})}{w_K |\Delta_K|^{\frac{n+1}{2}} \xi_K(n+1)}.$$

Choosing $V = K^{\oplus(n+1)}$ and $\overline{E} = \overline{\mathcal{O}_K}^{\oplus(n+1)}$, in which case $\mathrm{N}(\overline{E}) = 1$, we obtain the following corollary as an application of Theorem 27.

Corollary 66 (Schanuel's estimate)**.** *Let $N(\mathbb{P}^n, B) := \#\{P \in \mathbb{P}^n(K) : H_{\mathbb{P}^n}(P) \leq B\}$. Then*

$$N(\mathbb{P}^n, B) \sim C B^{n+1} \quad \text{as } B \to \infty,$$

with

$$C := \frac{R_K h_K}{(n+1) w_K |\Delta_K|^{\frac{n+1}{2}} \xi_K(n+1)}.$$

Remark 67. Note that the asymptotic constant given above is in general different from to the one obtained by Schanuel in [34]. This is due to the fact that Schanuel uses an ℓ^1 norm on the non-Archimedean places, while we use an ℓ^2 norm. Also, compare this result with the one obtained by Guignard in [14, Cor. 3.4.2], taking into account that Guignard defines $\mathrm{N}(\overline{E}) = e^{-\widehat{\deg}(\overline{E})}$.

2.10 Counting rational points on Hirzebruch–Kleinschmidt varieties

In this section, we construct an Arakelov height function H_L associated to a big line bundle class $L \in \mathrm{Pic}(X_d(a_1, \ldots, a_r))$, and describe the asymptotic growth of the number $N(U, H_L, B) := \#\{P \in U(K) : H_L(P) \leq B\}$, where $U = U_d(a_1, \ldots, a_r)$ is the good open subset of $X_d(a_1, \ldots, a_r)$, as defined in the Introduction. The main results are Theorems 70 and 76, which are used in Section 2.10.2 to prove Theorem 2. Finally, in Section 2.10.3, we briefly discuss subvarieties that accumulate more rational points than others and provide criteria to determine when this occurs.

2.10.1 Heights induced by big line bundles

Let $X = X_d(a_1, \ldots, a_r)$ be a Hirzebruch–Kleinschmidt variety of dimension $d = r + t - 1$ defined over the number field K (see Definition 28). Recall that

$$\pi : X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1 - a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r)) \to \mathbb{P}^{t-1}$$

is a projective vector bundle over $\mathbb{P}^{t-1}$.

Let us define

$$W := \mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r - a_{r-1}), \tag{2.10.1}$$

and recall that for $P \in X(K)$ the fiber

$$\mathcal{O}_X(-1)_P = \ell \subset (\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1 - a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r))^{\vee}_{\pi(P)},$$

is given by the one-dimensional subspace ℓ of the $(r+1)$-dimensional vector space

$$(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r))^{\vee}_{\pi(P)} = W_{\pi(P)},$$

corresponding to the point P in $\pi^{-1}(\pi(P)) = \mathbb{P}(W_{\pi(P)})$. The vector space $W_{\pi(P)}$ contains the Hermitian vector bundle

$$\overline{(\mathcal{O}_{\mathbb{P}^{t-1}})_{\pi(P)}} \oplus \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(a_r)_{\pi(P)}} \oplus \cdots \oplus \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(a_r - a_{r-1})_{\pi(P)}},$$

which we denote by $\overline{W_{\pi(P)}}$ for simplicity. By endowing $\ell \cap \overline{W_{\pi(P)}}$ with the restriction of the Hermitian forms on $\overline{W_{\pi(P)}}$, we obtain the Hermitian line bundle $\overline{\mathcal{O}_X(-1)_P}$. As usual, by taking duals and tensor powers we define $\overline{\mathcal{O}_X(a)_P}$ for any $a \in \mathbb{Z}$.

Given $L = \lambda h + \mu f \in \mathrm{Pic}(X)$ big and $P \in X(K)$, we put

$$\overline{L_P} := \overline{\mathcal{O}_X(\lambda)_P} \otimes \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(\mu)}_{\pi(P)}.$$

This induces an *adelic metric* on L as defined in [27, *Définition 1.4*]. We refer to this as the *standard metric* on L.

We can now define the *standard height function* H_L over $X(K)$ associated to L as

$$H_L(P) := \mathrm{N}(\overline{L_P}).$$

More explicitly, we have

$$H_L(P) = e^{-\widehat{\deg}(\overline{\mathcal{O}_X(1)_P})^{\lambda}} e^{-\widehat{\deg}(\overline{\mathcal{O}_{\mathbb{P}^{t-1}}(1)}_{\pi(P)})^{\mu}} = H_{\mathbb{P}(W_{\pi(P)})}(P)^{\lambda} H_{\mathbb{P}^{t-1}}(\pi(P))^{\mu}. \tag{2.10.2}$$

Associated to $L = \lambda h + \mu f$ as above, we define

$$\lambda_L := \frac{r+1}{\lambda}, \quad \mu_L := \frac{(r+1)a_r + t - |a|}{\mu}. \tag{2.10.3}$$

Then, it easily follows from Proposition 30 that

$$a(L) = \max\{\lambda_L, \mu_L\} \quad \text{and} \quad b(L) = \begin{cases} 2 & \text{if } \lambda_L = \mu_L, \\ 1 & \text{if } \lambda_L \neq \mu_L. \end{cases} \tag{2.10.4}$$

As in the Introduction, we restrict our attention to rational points in a specific open subset $U \subseteq X$. This is done in order to ensure that $N(U, H_L, B)$ is finite for all $B > 0$, and to avoid possible proper subvarieties with too many rational points.

Recall that in Section 2.3 we defined the projective subbundle $F = \mathbb{P}(Y) \subseteq X_d(a_1, \dots, a_r)$ where

$$Y = \mathcal{O}_{\mathbb{P}^{t-1}}(-a_r) \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1 - a_r) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_{r-1} - a_r).$$

The following definition was given in the Introduction.

Definition 68. Given integers $r \geq 1$, $t \geq 2$ and $0 \leq a_1 \leq \cdots \leq a_r$, we define the *good open subset* of $X = X_d(a_1, \dots, a_d)$ as

$$U_d(a_1, \dots, a_r) := X_d(a_1, \dots, a_r) \setminus F.$$

Remark 69. Given a big line bundle class $L \in \mathrm{Pic}(X)$, it is know that there exists a dense open subset $U_L \subseteq X$ such that $N(U_L, H_L, B)$ is finite for every $B > 0$ (see e.g. [30, Proposition 2.12]). Our good open subset U serves as such a dense open U_L for every big L.

The first main result of this section is the following theorem, where we assume $a_r > 0$. The easier case when $a_r = 0$ is presented later in this section (see Theorem 76). Recall that, for $m \geq 1$, we defined $Z_{\mathbb{P}^m}(s)$ as the height zeta function of the projective space $\mathbb{P}^m$ with respect to the standard height function (see Section 2.9). Here, we extend this definition by putting $Z_{\mathbb{P}^m}(s) := 1$ (resp. 0) if $m = 0$ (resp. $m = -1$).

Theorem 70. *Let $X = X_d(a_1, \dots, a_r)$ be a Hirzebruch–Kleinschmidt variety over the number field K of dimension $d = r + t - 1$, and let $L = \lambda h + \mu f \in \operatorname{Pic}(X)$ big. Assume $a_r > 0$. Then, we have*

$$N(U, H_L, B) \sim C_{L,K} B^{a(L)} \log(B)^{b(L)} \quad \text{as } B \to \infty,$$

with $C_{L,K}$ given by

$$\begin{cases} \dfrac{R_K^2 h_K^2 |\Delta_K|^{-\frac{(d+2)}{2}}}{w_K^2 (r+1)\,\zeta_K(r+1)\,\zeta_K(t)} & \text{if } \lambda_L = \mu_L, \\ \dfrac{R_K h_K |\Delta_K|^{-\frac{r+1}{2}}}{w_K (r+1)\,\zeta_K(r+1)} Z_{\mathbb{P}^{t-1}}(\mu\lambda_L + |\mathbf{a}| - (r+1)a_r) & \text{if } \lambda_L > \mu_L, \\ \dfrac{R_K h_K |\Delta_K|^{-\frac{t-N_X+(r+1)}{2}}\,\zeta_K(-\ell_L + N_X - (r+1))}{w_K((r+1)a_r + t - |\mathbf{a}|)\,\zeta_K(-\ell_L)\,\zeta_K(t)} \\ \quad \cdot \left(Z_{\mathbb{P}^{N_X-1}}(\lambda\mu_L + N_X - (r+1)) - Z_{\mathbb{P}^{N_X-2}}(\lambda\mu_L + N_X - (r+1))\right) & \text{if } \lambda_L < \mu_L, \end{cases}$$

where $N_X := \#\{i \in \{1, \dots, r\} : a_i = a_r\}$.

In the proof of Theorem 70 below, we study the analytic properties of the height zeta function

$$Z_{U,L}(s) := \sum_{P \in U(K)} H_L(P)^{-s}$$

associated to the big line bundle class L and the good open subset $U := U_d(a_1, \dots, a_r)$, and we make use of the following three lemmas.

Lemma 71. *For an integer $m \geq 0$, define*

$$\varphi_m(D) := \begin{cases} \varphi(D) & \text{if } m = 0, \\ \varphi(D)\varphi(D^{-m}) & \text{if } m \geq 1. \end{cases} \tag{2.10.5}$$

Then, the following properties hold:

1. *For $D \in \operatorname{Pic}(K)$ we have $\varphi(D^{-m}) = O(1)$, with an implicit constant depending only on m and on the base field K.*
2. *The integral*
$$\int_{\operatorname{Pic}(K)} \mathrm{N}(D)^{-s} \varphi_m(D)\, \mathrm{d}D$$
converges absolutely and uniformly for s in compact subsets of $\mathbb{C}$.

Proof. Item (1) follows from Proposition 60 since $\varphi(D^{-n}) = \varphi(D^{-1} \otimes \overline{\mathcal{O}_K}^{\,n})$ is bounded above, for $D \in \operatorname{Pic}(K)$, by a constant depending only on K and n, hence we can use this fact for $n = 1$ and $n = m$. Item (2) follows directly from Proposition 63(1) and the fact that $\varphi(D^{-m})$ is bounded. This proves the lemma. □

Lemma 72. *Given positive integers $0 < b_1 \le b_2 \le \cdots \le b_n$ and $Q \in \mathbb{P}^{t-1}$, define*

$$\overline{E_Q} := \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(b_1)_Q} \oplus \cdots \oplus \overline{(\mathcal{O}_{\mathbb{P}^{t-1}}(b_n))_Q},$$

and $|\mathbf{b}| := \sum_{i=1}^{n} b_i$. Then, for every integer $m \ge 0$, every compact subset $\mathcal{K} \subset \mathbb{C}$ and every $s \in \mathcal{K}$, we have

$$\int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\varphi_m(D)\varphi\left(D \otimes \overline{E_Q}\right) \mathrm{d}D = |\Delta_K|^{-\frac{n}{2}} H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{n-s}\varphi_m(D)\,\mathrm{d}D + O(H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|-b_1}),$$

with an implicit constant depending only on $m, \mathcal{K}, b_1, \dots, b_n$ and on the base field K, where $\varphi_m(D)$ is defined in (2.10.5).

Proof. Let us define

$$\mathrm{I}_Q := \{D \in \mathrm{Pic}(K) : N(D)H_{\mathbb{P}^{t-1}}(Q)^{b_1} \le \sqrt{|\Delta_K|}\},$$
$$\mathrm{II}_Q := \{D \in \mathrm{Pic}(K) : \sqrt{|\Delta_K|} < N(D)H_{\mathbb{P}^{t-1}}(Q)^{b_1}\},$$

and for $i \in \{1, \dots, n\}$ put $\overline{E_Q^{(i)}} := \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(b_i)_Q}$. Fix a compact subset $\mathcal{K} \subset \mathbb{C}$ and assume $s \in \mathcal{K}$. In what follows, all terms of the form $O(\dots)$ are meant to have implicit constants depending only on $m, \mathcal{K}, b_1, \dots, b_n$ and the base field K.

First, from Lemma 71(1) with $m = 1$ it follows that there exists a constant $C_1 \ge 1$, depending only on the base field K, such that $\varphi(D) \le C_1$ for all $D \in \mathrm{Pic}(K)$. Equivalently, $\varphi\left(D \otimes \overline{E_Q^{(1)}}\right) \le C_1$ for all $D \in \mathrm{I}_Q$. Also, by Proposition 61 we have

$$\varphi\left(D \otimes \overline{E_Q^{(i)}}\right) \le e^{1+\frac{1}{2}\log(|\Delta_K|)+b_i\log(H_{\mathbb{P}^{t-1}}(Q))} = e|\Delta_K|^{\frac{1}{2}} H_{\mathbb{P}^{t-1}}(Q)^{b_i},$$

for all $D \in \mathrm{Pic}(K)$ and $i \in \{2, \dots, n\}$. Together with Corollary 56, these estimates imply

$$\varphi\left(D \otimes \overline{E_Q}\right) \le 2^n C_1 (e|\Delta_K|^{\frac{1}{2}})^{n-1} H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|-b_1},$$

for all $D \in \mathrm{I}_Q$. Hence, by Lemma 71(2) we conclude

$$\int_{\mathrm{I}_Q} \mathrm{N}(D)^{-s}\varphi_m(D)\varphi\left(D \otimes \overline{E_Q}\right) \mathrm{d}D = O(H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|-b_1}). \tag{2.10.6}$$

Now, by Proposition 58 there exists a constant $\beta \ge 1$, depending only on K, such that $\varphi(D) \le \beta e^{-n_K e^{-\frac{2}{n_K}\widehat{\deg}(D)}}$ for all $D \in \mathrm{Pic}(K)$, with $n_K = [K : \mathbb{Q}]$ as usual. Together with Lemma 71(1), this implies

$$\left|\int_{\mathrm{I}_Q} \mathrm{N}(D)^{n-s}\varphi_m(D)\,\mathrm{d}D\right| \le \beta' \int_{-\infty}^{\frac{1}{2}\log(|\Delta_K|)-b_1\log(H_{\mathbb{P}^{t-1}}(Q))} e^{x(n-\Re(s))-n_K e^{-\frac{2}{n_K}x}}\,\mathrm{d}x,$$

for some constant $\beta^0 > 0$ depending only on K and m. Put $C_2 := \frac{j\mathbf{b}j}{b_1}$, and let $T < 0$ such that

$$x(n \quad <(s)) \quad \pi n_K e^{\frac{2}{n_K}x} \quad C_2 x \quad \text{for all } x\ 2] \quad 1, T] \text{ and } s\ 2\ K\ .$$

If $\frac{1}{2}\log(j \quad _K j) \quad b_1 \log(H_{\mathrm{P}t \quad 1}(Q)) \quad T$, then

$$\left|\int_{\mathrm{I}_Q} \mathrm{N}(D)^{n \quad s} \varphi_m(D)\, \mathrm{d}D\right| \quad \beta^0 \int_{1}^{\frac{1}{2}\log(j \quad _K j) \quad b_1 \log(H_{\mathrm{P}t \quad 1}(Q))} e^{C_2 x} \mathrm{d}x$$
$$= \frac{\beta^0 (j \quad _K j)^{\frac{C_2}{2}}}{C_2 H_{\mathrm{P}t \quad 1}(Q)^{C_2 b_1}}.$$

Hence, on the one hand, assuming $\frac{1}{2}\log(j \quad _K j) \quad b_1 \log(H_{\mathrm{P}t \quad 1}(Q)) \quad T$ we get

$$H_{\mathrm{P}t \quad 1}(Q)^{j\mathbf{b}j} \left|\int_{\mathrm{I}_Q} \mathrm{N}(D)^{n \quad s} \varphi_m(D)\, \mathrm{d}D\right| = O(1). \tag{2.10.7}$$

On the other hand, if $\frac{1}{2}\log(j \quad _K j) \quad b_1 \log(H_{\mathrm{P}t \quad 1}(Q)) \quad T$ then $H_{\mathrm{P}t \quad 1}(Q)$ is bounded above by a constant that depends only on K and T, hence in that case (2.10.7) also holds thanks to Lemma 71(2).

Now we look at integrals over II_Q. First, we use Theorem 57 to get

$$\varphi\left(D \quad \overline{E_Q}\right) = \left(\varphi\left(D^- \quad \overline{E_Q}^- \quad \overline{\omega_{O_K}}\right) + 1\right) \mathrm{N}(D)^n H_{\mathrm{P}t \quad 1}(Q)^{j\mathbf{b}j} j \quad _K j^{\ \frac{n}{2}} \quad 1$$
$$= \mathrm{N}(D)^n H_{\mathrm{P}t \quad 1}(Q)^{j\mathbf{b}j} j \quad _K j^{\ \frac{n}{2}} \quad 1 + \mathrm{N}(D)^n H_{\mathrm{P}t \quad 1}(Q)^{j\mathbf{b}j} j \quad _K j^{\ \frac{n}{2}} \varphi\left(D^- \quad \overline{E_Q}^- \quad \overline{\omega_{O_K}}\right).$$

This implies

$$\begin{aligned}
&\int_{\mathrm{II}_Q} \mathrm{N}(D)^{\ \ s} \varphi_m(D) \varphi\left(D \quad \overline{E_Q}\right) \mathrm{d}D \\
&= j \quad _K j^{\ \frac{n}{2}} H_{\mathrm{P}t \quad 1}(Q)^{j\mathbf{b}j} \int_{\mathrm{II}_Q} \mathrm{N}(D)^{n \quad s} \varphi_m(D) \varphi\left(D^- \quad \overline{E_Q}^- \quad \overline{\omega_{O_K}}\right) \mathrm{d}D \\
&\quad + j \quad _K j^{\ \frac{n}{2}} H_{\mathrm{P}t \quad 1}(Q)^{j\mathbf{b}j} \int_{\mathrm{II}_Q} \mathrm{N}(D)^{n \quad s} \varphi_m(D)\, \mathrm{d}D \\
&\qquad \int_{\mathrm{II}_Q} \mathrm{N}(D)^{\ \ s} \varphi_m(D)\, \mathrm{d}D.
\end{aligned} \tag{2.10.8}$$

From Lemma 71(2) we know that

$$\int_{\mathrm{II}_Q} \mathrm{N}(D)^{\ \ s} \varphi_m(D)\, \mathrm{d}D = O(1). \tag{2.10.9}$$

Now, note that for $D\ 2\ \mathrm{II}_Q$ we have $D^- \quad (\overline{E_Q^{(i)}})^- \quad \overline{\omega_{O_K}}\ 2\ \mathrm{Pic}(K)$ for all $i\ 2\ f1, \dots, ng$, hence by Proposition 58 we get

$$\varphi(D^- \quad (\overline{E_Q^{(i)}})^- \quad \overline{\omega_{O_K}}) \quad \beta e^{\ \ n_K (j \quad _K j^{\ \ 1} \mathrm{N}(D) H_{\mathrm{P}t \quad 1}(Q)^{b_i})^{\frac{2}{n_K}}} \quad \beta e^{\ \ C_3 (\mathrm{N}(D) H_{\mathrm{P}t \quad 1}(Q)^{b_1})^{\frac{2}{n_K}}},$$

where $\beta \geq 1$ is the same constant as before and $C_3 := \pi n_K |\Delta_K|^{-\frac{2}{n_K}}$. Combined with Corollary 56, we conclude

$$\varphi(D^{-} \otimes \overline{E_Q}^{-} \otimes \overline{\omega_{O_K}}) \leq 2^n \beta^n e^{-nC_3(\mathrm{N}(D) H_{\mathbb{P}^{t-1}}(Q)^{b_1})^{\frac{2}{n_K}}},$$

for all $D \in \mathrm{I}_Q$. Letting $C_4 > 0$ be such that $x e^{-nC_3 x^{\frac{2}{n_K}}} \leq C_4$ for all $x \geq 0$, we get

$$\mathrm{N}(D) H_{\mathbb{P}^{t-1}}(Q)^{b_1} e^{-nC_3(\mathrm{N}(D) H_{\mathbb{P}^{t-1}}(Q)^{b_1})^{\frac{2}{n_K}}} \leq C_4,$$

thus

$$H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|} \int_{\mathrm{I}_Q} \mathrm{N}(D)^{n-s} \varphi_m(D) \varphi\left(D^{-} \otimes E_{\bar{Q}} \otimes \overline{\omega_{O_K}}\right) \mathrm{d}D$$
$$\leq C_4 2^n \beta^n H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|-b_1} \int_{\mathrm{I}_Q} \mathrm{N}(D)^{n-1-s} \varphi_m(D)\, \mathrm{d}D = O(H_{\mathbb{P}^{t-1}}(Q)^{|\mathbf{b}|-b_1}), \tag{2.10.10}$$

by Lemma 71(2). The desired result then follows from (2.10.6), (2.10.7), (2.10.8), (2.10.9) and (2.10.10). This completes the proof of the lemma. □

Lemma 73. *Given negative integers $0 > b_1 \geq b_2 \geq \dots \geq b_n$ and $Q \in \mathbb{P}^{t-1}$, define*

$$\overline{E_Q} := \overline{O_{\mathbb{P}^{t-1}}(b_1)_Q} \oplus \dots \oplus \overline{(O_{\mathbb{P}^{t-1}}(b_n))_Q}.$$

Then, there exist $\beta, \gamma > 0$, depending only on K and n, such that for every $D \in \mathrm{Pic}(K)$ we have

$$\varphi(D \otimes \overline{E_Q}) \leq \beta e^{-\gamma e^{-\frac{2}{n_K}\widehat{\deg}(D)}} e^{-\gamma H_{\mathbb{P}^{t-1}}(Q)^{\frac{2}{n_K}}},$$

where $n_K = [K : \mathbb{Q}]$ as usual.

Proof. By Corollary 56 it is enough to prove the lemma in the case $n = 1$. Since

$$\widehat{\deg}(D \otimes \overline{O_{\mathbb{P}^{t-1}}(b_1)_Q}) = \widehat{\deg}(D) + b_1 \log(H_{\mathbb{P}^{t-1}}(Q)) - \frac{1}{2}\log(|\Delta_K|),$$

we can use Proposition 60. This shows that there exists a constant $\beta > 0$, depending only on K, such that

$$\varphi(D \otimes \overline{O_{\mathbb{P}^{t-1}}(b_1)_Q}) \leq \beta e^{-\pi n_K e^{-\frac{2}{n_K}(\widehat{\deg}(D) + b_1 \log(H_{\mathbb{P}^{t-1}}(Q)))}} = \beta e^{-\pi n_K e^{-\frac{2}{n_K}\widehat{\deg}(D)} H_{\mathbb{P}^{t-1}}(Q)^{\frac{2}{n_K}}},$$

where in the last inequality we used that $b_1 \leq -1$. Putting $A := e^{-\frac{2}{n_K}\widehat{\deg}(D)}$ and $B := H_{\mathbb{P}^{t-1}}(Q)^{\frac{2}{n_K}}$, we see that $A \geq |\Delta_K|^{-\frac{1}{n_K}}$ and $B \geq 1$, hence there exists a constant $\rho > 0$, depending only on K, such that $AB \geq \rho(A + B)$. This implies

$$\varphi(D \otimes \overline{O_{\mathbb{P}^{t-1}}(b_1)_Q}) \leq \beta e^{-\gamma e^{-\frac{2}{n_K}\widehat{\deg}(D)}} e^{-\gamma H_{\mathbb{P}^{t-1}}(Q)^{\frac{2}{n_K}}},$$

with $\gamma = \pi n_K \rho$. This proves the lemma. □

Notation 74. In the proof of Theorem 70 below, we put $\varphi(D^{-m}) := 0$ when $m = 0$.

Proof of Theorem 70. Given $P \in X(K)$ we put $Q := \pi(P)$. Then, we have $P \in \mathbb{P}(W_Q)(K)$ with W defined in (2.10.1). Moreover, if $P \in F(K)$ then $P \in \mathbb{P}(Y_{\bar{Q}})(K)$ and $H_{\mathbb{P}(W_Q)}(P) = H_{\mathbb{P}(Y_{\bar{Q}})}(P)$. Indeed, this follows from the fact that $Y_{\bar{Q}} \subset W_Q$, which implies that $\overline{\mathcal{O}_{Y_{\bar{Q}}}(-1)_P} = \overline{\mathcal{O}_{W_Q}(-1)_P}$. Then, taking duals and norms leads to the equality of heights. From this and (2.10.2), we get

$$
\begin{aligned}
Z_{U;L}(s) &= \sum_{P \in U(K)} \left(H_{\mathbb{P}(W_{\pi(P)})}(P)\, H_{\mathbb{P}^{t-1}}(\pi(P)) \right)^{-s} \\
&= \sum_{Q \in \mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s} \left(\sum_{P \in \mathbb{P}(W_Q)(K)} H_{\mathbb{P}(W_Q)}(P)^{-s} - \sum_{P \in \mathbb{P}(Y_{\bar{Q}})(K)} H_{\mathbb{P}(W_Q)}(P)^{-s} \right) \\
&= \sum_{Q \in \mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s} \left(Z_{\mathbb{P}(W_Q)}(\lambda s) - Z_{\mathbb{P}(Y_{\bar{Q}})}(\lambda s) \right).
\end{aligned}
$$

Now, fixing $Q \in \mathbb{P}^{t-1}(K)$ and using Proposition 63(2), we have

$$
\begin{aligned}
&w_K \xi_K(\lambda s)\, Z_{\mathbb{P}(W_Q)}(\lambda s) \\
&= R_K h_K |\Delta_K|^{-\frac{s}{2}} \left(\frac{N(\overline{W_Q})}{\lambda s - (r+1)} - \frac{1}{\lambda s} \right) + \int_{\mathrm{Pic}(K)} N(D)^{-s} \varphi(D \otimes \overline{W_Q})\, \mathrm{d}D \\
&\qquad + N(\overline{W_Q}) |\Delta_K|^{\frac{r+1}{2} - s} \int_{\mathrm{Pic}(K)} N(D)^{s-(r+1)} \varphi(D \otimes \overline{W_Q}^{\vee})\, \mathrm{d}D.
\end{aligned}
$$

Similarly, since $Y_{\bar{Q}}$ has rank r, we have

$$
\begin{aligned}
&w_K \xi_K(\lambda s)\, Z_{\mathbb{P}(Y_{\bar{Q}})}(\lambda s) \\
&= R_K h_K |\Delta_K|^{-\frac{s}{2}} \left(\frac{N(\overline{Y_{\bar{Q}}})}{\lambda s - r} - \frac{1}{\lambda s} \right) + \int_{\mathrm{Pic}(K)} N(D)^{-s} \varphi(D \otimes \overline{Y_{\bar{Q}}})\, \mathrm{d}D \\
&\qquad + N(\overline{Y_{\bar{Q}}}) |\Delta_K|^{\frac{r}{2} - s} \int_{\mathrm{Pic}(K)} N(D)^{s-r} \varphi(D \otimes \overline{Y_Q})\, \mathrm{d}D.
\end{aligned}
$$

Hence, we can write

$$
w_K \xi_K(\lambda s)\, Z_{U;L}(s) = \sum_{j=1}^{5} F_j(s),
$$

where

$$F_1(s) := \frac{R_K h_K |\Delta_K|^{-\frac{s}{2}}}{\lambda s - (r+1)} \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s}\,\mathrm{N}(\overline{W_Q}),$$

$$F_2(s) := -\frac{R_K h_K |\Delta_K|^{-\frac{s}{2}}}{\lambda s - r} \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s}\,\mathrm{N}(\overline{Y_{\bar{Q}}}),$$

$$F_3(s) := \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\left(\varphi(D-\overline{W_Q}) - \varphi(D-\overline{Y_{\bar{Q}}})\right) \mathrm{d}D,$$

$$F_4(s) := |\Delta_K|^{\frac{r+1}{2}-s} \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s}\,\mathrm{N}(\overline{W_Q}) \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s-(r+1)}\varphi(D-\overline{W_Q}^{-})\,\mathrm{d}D,$$

$$F_5(s) := -|\Delta_K|^{\frac{r}{2}-s} \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s}\,\mathrm{N}(\overline{Y_{\bar{Q}}}) \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s-r}\varphi(D-\overline{Y_Q})\,\mathrm{d}D.$$

We are going to analyze each of these functions separately. First, we compute

$$\begin{aligned}\mathrm{N}(\overline{Y_{\bar{Q}}}) = \mathrm{N}(\overline{W_Q}) &= \mathrm{N}(\overline{(\mathcal{O}_{\mathbb{P}^{t-1}}(-1))_Q} \otimes \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(a_r)_Q} \otimes \cdots \otimes \overline{\mathcal{O}_{\mathbb{P}^{t-1}}(a_r - a_{r-1})_Q}) \\ &= H_{\mathbb{P}^{t-1}}(Q)^{(r+1)a_r - |a|}.\end{aligned} \tag{2.10.11}$$

This implies

$$F_1(s) = \frac{R_K h_K |\Delta_K|^{-\frac{s}{2}}}{\lambda s - (r+1)} Z_{\mathbb{P}^{t-1}}\left(\mu s + |a| - (r+1)a_r\right),$$

hence by Theorem 65 the function $F_1(s)$ is holomorphic in $\Re(s) > \frac{1+(r+1)a_r-|a|}{\mu}$, $s \neq \lambda_L$, $s \neq \mu_L$. Similarly,

$$F_2(s) = -\frac{R_K h_K |\Delta_K|^{-\frac{s}{2}}}{\lambda s - r} Z_{\mathbb{P}^{t-1}}\left(\mu s + |a| - (r+1)a_r\right),$$

hence $F_2(s)$ is holomorphic in $\Re(s) > \frac{1+(r+1)a_r-|a|}{\mu}$, $s \neq \frac{r}{\lambda}$, $s \neq \mu_L$.

We now focus on the function $F_3(s)$. First, note that $\overline{W_Q} = \overline{Y_{\bar{Q}}} \otimes \overline{(\mathcal{O}_{\mathbb{P}^{t-1}}(-1))_Q}$ and hence, using Lemma 55(2), we can compute

$$\begin{aligned}\varphi(D-\overline{W_Q}) &= \varphi(D-(\overline{Y_{\bar{Q}}} \otimes \overline{(\mathcal{O}_{\mathbb{P}^{t-1}}(-1))_Q})) \\ &= \varphi(D-\overline{Y_{\bar{Q}}}) + \varphi(D-\overline{(\mathcal{O}_{\mathbb{P}^{t-1}}(-1))_Q}) + \varphi(D-\overline{Y_{\bar{Q}}})\varphi(D-\overline{(\mathcal{O}_{\mathbb{P}^{t-1}}(-1))_Q}) \\ &= \varphi(D-\overline{Y_{\bar{Q}}}) + \varphi(D) + \varphi(D-\overline{Y_{\bar{Q}}})\varphi(D).\end{aligned}$$

We deduce that $F_3(s) = G_1(s) + G_2(s)$, where

$$\begin{aligned}G_1(s) &:= \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\varphi(D)\,\mathrm{d}D \\ &= Z_{\mathbb{P}^{t-1}}(\mu s) \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\varphi(D)\,\mathrm{d}D,\end{aligned}$$

and

$$G_2(s) := \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s}\varphi(D\otimes\overline{Y_{\bar Q}})\varphi(D)\,\mathrm{d}D.$$

By Lemma 71(1) with $m = 0$, together with Theorem 65, the function $G_1(s)$ is holomorphic in $\Re(s) > \frac{t}{}$. In order to analyze the function $G_2(s)$, put $a_0 := 0$ an recall that $N_X = \#\{i \in \{1,\dots,r\} : a_i = a_r\}$, so we can write

$$\overline{Y_{\bar Q}} = \overline{(\mathcal{O}_{\mathbb{P}^{t-1}})_Q}^{\oplus(N_X-1)} \oplus \overline{E_Q}$$

where $\overline{E_Q}$ is the direct sum of the line bundles $\overline{\mathcal{O}_{\mathbb{P}^{t-1}}(a_r-a_i))_Q}$ over $i \in \{0,\dots,r-1\}$ with $a_i < a_r$. Using Lemma 55(2) again, we have

$$\varphi(D\otimes\overline{Y_{\bar Q}}) = \varphi(D^{\oplus(N_X-1)}) + \varphi(D\otimes\overline{E_Q}) + \varphi(D^{\oplus(N_X-1)})\varphi(D\otimes\overline{E_Q}).$$

We now use Lemma 72 with $n = r-(N_X-1)$, $b_1 = a_r - a_{r-N_X}$, $b_2 = a_r - a_{r-N_X-1}, \dots, b_n = a_r$, and $m = 0$, and also with $m = N_X - 1$ if $N_X > 1$, in order to write

$$\begin{aligned} G_2(s) &= \widetilde{G_2}(s) + |\Delta_K|^{\frac{N_X-(r+1)}{2}} \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s+(r+1)a_r-|\mathbf{a}|} \\ &\qquad \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{r-(N_X-1)-s}\varphi(D)(1+\varphi(D^{\oplus(N_X-1)}))\,\mathrm{d}D \\ &= \widetilde{G_2}(s) + |\Delta_K|^{\frac{N_X-(r+1)}{2}} Z_{\mathbb{P}^{t-1}}(\mu s + |\mathbf{a}| - (r+1)a_r) \\ &\qquad \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{r-(N_X-1)-s}\varphi(D)(1+\varphi(D^{\oplus(N_X-1)}))\,\mathrm{d}D, \end{aligned}$$

with $\widetilde{G_2}(s)$ an analytic function on $\Re(s) > \frac{t+(r+1)a_r-|\mathbf{a}|-b_1}{} = \frac{t+ra_r+a_{r-N_X}-|\mathbf{a}|}{}$. Hence,

$$\begin{aligned} F_3(s) = G_1(s) + \widetilde{G_2}(s) + |\Delta_K|^{\frac{N_X-(r+1)}{2}} Z_{\mathbb{P}^{t-1}}(\mu s + |\mathbf{a}| - (r+1)a_r) \\ \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{r-(N_X-1)-s}\varphi(D)(1+\varphi(D^{\oplus(N_X-1)}))\,\mathrm{d}D, \end{aligned}$$

with $G_1(s) + \widetilde{G_2}(s)$ analytic in $\Re(s) > \frac{t+ra_r+a_{r-N_X}-|\mathbf{a}|}{}$.

In order to analyze the function $F_4(s)$, we start by using (2.10.11) to write

$$\begin{aligned} F_4(s) = |\Delta_K|^{\frac{r+1}{2}-s} \sum_{Q\in\mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s+(r+1)a_r-|\mathbf{a}|} \\ \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s-(r+1)}\varphi(D\otimes\overline{W_{\bar Q}})\,\mathrm{d}D. \end{aligned}$$

Now we write

$$\overline{W_{\bar Q}} = \overline{(\mathcal{O}_{\mathbb{P}^{t-1}})_Q}^{\oplus N_X} \oplus \overline{E_{\bar Q}}$$

with $\overline{E_{\overline{Q}}}$ the sum of the line bundles $\overline{O_{\mathbb{P}^{t-1}}(a_i - a_r)_Q}$ over $i \in \{0, \dots, r-1\}$ with $a_i < a_r$. By Lemma 55(2) we have

$$\varphi(D - \overline{W_{\overline{Q}}}) = \varphi(D - N_x) + \varphi(D - \overline{E_{\overline{Q}}}) + \varphi(D - N_x)\varphi(D - \overline{E_{\overline{Q}}}).$$

Hence, we can write $F_4(s) = G_3(s) + G_4(s)$ where

$$G_3(s) := |\Delta_K|^{\frac{r+1}{2} - s} Z_{\mathbb{P}^{t-1}}(\mu s + |a| - (r+1)a_r) \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s-(r+1)} \varphi(D - N_x)\,\mathrm{d}D,$$

$$G_4(s) := |\Delta_K|^{\frac{r+1}{2} - s} \sum_{Q \in \mathbb{P}^{t-1}(K)} H_{\mathbb{P}^{t-1}}(Q)^{-s+(r+1)a_r - |a|}$$
$$\int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s-(r+1)} \left(\varphi(D - \overline{E_{\overline{Q}}}) + \varphi(D - N_x)\varphi(D - \overline{E_{\overline{Q}}})\right) \mathrm{d}D.$$

On the one hand, by Theorem 65 and Proposition 63(1), the series $G_3(s)$ extends to a holomorphic function in $\Re(s) > \frac{1+(r+1)a_r - |a|}{\mu}, s \neq \mu_L$. On the other hand, using Lemmas 73 and 71(1) to bound $\varphi(D - \overline{E_{\overline{Q}}})$ and $\varphi(D - N_x)$, respectively, we see that the series $G_4(s)$ converges absolutely and uniformly for s in compact subsets of $\mathbb{C}$, hence $G_4(s)$ extends to an entire function. We conclude that $F_4(s)$ is holomorphic in $\Re(s) > \frac{1+(r+1)a_r - |a|}{\mu}, s \neq \mu_L$.

Finally, the analysis of the function $F_5(s)$ is analogous to that of $F_4(s)$, and we get that $F_5(s) = G_5(s) + G_6(s)$ with

$$G_5(s) := -|\Delta_K|^{\frac{r}{2} - s} Z_{\mathbb{P}^{t-1}}(\mu s + |a| - (r+1)a_r) \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-s-r} \varphi(D - (N_x - 1))\,\mathrm{d}D,$$

and $G_6(s)$ entire. In particular, $F_5(s)$ is holomorphic in $\Re(s) > \frac{1+(r+1)a_r - |a|}{\mu}, s \neq \mu_L$.

Putting everything together, and recalling that $t \geq 2$ and $a_r \geq 1$, we obtain the following:

1. If $\lambda_L = \mu_L$, then $Z_{U;L}(s)$ is holomorphic in

$$\Re(s) > \max\left\{\frac{1 + (r+1)a_r - |a|}{\mu}, \frac{r}{\lambda}, \frac{t + ra_r + a_{r-N_x} - |a|}{\mu}\right\}, s \neq \lambda_L$$

and it has a pole of order two at $s = \lambda_L$ (coming from F_1) with

$$\lim_{s \to \lambda_L} (s - \lambda_L)^2 Z_{U;L}(s) = \frac{R_K h_K |\Delta_K|^{-\frac{r+1}{2}}}{w_K \lambda \mu \xi (r+1)} \mathrm{Res}_{s=t} Z_{\mathbb{P}^{t-1}}(s) = \frac{R_K^2 h_K^2 |\Delta_K|^{-\frac{(d+2)}{2}}}{w_K^2 \lambda \mu \xi_K (r+1) \xi_K(t)}.$$

2. If $\lambda_L > \mu_L$, then $Z_{U;L}(s)$ has holomorphic continuation to

$$\Re(s) > \max\left\{\frac{r}{\lambda}, \mu_L\right\}, s \neq \lambda_L$$

and it has a simple pole at $s = \lambda_L$ (coming from F_1) with

$$\lim_{s \to \lambda_L} (s - \lambda_L) Z_{U;L}(s) = \frac{R_K h_K |\Delta_K|^{-\frac{r+1}{2}}}{w_K \lambda \xi (r+1)} Z_{\mathbb{P}^{t-1}}(\mu \lambda_L + |a| - (r+1)a_r).$$

3. If $\lambda_L < \mu_L$ then $Z_{U,L}(s)$ is holomorphic in

$$\Re(s) > \max\left\{\lambda_L, \frac{1+(r+1)a_r - |a|}{\mu}, \frac{t + ra_r + a_r - N_X - |a|}{\mu}\right\}, s \neq \mu_L$$

and it has a possible singularity at $s = \mu_L$ (coming from F_1, F_2, F_3, G_3 and G_5) with

$$\lim_{s\to\mu_L}(s-\mu_L)Z_{U,L}(s) = \frac{R_K h_K |\Delta_K|^{-\frac{t}{2}}}{w_K^2 \mu \xi_K(\lambda\mu_L)\xi_K(t)}\Bigg(\frac{R_K h_K|\Delta_K|^{-\frac{\ell_L}{2}}}{\lambda\mu_L-(r+1)} - \frac{R_K h_K|\Delta_K|^{-\frac{\ell_L}{2}}}{\lambda\mu_L - r}$$
$$+ |\Delta_K|^{\frac{N_X-(r+1)}{2}} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{r-(N_X-1)-\ell_L}\varphi(D)(1+\varphi(D^{-(N_X-1)}))\,\mathrm{d}D$$
$$+ |\Delta_K|^{\frac{r+1}{2}-\ell_L} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-\ell_L-(r+1)}\varphi(D^{-N_X})\,\mathrm{d}D$$
$$- |\Delta_K|^{\frac{r}{2}-\ell_L} \int_{\mathrm{Pic}(K)} \mathrm{N}(D)^{-\ell_L-r}\varphi(D^{-(N_X-1)})\,\mathrm{d}D\Bigg).$$

Furthermore, in this case we can write (using Lemma 55(2))

$$\varphi(D)(1+\varphi(D^{-(N_X-1)})) = \varphi(D^{-N_X}) - \varphi(D^{-(N_X-1)}),$$

and use Proposition 63(2), together with formula (2.9.3) when $N_X = 1$ or 2, to get

$$\lim_{s\to\mu_L}(s-\mu_L)Z_{U,L}(s) = \frac{R_K h_K |\Delta_K|^{-\frac{t}{2}}}{w_K^2 \mu \xi_K(\lambda\mu_L)\xi_K(t)} w_K |\Delta_K|^{\frac{N_X-(r+1)}{2}} \xi_K(\lambda\mu_L + N_X - (r+1))$$
$$\left(Z_{\mathbb{P}^{N_X-1}}(\lambda\mu_L + N_X - (r+1)) - Z_{\mathbb{P}^{N_X-2}}(\lambda\mu_L + N_X - (r+1))\right).$$

Since this value is positive, we conclude that $Z_{U,L}(s)$ has a simple pole at $s = \mu_L$ in this case.

Then, the asymptotic formula for $N(U, H_{L}, B)$ follows from these properties, together with (2.10.4) and Theorem 27. This completes the proof of the theorem. □

Remark 75. The different cases that appear in Theorem 70 give a subdivision of the *big cone* of X, i.e., the interior of Λ_{eff} (see Figure 2.1 for an illustration). The line bundles L contained in the ray passing through the anticanonical class have height zeta functions with a double pole at $s = \lambda_L = \mu_L$, while line bundles outside this ray have $\lambda_L \neq \mu_L$ and have height zeta functions with a simple pole at $s = \max\{\lambda_L, \mu_L\}$.

In Theorem 70 we have omitted the case when $a_r = 0$. This is because in the case $a_r = 0$ we have $X_d(a_1,\ldots,a_r) \simeq \mathbb{P}^{t-1} \times \mathbb{P}^r$, and there is no need to remove a proper subvariety of X to obtain the "correct" growth of the number of rational points of bounded height. Note that, in this case, we have

$$\lambda_L = \frac{r+1}{\lambda}, \quad \mu_L = \frac{t}{\mu}.$$

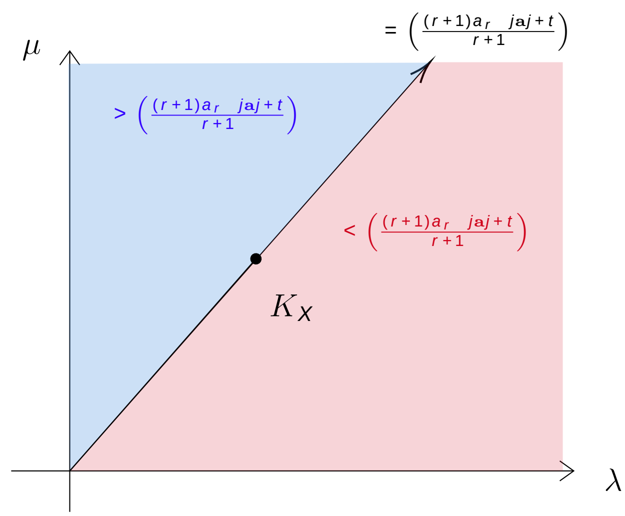

Figure 2.1: Subdivision of the big cone of X.

Theorem 76. *Let $X \simeq \mathbb{P}^{t-1} \times \mathbb{P}^r$ be a Hirzebruch–Kleinschmidt variety over the number field K with $a_r = 0$, and let $L = \lambda h + \mu f$ be a big line bundle class in $\mathrm{Pic}(X)$. Then, we have*

$$N(X, H_L, B) \sim C_{L;K} B^{a(L)} \log(B)^{b(L)} \quad \text{as } B \to \infty,$$

with $C_{L;K}$ given by

$$\begin{cases} \frac{R_K^2 h_K^2 |\Delta_K|^{-\frac{(d+2)}{2}}}{w_K^2 (r+1) \zeta_K(r+1) \zeta_K(t)} & \text{if } \lambda_L = \mu_L, \\ \frac{R_K h_K |\Delta_K|^{-\frac{r+1}{2}}}{w_K (r+1) \zeta_K(r+1)} Z_{\mathbb{P}^{t-1}}(\mu \lambda_L) & \text{if } \lambda_L > \mu_L, \\ \frac{R_K h_K |\Delta_K|^{-\frac{t}{2}}}{w_K t \zeta_K(t)} Z_{\mathbb{P}^r}(\lambda \mu_L) & \text{if } \lambda_L < \mu_L. \end{cases}$$

One can adapt the proof of Theorem 70 to give a proof of Theorem 76. Instead of doing that, we present a simpler argument based only on the analytic properties of the height zeta functions of the projective spaces $\mathbb{P}^{t-1}$ and $\mathbb{P}^r$.

Proof. We have

$$Z_{X;L}(s) := \sum_{P \in X(K)} H_L(P)^{-s} = Z_{\mathbb{P}^r}(\lambda s) Z_{\mathbb{P}^{t-1}}(\mu s).$$

Then, using Theorem 65 we see that $Z_{X;L}(s)$ is holomorphic in

$$\Re(s) > \max\left\{\frac{1}{\lambda}, \frac{1}{\mu}\right\}, s \neq \lambda_L, s \neq \mu_L,$$

and it has a double pole at $s = \lambda_L$ if $\lambda_L = \mu_L$, and a simple pole at $s = \max\{\lambda_L, \mu_L\}$ if $\lambda_L \neq \mu_L$. Then, the result follows by using Theorem 65 to compute $\lim_{s \to a(L)} (s - a(L))^{b(L)} Z_{X;L}(s)$, and using Theorem 27. □

Example 77. Consider the variety $X_2(0) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $L = 3h + f$, so that $\lambda_L = \frac{2}{3}$ and $\mu_L = 2$. Then, we get

$$N(X, H_L, B) \sim C_{L;K} B^2 \quad \text{as } B \to \infty,$$

with $C_{L;K} = \frac{R_K h_K |\Delta_K|^{-1}}{w_K 2 \zeta_K(2)} Z_{\mathbb{P}^1}(6)$. In the case $K = \mathbb{Q}$, a simple computation gives

$$Z_{\mathbb{P}^1}(s) = 2 + 2\frac{\zeta(s/2)}{\zeta(s)} L_{-4}(s/2), \text{ where } L_{-4}(s) := \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n^{-s}.$$

Hence, using that $L_{-4}(3) = \frac{\pi^3}{32}$ (see e.g [10, p. 189]) and $\zeta(6) = \frac{\pi^6}{945}$, we get

$$C_{L;\mathbb{Q}} = \frac{6}{\pi}\left(1 + \frac{945\zeta(3)}{32\pi^3}\right) = 4.09640530\ldots.$$

Note that $C_{L;\mathbb{Q}} = C' + C''$ where C', C'' are the constants appearing at the end of Example 4 in the Introduction (because $X_2(0) \simeq U' \sqcup F'$ in the notation used there).

2.10.2 The anticanonical height

In the case $\lambda = r + 1$ and $\mu = (r + 1)a_r + t - |a|$ we get $L = -K_X$ by Proposition 30(2), hence H_L is the anticanonial height function $H = H_{-K_X}$. Moreover, $\lambda_L = \mu_L = 1$ according to (2.10.3). When $a_r > 0$, Theorem 70 gives the asymptotic formula

$$N(U, H, B) \sim CB \log(B) \quad \text{as } B \to \infty,$$

with C given by (0.0.5). Assume $a_r = 0$. Then $X \simeq \mathbb{P}^{t-1} \times \mathbb{P}^r$ and $F \simeq \mathbb{P}^{t-1} \times \mathbb{P}^{r-1}$ if $r \geq 2$, while $F \simeq \mathbb{P}^{t-1}$ if $r = 1$. By Lemmas 33 and 34, together with Theorem 76 and Corollary 66, we have

$$\begin{aligned} N(X, H, B) &\sim CB \log(B), \\ N(F, H, B) &\sim C_r B \end{aligned}$$

as $B \to \infty$, with the same C as before and C_r another explicit constant[3]. In any case, this implies

$$N(U, H, B) = N(X, H, B) - N(F, H, B) \sim CB \log(B) \quad \text{as } B \to \infty.$$

This proves Theorem 2.

[3]The exact value is $C_r = \frac{R_K h_K |\Delta_K|^{-\frac{t}{2}}}{w_K t \zeta_K(t)} Z_{\mathbb{P}^{r-1}}(r + 1)$ if $r \geq 2$, and $\frac{R_K h_K |\Delta_K|^{-\frac{t}{2}}}{w_K t \zeta_K(t)}$ if $r = 1$.

2.10.3 Accumulation of rational points

In the literature, there are different notions that capture the idea of subvarieties having too many rational points. Since our aim in this chapter is to give explicit asymptotic formulas that allow for quantitative comparisons, we introduce the following relative notion of subvarieties accumulating more rational points than others.

Definition 78. Let $Y_1, Y_2 \subset X$ be two subvarieties of a Hirzebruch–Kleinschmidt variety X, and let $L \in \mathrm{Pic}(X)$ be a big line bundle class. Assume that $\#Y_1(K) = \#Y_2(K) = \infty$ and that $N(Y_1, H_L, B)$ and $N(Y_2, H_L, B)$ are both finite for every $B > 0$. Then, we say that Y_1 *strongly accumulates more rational points of bounded H_L-height than* Y_2 if

$$\lim_{B \to \infty} \frac{N(Y_2, H_L, B)}{N(Y_1, H_L, B)} = 0. \tag{2.10.12}$$

Theorems 70 and 76 lead to the following corollary.

Corollary 79. *Let $X = X_d(a_1, \ldots, a_r)$ be a Hirzebruch–Kleinschmidt variety over the number field K with $a_r > 0$ and good open subset $U = U_d(a_1, \ldots, a_r)$, and let $L = \lambda h + \mu f$ be a big line bundle class in $\mathrm{Pic}(X)$. Then, the following properties hold:*

1. *If $r > 1$, assume $\mu > \lambda(a_r - a_{r-1})$ so that $L|_F$ is big on the subvariety $F \cong X_{d-1}(a_1, \ldots, a_{r-1})$ of X. Then, the the good open subset $U' \cong U_{d-1}(a_1, \ldots, a_{r-1})$ of F strongly accumulates more rational points of bounded H_L-height than U if and only if*
$$\max\{\lambda_L, \mu_L\} < \mu_{L|_F}, \tag{2.10.13}$$
where $\mu_{L|_F} = \frac{r a_{r-1} + t - |a| + a_r}{(a_r - a_{r-1})}$.

2. *If $r = 1$, assume $\mu > \lambda a_1$ so that $L|_F$ is big on $F \cong \mathbb{P}^{t-1}$. Then, F strongly accumulates more rational points of bounded H_L-height than U if and only if (2.10.13) holds, where $\mu_{L|_F} = \frac{t}{a_1}$.*

Proof. Assume $r > 1$ and $\mu > \lambda(a_r - a_{r-1})$. By Theorem 70 the good open subset U' of F strongly accumulates more rational points of bounded H_L-height than U if and only if

$$\max\{\lambda_L, \mu_L\} < \max\{\lambda_{L|_F}, \mu_{L|_F}\}, \text{ or } \max\{\lambda_L, \mu_L\} = \lambda_{L|_F} = \mu_{L|_F} \text{ and } \lambda_L \neq \mu_L.$$

Since $\lambda_{L|_F} = \frac{r}{\lambda} < \frac{r+1}{\lambda} = \lambda_L$, we see that the second case cannot occur. This proves (1).

Now, if $r = 1$ and $\mu > \lambda a_1$, then by Lemma 34 the height function H_L restricted to F corresponds to the power $H_{\mathbb{P}^{t-1}}{}^{a_1}$ of the standard height function of $\mathbb{P}^{t-1}$. By Theorems 70 and Corollary 66 we see that F strongly accumulates more rational points of bounded H_L-height than U if and only if (2.10.13) holds. This completes the proof of the corollary. □

Remark 80. One can also define a weaker relative notion of subvarieties accumulating more rational points than others, by replacing condition (2.10.12) with

$$0 < \limsup_{B \to \infty} \frac{N(Y_2, H_L, B)}{N(Y_1, H_L, B)} < 1.$$

Then, in all possible cases, one can use the explicit formulas in Theorems 70, 76 and Corollary 66 to decide when U^0 (resp. F) weakly accumulates more rational points of bounded H_L-height than U. For instance, in the case $(a_1, a_2) = (0, 1)$ of Example 4, we saw that U^0 weakly accumulates more rational points of bounded anticanonical height than F^0.

2.10.4 Example: Hirzebruch surfaces

For an integer $a > 0$ consider the *Hirzebruch surface*

$$X = X_2(a) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)),$$

which we consider as a variety over $K = \mathbb{Q}$ for simplicity. In the basis $\{h, f\}$ of $\mathrm{Pic}(X)$ given in Proposition 30, choose a big line bundle class $L = \lambda h + \mu f$. We then have

$$\lambda_L = \frac{2}{\lambda}, \quad \mu_L = \frac{a+2}{\mu}.$$

We write $U = U_2(a)$ and $F = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a))$, so that

$$X = U \sqcup F \cong U_2(a) \sqcup \mathbb{P}^1.$$

Using that $\xi_{\mathbb{Q}}(s) = (2\pi^{s/2})^{-1}\Gamma(s/2)\zeta(s)$ and $\zeta(2) = \frac{\pi^2}{6}$, we get by Theorem 70 the asymptotic formula

$$N(U, H_L, B) \sim C_L \begin{cases} B^{\lambda_L}\log(B), & \text{if } \lambda_L = \mu_L, \\ B^{\max\{\lambda_L, \mu_L\}}, & \text{if } \lambda_L \neq \mu_L, \end{cases}$$

as $B \to \infty$, where

$$C_L = \begin{cases} \frac{18}{\pi^2} & \text{if } \lambda_L = \mu_L, \\ \frac{3}{\pi} Z_{\mathbb{P}^1}(\mu\lambda_L - a) & \text{if } \lambda_L > \mu_L, \\ \frac{6\,\xi_{\mathbb{Q}}(\mu_L - 1)}{(a+2)\,\xi_{\mathbb{Q}}(\mu_L)} & \text{if } \lambda_L < \mu_L. \end{cases} \tag{2.10.14}$$

Now, the restriction $L|_F$ of the line bundle class L to $F \cong \mathbb{P}^1$ is big if and only if $\mu > \lambda a$. In this case, by Corollary 66, we have

$$N(F, H_L, B) \sim \frac{3}{\pi} B^{\frac{2}{\mu - \lambda a}} \quad \text{as } B \to \infty.$$

As in Corollary 79(2), we have that F strongly accumulates more points of bounded H_L-height than U if and only if

$$\max\left\{\frac{2}{\lambda}, \frac{a+2}{\mu}\right\} < \frac{2}{\mu - \lambda a},$$

which is easily seen to be equivalent

$$\lambda a < \mu < \lambda(a+1).$$

Finally, we turn our attention to the numerical value of the constant C_L in (2.10.14). Assuming $a = 1$ for simplicity, we get the following first values of C_L depending on the choice of $L = \lambda h + \mu f$. Here, as in Example 77, we use the Dirichlet L-function $L_{-4}(s) := \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n^{-s}$.

λ	μ	**Case**	C_L
1	1	$\lambda_L < \mu_L$	$\frac{2\ _{Q}(2)}{_{Q}(3)} = \frac{2}{3\ (3)} = 1.74234272\ldots$
1	2	$\lambda_L > \mu_L$	$\frac{3}{} Z_{P^1}(3) = \frac{6}{}\left(1 + \frac{(3=2)L\ \ _4(3=2)}{(3)}\right) = 5.49807267\ldots$
1	3	$\lambda_L > \mu_L$	$\frac{3}{} Z_{P^1}(5) = \frac{6}{}\left(1 + \frac{(5=2)L\ \ _4(5=2)}{(5)}\right) = 4.25372490\ldots$
2	1	$\lambda_L < \mu_L$	$\frac{2\ _{Q}(5)}{_{Q}(6)} = \frac{2835\ (5)}{4\ \ ^6} = 0.76443811\ldots$
2	2	$\lambda_L < \mu_L$	$\frac{2\ _{Q}(2)}{_{Q}(3)} = \frac{2}{3\ (3)} = 1.74234272\ldots\ldots$
2	3	$\lambda_L = \mu_L$	$\frac{6}{\ ^2} = 0.60792710\ldots$
3	1	$\lambda_L < \mu_L$	$\frac{2\ _{Q}(8)}{_{Q}(9)} = \frac{32\ \ ^7}{165375\ (9)} = 0.58325419\ldots$
3	2	$\lambda_L < \mu_L$	$\frac{2\ _{Q}(7=2)}{_{Q}(9=2)} = \not{p}\frac{2\ (7=2)\ \ (7=4)}{(9=2)\ \ (9=4)} = 0.97781868\ldots$
3	3	$\lambda_L < \mu_L$	$\frac{2\ _{Q}(2)}{_{Q}(3)} = \frac{2}{3\ (3)} = 1.74234272\ldots\ldots$

Remark 81. The case $X_2(1) = P(O_{P^1} \quad O_{P^1}(\ \ 1))$ was already studied by Serre (see [35, Section 2.12]), and revisited by Batyrev and Manin in [2, Section 1.6] and by Peyre in [27, *Proposition* 2.7]. Following the notation in [27], one realizes $X_2(1)$ as the variety

$$V = \left\{([y_0, y_1, y_2], [z_0, z_1])\ 2\ P^2 \quad P^1 : y_0 z_1 = y_1 z_0\right\}.$$

Then, for integers r, s, the height function $H_{r;s}$ on $V(Q)$ defined by

$$H_{r;s}(([y_0, y_1, y_2], [z_0, z_1])) := \sqrt{y_0^2 + y_1^2 + y_2^2}^{\,r+s} \sqrt{z_0^2 + z_1^2}^{\quad s},$$

for $(y_0, y_1, y_2)\ 2\ Z^3$ and $(z_0, z_1)\ 2\ Z^2$ primitive, corresponds to our height function H_L with $L = (r + s)h + rf$. Moreover, the divisor $E \quad V$ defined in loc. cit. by $y_0 = y_1 = 0$ corresponds to our subbundle F. In the particular case of $r = 1$ and $s = 0$, corresponding to $\lambda = \mu = 1$, we see that

$$N(V\ n\ E, H_{1;0}, B) \quad \frac{1}{2}\#\left\{(y_0, y_1, y_2)\ 2\ Z^3 \text{ primitive, such that } \sqrt{y_0^2 + y_1^2 + y_2^2} \quad B\right\}$$
$$\frac{2\pi}{3\zeta(3)}B^3 \quad \text{as } B\ !\ \ 1$$

(see, e.g. [9]). This matches our computations, since in the case $\lambda = \mu = 1$ we get

$$N(U, H_L, B) \quad C_L B^3, \quad C_L = \frac{2\xi_Q(2)}{\pi\xi_Q(3)} = \frac{2\pi}{3\zeta(3)}.$$

Chapter 3

Global function fields case

3.1 Basic notation

Let C be a projective, smooth, geometrically irreducible curve of genus g defined over the finite field $\mathbb{F}_q$ of q elements (as usual, q is a positive power of a rational prime). Throughout this article we let K denote a finite extension of the rational function field $\mathbb{F}_q(C)$ of C. Associated to K we have the following objects (see e.g. [33] for details):

- The set of valuations $\mathrm{Val}(K)$ of K, which is in bijection with the set of closed points of C.
- For each $v \in \mathrm{Val}(K)$, let O_v be the associated valuation ring and m_v its maximal ideal, $k_v := O_v/m_v$ and $f_v := [k_v : \mathbb{F}_q]$, so we have $\# k_v = q^{f_v}$. We define for $x \in K$, $|x|_v := q^{-f_v v(x)}$, so that for every $x \in K, x \neq 0$ we have the product formula
$$\prod_v |x|_v = 1. \tag{3.1.1}$$
- The free abelian group $\mathrm{Div}(K)$ generated by $\mathrm{Val}(K)$. The elements of $\mathrm{Div}(K)$ are finite sums of the form $\sum_{v \in V_K} n_v v$ with $n_v \in \mathbb{Z}$ and $n_v = 0$ for all but finitely many $v \in \mathrm{Val}(K)$.
- For $D = \sum_{v \in \mathrm{Val}(K)} n_v v \in \mathrm{Div}(K)$ and $v \in \mathrm{Val}(K)$ define $v(D) := n_v$ and set
$$\mathrm{Div}^+(K) := \{D \in \mathrm{Div}(K) : v(D) \geq 0, \forall v \in \mathrm{Val}(K)\}.$$
Moreover, for $D_1, \dots, D_n \in \mathrm{Div}(K)$, we put
$$\sup(D_1, \dots, D_n) := \sum_{v \in \mathrm{Val}(K)} \sup\{v(D_1), \dots, v(D_n)\} v.$$
- For $x \in K^*$, define the divisor of x as $(x) := (x)_0 - (x)_\infty$ where
$$(x)_0 := \sum_{\substack{v \in \mathrm{Val}(K) \\ v(x) > 0}} v(x) v, \quad (x)_\infty := -\sum_{\substack{v \in \mathrm{Val}(K) \\ v(x) < 0}} v(x) v.$$

We also put $(0) = (0)_0 = (0)_\infty = 0$.

- The degree function $\deg : \mathrm{Div}(K) \to \mathbb{Z}$ defined by $\deg(D) := \sum_{v \in \mathrm{Val}(K)} f_v v(D)$. We have $\deg((x)) = 0$ for all $x \in K$ (by the product formula (3.1.1) in the case $x \neq 0$).
- The class number $h_K = \#(\mathrm{Div}^0(K)/(K))$, where $\mathrm{Div}^0(K) := \{D \in \mathrm{Div}(K) : \deg(D) = 0\}$ and $(K) := \{(x) : x \in K\}$.

For a vector $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}_0^r$, we write $|\mathbf{a}| = \sum_{i=1}^r a_i$.

3.2 Arithmetic tools

Let $\ell(D) = \dim_{\mathbb{F}_q} H^0(C, \mathcal{O}_C(D))$, where

$$H^0(C, \mathcal{O}_C(D)) = \{x \in K^* : (x) + D \geq 0\} \cup \{0\}.$$

Note that $\ell(D) = 0$ if $\deg(D) < 0$ and if $D \geq 0$ then $\#\{x \in K : (x)_\infty \leq D\} = q^{\ell(D)}$. As a consequence of the Riemann–Roch theorem for curves we have that

$$\ell(D) - \ell(K_C - D) = \deg(D) + 1 - g,$$

and if $\deg(D) > 2g - 2 = \deg(K_C)$ then

$$\ell(D) = \deg(D) + 1 - g,$$

where K_C is a canonical divisor of the curve C.

We also define the zeta function

$$Z_K(T) = \sum_{D \geq 0} T^{\deg(D)} = \prod_{v \in \mathrm{Val}(K)} \left(1 - T^{f_v}\right)^{-1},$$

and define the Dedekind zeta function of K by mean of

$$\zeta_K(s) = Z_K(q^{-s}).$$

Then $\zeta_K(s)$ converges for $\Re(s) > 1$ (see e.g. [38, Chapter VII, Theorem 4]) and has meromorphic continuation with a simple pole at $s = 1$ and residue

$$\frac{hq^{1-g}}{(q-1)\log(q)},$$

where $h = \#\{D \in \mathrm{Div}(K) : \deg(D) = 0\}$.

In the case of positive characteristic, we will make use of the equations, so we will work with the following definition motivated by Theorem 12.

Definition 82. Given integers $r \geq 1$, $t \geq 2$ and $0 \leq a_1 \leq \cdots \leq a_r$, the **Hirzebruch–Kleinschmidt** variety $X_d(a_1, \ldots, a_r)$ is defined as the subvariety of $\mathbb{P}^{tr} \times \mathbb{P}^{t-1}$ given in homogeneous coordinates $([x_0 : (x_{ij})_{i \in I_t, j \in I_r}], [y_1 : \ldots : y_t])$ by the equations

$$x_{mj} y_n^{a_j} = x_{nj} y_m^{a_j}, \quad \text{for all } j \in I_r \text{ and all } m, n \in I_t \text{ with } m \neq n, \tag{3.2.1}$$

where $d = \dim(X_d(a_1, \ldots, a_r)) = r + t - 1$ and $I_k = \{1, \ldots, k\} \subseteq \mathbb{Z}$.

3.3 Effective divisors

The following description of the cone of effective divisors on $X_d(a_1, \dots, a_r)$ is an adaptation of Proposition 30, since in this chapter we have

$$X_d(a_1, \dots, a_r) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{t-1}}(a_r)).$$

Proposition 83. *Let $X = X_d(a_1, \dots, a_r)$ be a Hirzebruch–Kleinschmidt variety and let us denote by f the class of $\pi^* \mathcal{O}_{\mathbb{P}^{t-1}}(1)$ and by h the class of $\mathcal{O}_X(1)$, both in $\mathrm{Pic}(X)$. Then:*

1. $\mathrm{Pic}(X) \simeq \mathbb{Z}h \oplus \mathbb{Z}f$.
2. *The anti-canonical divisor class of X is given by*

   $$-K_X = (r+1)h + (t - |a|) f,$$

   where $|a| = \sum_{i=1}^r a_i$.
3. *The cone of effective divisors of X is given by*

   $$\Lambda_{\mathrm{eff}}(X) = \{\gamma h + \xi f : \gamma \geq 0, \xi \geq -\gamma a_r\} \subset \mathrm{Pic}(X)_{\mathbb{R}}$$

   where $\mathrm{Pic}(X)_{\mathbb{R}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

In particular, $L = \gamma h + \xi f \in \mathrm{Pic}(X)$ is big if and only if $\gamma > 0$ and $f > -\gamma a_r$.

3.4 Height zeta function of projective space

For an integer $n \geq 1$, as we mention before, consider in the projective space $\mathbb{P}^n$ over K, with the *naive height function*

$$H_{\mathbb{P}^n}([x_0, \dots, x_n]) := \prod_{v \in \mathrm{Val}(K)} \sup\{|x_0|_v, |x_1|_v, \dots, |x_n|_v\},$$

and the associated *height zeta function*

$$\zeta_{\mathbb{P}^n}(s) := \sum_{P \in \mathbb{P}^n(K)} H_{\mathbb{P}^n}(P)^{-s} \quad \text{for } s \in \mathbb{C} \text{ with } \Re(s) \gg 0.$$

The goal of this section is to state the following well-known theorem, for which we give a proof for convenience of the reader.

Theorem 84. *The height zeta function $\zeta_{\mathbb{P}^n}(s)$ converges absolutely on $\Re(s) > n+1$, and it is a rational function on q^{-s}. Moreover, it has a simple pole at $s = n+1$ with residue*

$$\mathrm{Res}_{s=n+1}\, \zeta_{\mathbb{P}^n}(s) = \frac{h_K q^{(n+1)(1-g)}}{\zeta_K(n+1)(q-1)\log(q)}.$$

Proof. The fact that $\zeta_{\mathbb{P}^n}(s)$ is a rational function on q^{-s} follows from [37, Theorem 3.2]. Now, for $m \geq 1$ put $N(m) := \#\{P \in \mathbb{P}^n(K) : H(P) = m\}$. By definition of the standard height function, we have $N(m) = 0$ if m is not of the form $m = q^d$ with $d \geq 0$ an integer. Hence

$$\zeta_{\mathbb{P}^n}(s) = \sum_{d \geq 0} N(q^d) q^{-ds}.$$

Now, by [35, Section 2.5], we have the estimate

$$N(q^d) = \frac{h_K q^{(n+1)(1-g)}}{\zeta_K(n+1)(q-1)} q^{d(n+1)} + O(q^{dn}) \quad \text{as } d \to \infty. \tag{3.4.1}$$

This implies that $\zeta_{\mathbb{P}^n}(s)$ converges absolutely on $\Re(s) > n+1$, and moreover

$$\zeta_{\mathbb{P}^n}(s) = \frac{h_K q^{(n+1)(1-g)}}{\zeta_K(n+1)(q-1)} \left(\frac{1}{1 - q^{(n+1)-s}} \right) + F(s),$$

with $F(s)$ an analytic function on $\Re(s) > n$. Hence, we also get

$$\mathrm{Res}_{s=n+1} \zeta_{\mathbb{P}^n}(s) = \frac{h_K q^{(n+1)(1-g)}}{\zeta_K(n+1)(q-1)} \left(\lim_{s \to n+1} \frac{s-(n+1)}{1 - q^{(n+1)-s}} \right) = \frac{h_K q^{(n+1)(1-g)}}{\zeta_K(n+1)(q-1)\log(q)}.$$

This shows the desired results and completes the proof of the theorem. □

Remark 85.
1. A better estimate for the error term in (3.4.1) is given in [37, Corollary 4.3], which is related to the fact that $\zeta_{\mathbb{P}^n}(s)$ is analytic in $\Re(s) \in \left]\frac{1}{2}, n+1\right[$. For our purposes, we do not require such a strong result.
2. Theorem 84 also follows from [29, *Théorème* 3.11].

3.5 Counting rational points on Hirzebruch–Kleinschmidt varieties

In this section, we use the explicit equations for Hirzebruch–Kleinschmidt varieties to count rational points on them. To do this, we first present some lemmas concerning Möbius inversion and the calculation of the supremum of divisors.

In the notation of Proposition 83, let $L = \gamma h + \xi f$ be the class of a big line bundle. Then, the height function induced by L on the set of rational points of $X_d(a_1, \ldots, a_r)$ is

$$H_L\left(([x_0 : (x_{ij})_{i \in I_t; j \in I_r}], [y_1 : \ldots : y_t])\right) := \prod_{v \in \mathrm{Val}(K)} \sup_{i \in I_t; j \in I_r} \{|x_0|_v, |x_{ij}|_v\} \prod_{v \in \mathrm{Val}(K)} \sup_{i \in I_t} \{|y_i|_v\}. \tag{3.5.1}$$

It is easy to see that this height function is associated to an adelic metrization of L in the sense of [29, Secion 1.2]. We refer to this as the "standard metrization" of L.

To carry out the counting of rational points, we find it necessary to remove a closed set in order to ensure the convergence of the zeta functions.

More precisely, if we consider the equations defining the variety $X_d(a_1, \dots, a_r)$ in homogeneous coordinates $([x_0 : (x_{ij})_{i \in I_t; j \in I_r}], [y_1 : \dots : y_t])$ of $\mathbb{P}^{rt} \times \mathbb{P}^{t-1}$ (see Definition 82), we need to remove the closed subvariety defined by the single equation $\{x_{tr} = 0\}$.

Definition 86. Given integers $r \geq 1$, $t \geq 2$ and $0 \leq a_1 \leq \dots \leq a_r$ defining the Hirzebruch–Kleinschmidt variety as the subvariety of $\mathbb{P}^{rt} \times \mathbb{P}^{t-1}$ given by the equations

$$x_{mj} y_n^{a_j} = x_{nj} y_m^{a_j}, \quad \text{for all } j \in I_r \text{ and all } m, n \in I_t \text{ with } m \neq n, \tag{3.5.2}$$

where $d = \dim(X_d(a_1, \dots, a_r)) = r + t - 1$, we define the **good open subset** $U_d(a_1, \dots, a_r)$ of $X_d(a_1, \dots, a_d)$ as the complement of the closed subvariety defined by the equation $x_{tr} = 0$.

3.5.1 Decomposition of Hirzebruch–Kleinschmidt varieties

Note that for $1 \leq r' \leq r$ and $2 \leq t' \leq t$, there is a natural embedding

$$X_{t'+r'-1}(a_1, \dots, a_{r'}) \hookrightarrow X_d(a_1, \dots, a_r), \tag{3.5.3}$$

which is given in coordinates by

$$([x_0 : (x_{ij})_{i \in I_{t'}; j \in I_{r'}}], [y_1 : \dots : y_{t'}]) \mapsto ([x_0 : (\mathfrak{x}_{ij})_{i \in I_t; j \in I_r}], [y_1 : \dots : y_{t'} : 0 : \dots : 0])$$

where

$$\mathfrak{x}_{ij} = \begin{cases} x_{ij} & \text{if } i \leq t' \text{ and } j \leq r', \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we enlarge the matrix (x_{ij}) of size $t' \times r'$ to a matrix of size $t \times r$ by simply adding zero columns on the right and zero rows on the bottom, and similarly we enlarge $[y_1 : \dots : y_{t'}]$ by adding zeroes on the right.

Similarly, we consider the natural embeddings

$$\begin{aligned} \mathbb{P}^r &\hookrightarrow X_d(a_1, \dots, a_r), \quad [z_0 : \dots : z_r] \mapsto ([z_0 : (\mathfrak{z}_{ij})_{i \in I_t; j \in I_r}], [1 : 0 : \dots : 0]), \\ \mathbb{P}^{t-1} &\hookrightarrow X_d(a_1, \dots, a_r), \quad [y_1 : \dots : y_t] \mapsto ([1 : (0)_{i \in I_t; j \in I_r}], [y_1 : \dots : y_t]), \end{aligned} \tag{3.5.4}$$

where

$$\mathfrak{z}_{ij} = \begin{cases} z_j & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

Note that the images of $\mathbb{P}^{t-1}$ and $\mathbb{P}^r$ inside $X_d(a_1, \dots, a_r)$ intersect exactly at the point $([1 : (0)_{i \in I_t; j \in I_r}], [1 : 0 : \dots : 0])$.

For $1 \leq r' \leq r$ and $2 \leq t' \leq t$, let us identify $U_{t'+r'-1}(a_1, \dots, a_{r'})$ with its image under (3.5.3). Similarly, let us identify $\mathbb{P}^r$ and $\mathbb{P}^{t-1}$ with their images under (3.5.4) the we denote $\mathbb{P}_1^r$ and $\mathbb{P}_2^{t-1}$, respectively.

Lemma 87. *Let $X_d(a_1, \dots, a_r)$ be a Hirzebruch–Kleinschmidt variety. If $r \geq 2$, then we have the disjoint union decomposition*

$$X_d(a_1, \dots, a_r) = X_{d-1}(a_1, \dots, a_{r-1}) \sqcup (\mathbb{P}_1^r \setminus \mathbb{P}_1^{r-1}) \sqcup \left(\bigsqcup_{2 \leq t' \leq t} U_{t'+r-1}(a_1, \dots, a_r) \right). \tag{3.5.5}$$

If $r = 1$, then we have the disjoint union decomposition

$$X_d(a_1) = \mathbb{P}_2^{t-1} \sqcup (\mathbb{P}_1^1 \setminus \{P_0\}) \sqcup \left(\bigsqcup_{2 \leq t' \leq t} U_{t'}(a_1) \right), \tag{3.5.6}$$

where $P_0 = ([1 : (0)_{i \in I_t, j \in I_r}], [1 : 0 : \ldots : 0])$. Moreover, given $L = \gamma h + \xi f \in \mathrm{Pic}(X)$ big, we have the following properties:

1. *For all $1 \leq r' \leq r$, the restriction of H_L to $\mathbb{P}_1^{r'} \cong \mathbb{P}^{r'}$ corresponds to $H_{\mathbb{P}^{r'}}$.*
2. *The restriction of H_L to $\mathbb{P}_2^{t-1} \cong \mathbb{P}^{t-1}$ corresponds to $H_{\mathbb{P}^{t-1}}$.*
3. *For $1 \leq r' \leq r$ and $2 \leq t' \leq t$, the restriction of H_L to $U_{t'+r'-1}(a_1, \ldots, a_{r'})$ equals $H_{L'}$ where $L' := \gamma h' + \xi f'$ with $\{h', f'\}$ the basis of $\mathrm{Pic}(X_{t'+r'-1}(a_1, \ldots, a_{r'}))$ given in Proposition 30.*

Proof. Assume $r \geq 2$ and let $x = ([x_0 : (x_{ij})_{i \in I_t, j \in I_r}], [y_1 : \ldots : y_t]) \in X_d(a_1, \ldots, a_r)$. If $x_{ir} = 0$ for all $i \in I_t$, then $x \in X_{d-1}(a_1, \ldots, a_{r-1})$. Assume this is not the case, and let $t' := \max\{i \in I_t : y_i \neq 0\}$. From equation (3.5.2) with $n = t'$ and $m = i > t'$ we see that

$$x_{ij} = 0 \text{ for all } i \in I_t \text{ with } i > t' \text{ and all } j \in I_r.$$

Assume $t' = 1$. Then, $x_{ij} = 0$ for all $i \in I_t$ with $i > 1$ and all $j \in I_r$. Since $x \notin X_{d-1}(a_1, \ldots, a_{r-1})$, we have $x_{1r} \neq 0$, and this implies that $x \in \mathbb{P}_1^r \setminus \mathbb{P}_1^{r-1}$. Finally, assume $t' > 1$. Since $y_{t'} \neq 0$, having $x_{t'r} = 0$ would imply $x_{ir} = 0$ also for all $i \neq t'$ (because of equation (3.5.2) with $n = t'$ and $j = r$), but this contradicts the fact that $x \notin X_{d-1}(a_1, \ldots, a_{r-1})$. So, $x_{t'r} \neq 0$ and we have in this case that $x \in U_{t'+r-1}(a_1, \ldots, a_r)$. This proves that

$$X_d(a_1, \ldots, a_r) = X_{d-1}(a_1, \ldots, a_{r-1}) \cup (\mathbb{P}_1^r \setminus \mathbb{P}_1^{r-1}) \cup \left(\bigcup_{2 \leq t' \leq t} U_{t'+r-1}(a_1, \ldots, a_r) \right).$$

Moreover, these unions are disjoint by construction. This proves (3.5.5). The proof of the decomposition (3.5.6) in the case $r = 1$ is completely analogous. Finally, the statements about the various restrictions of the height function H_L follow directly from (3.5.1). This proves the lemma. □

3.5.2 Preliminary lemmas

Definition 88. Given functions $f, g : \mathrm{Div}^+(K) \to \mathbb{C}$, we define their convolution product as

$$(f \star g)(D) := \sum_{0 \leq D' \leq D} f(D') g(D - D').$$

Moreover, we define the functions $1, \delta, \mu : \mathrm{Div}^+(K) \to \mathbb{C}$ by

- $\mathbb{1}(D) := 1$ for every $D \in \mathrm{Div}^+(K)$.
- $\delta(D) := \begin{cases} 1 & \text{if } D = 0, \\ 0 & \text{otherwise.} \end{cases}$
- $\mu(D) := \begin{cases} 1 & \text{if } D = 0, \\ (-1)^{\sum_v v(D)} & \text{if } D \neq 0 \text{ and } v(D) = 0 \text{ or } 1 \text{ for all } v \in \mathrm{Val}(K), \\ 0 & \text{otherwise.} \end{cases}$

In particular, δ is a unit for the convolution product and $\mathbb{1} \star \mu = \mu \star \mathbb{1} = \delta$.

The following result, based on a straightforward computation, is a Möbius inversion formula in this context.

Lemma 89. *Let $D \in \mathrm{Div}^+(K)$ and let $f, g : \mathrm{Div}^+(K) \to \mathbb{C}$ be two functions. Then, the relation*

$$f(D) = \sum_{0 \leq D' \leq D} g(D')$$

is equivalent to

$$g(D) = \sum_{0 \leq D' \leq D} \mu(D - D') f(D').$$

This motivates the following definition, that will be useful for counting purposes.

Definition 90. Let $f, g : \mathrm{Div}^+(K) \to \mathbb{C}$ be functions. We say that the pair (f, g) forms a μ**-couple** if they satisfy some the equivalent relations in Lemma 89.

By definition of the function δ, the pair $(\delta, \mathbb{1})$ form a μ-couple and thus we have the following result.

Lemma 91. *We have the following equality of formal series*

$$\sum_{D \geq 0} \mu(D) T^{\deg(D)} = \frac{1}{Z_K(T)}.$$

Proof. Since $(\delta, \mathbb{1})$ is a μ-couple, we have that

$$\begin{aligned}
\sum_{D' \geq 0} \mu(D') T^{\deg(D')} Z_K(T) &= \sum_{D' \geq 0} \mu(D') T^{\deg(D')} \sum_{D \geq 0} T^{\deg(D)} \\
&= \sum_{D \geq 0} \sum_{D' \geq 0} \mu(D') T^{\deg(D + D')} \\
&= \sum_{D \geq 0} \sum_{D \geq D' \geq 0} \mu(D - D') T^{\deg(D)} \\
&= \sum_{D \geq 0} \delta(D) T^{\deg(D)} = 1,
\end{aligned}$$

where the last equality follows from the definition of the function δ. □

Corollary 92. *For every $s \in \mathbb{C}$ with $\Re(s) > 1$, we have*

$$\sum_{D \geq 0} \mu(D) q^{-s \deg(D)} = \frac{1}{\zeta_K(s)},$$

and the series on the left hand side converges absolutely and uniformly on compact subsets of the half-plane $\Re(s) > 1$.

For computations using the supremum of divisors, we will need the following result.

Lemma 93. *Let $x, y \in K$, then*

$$\sup\{(x)_\infty, (xy)_\infty, (y)_\infty\} = (x)_\infty + (y)_\infty.$$

Proof. Let $v \in \mathrm{Val}(K)$. We recall that for a divisor $D = \sum_{v \in \mathrm{Val}(K)} n_v v \in \mathrm{Div}(K)$, we defined $v(D) = n_v$. In particular, we have the following four cases to consider:

1. If $v(x), v(y) \leq 0$ then $v((xy)_\infty) = v((x)_\infty) + v((y)_\infty)$.
2. If $v(x) \leq 0$ and $v(y) > 0$ then $v((xy)_\infty) \leq v((x)_\infty)$ and $v((y)_\infty) = 0$.
3. If $v(x) > 0$ and $v(y) \leq 0$ then $v((xy)_\infty) \leq v((y)_\infty)$ and $v((x)_\infty) = 0$.
4. If $v(x), v(y) > 0$ then $v((xy)_\infty) = v((x)_\infty) = v((y)_\infty) = 0$.

And the result follows. □

Lemma 94. *The good open subset $U_d(a_1, \ldots, a_r)$ equals the image of the map*

$$(x_1, \ldots, x_d) \in \mathbb{A}^d \mapsto ([x_d, (\mathfrak{x}_{ij})_{i \in I_t; j \in I_r}], [x_1, \ldots, x_{t-1}, 1]) \in X_d(a_1, \ldots, a_r), \tag{3.5.7}$$

where

$$\mathfrak{x}_{ij} = \begin{cases} x_{t+j-1} x_i^{a_j} & \text{if } i < t, j < r, \\ x_{t+j-1} & \text{if } i = t, j < r, \\ x_i^{a_r} & \text{if } i < t, j = r, \\ 1 & \text{if } i = t, j = r, \end{cases}$$

Proof. Let $P := ([x_0 : (x_{ij})_{i \in I_t; j \in I_r}], [y_1 : \ldots : y_t])$ be a point in $U_d(a_1, \ldots, a_r)$, and note that the condition $x_{tr} \neq 0$ implies $y_t \neq 0$. Indeed, due to the equations

$$x_{tr} y_n^{a_r} = x_{nr} y_t^{a_r} \quad (n = 1, \ldots, t-1),$$

the vanishing of y_t would imply that all the coordinates of $[y_1, \ldots, y_t] \in \mathbb{P}^{t-1}$ would be zero, which is absurd. Hence, we can assume $y_t = x_{tr} = 1$. Putting $x_i := y_i$ for $1 \leq i < t$, $x_{t+j-1} := x_{tj}$ for $1 \leq j < r$, and $x_d := x_0$, we obtain a point $(x_1, \ldots, x_d) \in \mathbb{A}^d$ which defines coefficients $\mathfrak{x}_{ij}$ as above. By construction, we have $\mathfrak{x}_{ij} = x_{ij}$ for $i = t$ and for all $j \in I_r$. Now, it follows from the equations (3.5.2) with $n = t$ and $j < r$ that $x_{ij} = x_{tj} y_i^{a_j} = x_{t+j-1} x_i^{a_j} = \mathfrak{x}_{ij}$ for all i with $1 \leq i < t$. Similarly, using the equations (3.5.2) with $n = t$ and $j = r$, we get $x_{ir} = x_{tr} y_i^{a_r} = x_i^{a_r} = \mathfrak{x}_{ir}$ for all i with $1 \leq i < t$. This proves that the image of $(x_1, \ldots, x_d)$ under (3.5.7) equals P, and completes the proof of the lemma. □

3.5.3 Height zeta functions

The **height zeta function** over the good open subset $U := U_d(a_1, \dots, a_r)$ is given by

$$\zeta_{U,L}(s) := \sum_{x \in U(K)} H_L(x)^{-s}.$$

We also introduce the formal power series

$$\mathrm{Z}_{U,L}(T) = \sum_{(x_i) \in \mathbb{A}^d} T^{\mathbf{d}_L(x_i)}, \tag{3.5.8}$$

where

$$\mathbf{d}_L(x_i) := \gamma \deg\Big(\sup_{\substack{i \in I_{t-1} \\ j \in I_{r-1}}} \big((x_d)^{-1}, (x_{t-1+j})^{-1} + a_j(x_i)^{-1}, a_r(x_i)^{-1}\big)\Big) + \xi \deg\Big(\sup_{i \in I_{t-1}} \big((x_i)^{-1}\big)\Big).$$

As in the case of the Dedekind zeta function of the base field K, we have the following relation between the above series.

Lemma 95. *With the above notation, we have the following equality*

$$\zeta_{U,L}(s) = \mathrm{Z}_{U,L}(q^{-s})$$

as formal series.

Proof. Let $x = (x_i) \in \mathbb{A}^d$ and denote by P_x the image of x in U under the map defined in Lemma 94. Then, we have

$$H_L(P_x) = \prod_v \sup_{\substack{i \in I_{t-1} \\ j \in I_{r-1}}} \{|x_d|_v, |x_{t-1+j} x_i^{a_j}|_v, |x_{t-1+j}|_v, |x_i^{a_r}|_v, 1\} \prod_v \sup_{i \in I_{t-1}} \{|x_i|_v, 1\}$$

$$= \prod_{\substack{v \\ \inf_{\substack{i \in I_{t-1} \\ j \in I_{r-1}}} \{v(x_d), v(x_{t-1+j} x_i^{a_j}), v(x_{t-1+j}), v(x_i^{a_r})\} < 0}} q^{-f_v \inf_{\substack{i \in I_{t-1} \\ j \in I_{r-1}}} \{v(x_d), v(x_{t-1+j} x_i^{a_j}), v(x_{t-1+j}), v(x_i^{a_r})\}}$$

$$\prod_{\substack{v \\ \inf_{i \in I_{t-1}} \{v(x_i)\} < 0}} q^{-f_v \inf_{i \in I_{i-1}} \{v(x_i)\}},$$

where each product runs over all $v \in \mathrm{Val}(K)$. By taking $\log_q$ on both sides, we obtain

$$
\begin{aligned}
\log_q H_L(P_x) &= \sum_{\substack{v \\ \inf_{\substack{i\in I_{t-1}\\ j\in I_{r-1}}} \{v(x_d), v(x_{t-1+j}x_i^{a_j}), v(x_{t-1+j}), v(x_i^{a_r})\} < 0}} f_v \gamma \inf_{\substack{i\in I_{t-1}\\ j\in I_{r-1}}} \left\{v(x_d), v(x_{t-1+j}x_i^{a_j}), v(x_{t-1+j}), v(x_i^{a_r})\right\} \\
&\quad + \sum_{\substack{v \\ \inf_{i\in I_{t-1}} \{v(x_i)\} < 0}} f_v \xi \inf_{i\in I_{t-1}} \{v(x_i)\} \\
&= \gamma \deg\left(\sup_{\substack{i\in I_{t-1}\\ j\in I_{r-1}}} \left((x_d)_\infty, (x_{t-1+j}x_i^{a_j})_\infty, (x_{t-1+j})_\infty, a_r(x_i)_\infty\right)\right) + \xi \deg\left(\sup_{i\in I_{t-1}} ((x_i)_\infty)\right).
\end{aligned}
$$

Since $a_j(x_i)_\infty \leq a_r(x_i)_\infty$, we can add $a_i(x_i)_\infty$ to each set in the first sum, and use Lemma 93 to get

$$
\begin{aligned}
\log_q H_L(P_x) &= \gamma \deg\left(\sup_{\substack{i\in I_{t-1}\\ j\in I_{r-1}}} \left((x_d)_\infty, (x_{t-1+j})_\infty + a_j(x_i)_\infty, a_r(x_i)_\infty\right)\right) + \xi \deg\left(\sup_{i\in I_{t-1}} ((x_i)_\infty)\right) \\
&= \mathbf{d}_L(x_i).
\end{aligned}
$$

It follows that $H_L(P_x)^{-s} = q^{-s\mathbf{d}_L(x_i)}$, hence

$$
\zeta_{U;L}(s) = \sum_{x\in \mathbb{A}^d} H_L(P_x)^{-s} = \sum_{(x_i)\in \mathbb{A}^d} q^{-s\mathbf{d}_L(x_i)} = Z_{U;L}(q^{-s})
$$

by Lemma 94. This completes the proof of the lemma. □

Analogous to Bourqui in [6], we define the following counting functions $\mathrm{Div}^+(K) \to \mathbb{Z}$ by

$$
\begin{aligned}
\bar{R}_L(D) &:= \#\left\{(x_i) \in \mathbb{A}^d : \mathbf{d}_L(x_i) = D\right\}, \\
R_L(D) &:= \#\left\{(x_i) \in \mathbb{A}^d : \mathbf{d}_L(x_i) \leq D\right\}.
\end{aligned}
$$

Remark 96. Note that $R_L(D) = \sum_{0\leq D'\leq D} \bar{R}_L(D')$, so $(R_L, \bar{R}_L)$ form a μ-couple.

It follows from the definition of $Z_{U;L}(T)$ in (3.5.8) that

$$
Z_{U;L}(T) = \sum_{D\geq 0} \bar{R}_L(D) T^{\deg(D)}.
$$

The following result is crucial for the study of the series $Z_{U;L}(s)$, as it allows us to relate this series to the series that defines the Dedekind zeta function of the field K.

Lemma 97. *The following relation holds*

$$Z_{U;L}(T) = \left(\sum_{D \geq 0} R_L(D)T^{\deg(D)}\right)\frac{1}{Z_K(T)}.$$

Proof. Since $(R_L, \tilde{R}_L)$ is a μ-couple, using Lemma 89 , we have

$$\begin{aligned} Z_{U;L}(T) &= \sum_{D \geq 0} \tilde{R}_L(D)T^{\deg(D)} = \sum_{D \geq 0} \sum_{0 \leq D' \leq D} \mu(D - D')R_L(D')T^{\deg(D)} \\ &= \sum_{D \geq 0}\sum_{D' \geq 0} \mu(D)R_L(D')T^{\deg(D+D')} = \left(\sum_{D \geq 0} R_L(D)T^{\deg(D)}\right)\left(\sum_{D' \geq 0}\mu(D)T^{\deg(D')}\right) \\ &= \left(\sum_{D \geq 0} R_L(D)T^{\deg(D)}\right)\frac{1}{Z_K(T)}, \end{aligned}$$

where in the last equality we used Lemma 91. This proves the desired identity. □

Associated to $L = \gamma h + \xi f \in \mathrm{Pic}(X)$ big, we define

$$A_L := \frac{r+1}{\gamma}, \quad B_L := \frac{(r+1)a_r - |a| + t}{\gamma a_r + \xi}. \tag{3.5.9}$$

Note that by Proposition 30 we have

$$a(L) = \max\{A_L, B_L\} \quad \text{and} \quad b(L) = \begin{cases} 2 & \text{if } A_L = B_L, \\ 1 & \text{if } A_L \neq B_L. \end{cases} \tag{3.5.10}$$

We now present the main result of this section, which describes the analytical properties of the function $\zeta_{U;L}(s)$.

Theorem 98. *Let $X := X_d(a_1, \dots, a_r)$ be a Hirzebruch–Kleinschmidt variety over the global function field $K = \mathbb{F}_q(C)$ with $a_r > 0$. Moreover, let $U := U_d(a_1, \dots, a_r)$ be the good open subset of X, let $L = \gamma h + \xi f$ be a big line bundle class in $\mathrm{Pic}(X_d(a_1, \dots, a_r))$, and let $\zeta_{U;L}(s)$ be the associated height zeta function. Then, $\zeta_{U;L}(s)$ is a rational function in q^{-s}. Moreover, $\zeta_{U;L}(s)$ converges absolutely for $\Re(s) > a(L)$ and it has a pole of order $b(L)$ at $s = a(L)$ with*

$$\lim_{s \to a(L)} (s - a(L))^{b(L)}\zeta_{U;L}(s) = \begin{cases} \dfrac{q^{(d+2)(1-g)}h_K^2}{\zeta_K(t)\,\zeta_K(r+1)(\gamma a_r + \xi)\,(q-1)^2\log(q)^2} & \text{if } \xi = \left(\frac{t-|a|}{r+1}\right)\gamma, \\ \dfrac{q^{(d+2-N_X)(1-g)}h_K\mathbf{R}_K(1-N_X;\ B_L - r + N_X - 1)}{\zeta_K(t)\,\zeta_K(\ B_L)(\gamma a_r + \xi)(q-1)\log(q)} & \text{if } \xi < \left(\frac{t-|a|}{r+1}\right)\gamma, \\ \dfrac{q^{(r+1)(1-g)}h_K\mathbf{R}_K(1-t; A_L\ + |a|)}{\zeta_K(A_L\ + |a|)\,\zeta_K(r+1)\ (q-1)\log(q)} & \text{if } \xi > \left(\frac{t-|a|}{r+1}\right)\gamma, \end{cases}$$

where $N_X := \#\{j \in \{1, \dots, r\} : a_j = a_r\}$ and

$$\mathbf{R}_K(a, b) := \sum_{D \geq 0} q^{-(a\ell(D) + b\deg(D))} \quad \text{for } a, b \in \mathbb{C} \text{ with } \Re(a+b) > 1.$$

Remark 99. 1. The Riemann–Roch theorem (see Section 3.2) implies that

$$\mathbf{R}_K(a,b) = \sum_{\substack{D \geq 0 \\ \deg(D) \leq 2g-2}} q^{-(a\ell(D)+b\deg(D))} + q^{a(g-1)} \sum_{\substack{D \geq 0 \\ \deg(D) > 2g-2}} q^{-(a+b)\deg(D)}, \quad (3.5.11)$$

hence $\mathbf{R}_K(a,b) = q^{a(g-1)}\zeta_K(a+b) + \mathbf{S}_K(a,b)$ with the finite sum

$$\mathbf{S}_K(a,b) := \sum_{\substack{D \geq 0 \\ \deg(D) \leq 2g-2}} \left(q^{-(a\ell(D)+b\deg(D))} - q^{a(g-1)-(a+b)\deg(D)}\right).$$

In particular, $\mathbf{R}_K(a,b) = q^{-a}\zeta_K(a+b)$ if $g = 0$.

2. As in the case of Hirzebruch–Kleinschmidt varieties over number fields (see [16, Remark 6.8]), the different cases that appear in Theorem 98 give a subdivision of the *big cone* of X, i.e. the interior of $\Lambda_{\text{eff}}(X)$ (see Figure 3.1 for an illustration, where we assume $t > |a|$ for simplicity). It follows from Theorem 98 that the line bundles L contained in the ray passing through the anticanonical class have height zeta functions with a double pole at $s = A_L = B_L$, while line bundles outside this ray have height zeta functions with a simple pole at $s = \max\{A_L, B_L\}$.

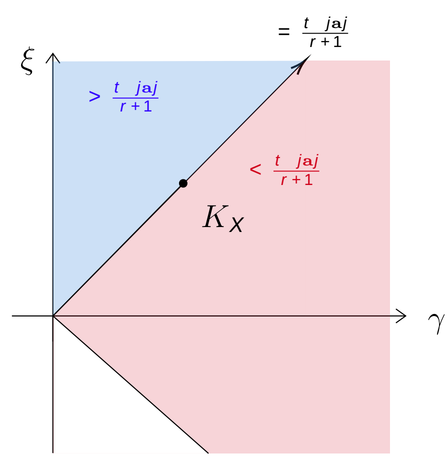

Figure 3.1: Subdivision of the big cone of X. The heights that come from line bundles contained in the ray passing through the anticanonical bundle induce height zeta functions having a double pole at $s = \frac{r+1}{\;}$.

Proof. For the sake of the reader's convenience, we divide the proof into four independent steps.

Step 1. Rewriting the function $R_L(D)$. For positive integers n, m and $D \in \mathrm{Div}^+(K)$, we define

$$N_m^n(D) := \#\left\{(x_i) \in \mathbb{A}^m : n \sup_{i \in I_m}(x_i)_1 \leq D\right\},$$

and

$$\bar{N}_m(D) := \#\left\{(x_i)\in \mathbb{A}^m : \sup_{i\in I_m}(x_i)_\infty = D\right\}.$$

Note that $(N_m^1, \bar{N}_m)$ forms a μ-couple and $N_m^1(D) = (N_1^1(D))^m = q^{m\ell(D)}$ (see Section 3.2).

For each integer $c \geq 1$, we also define the set

$$\mathrm{Div}_c := \{D \in \mathrm{Div}^+(K) : v(D) \leq c \text{ for all } v \in \mathrm{Val}(K)\},$$

and note that by the division algorithm all divisor $D \geq 0$ can be written uniquely as $D = (c+1)D_1 + D_2$, where $D_1 \geq 0$ and $D_2 \in \mathrm{Div}_c$. Observe also that

$$N_1^{c+1}((c+1)D_1 + D_2) = N_1^1(D_1) \quad \text{for all } D_1 \geq 0, D_2 \in \mathrm{Div}_c. \tag{3.5.12}$$

In particular, choosing $c = c_L - 1$ where

$$c_L := \gamma a_r + \xi,$$

we have that every $D \geq 0$ may be written uniquely as

$$D = c_L D_1 + D_2, \quad \text{with } D_1 \geq 0, D_2 \in \mathrm{Div}_{c_L - 1}, \tag{3.5.13}$$

and for every $D' \geq 0$, we have

$$c_L D' \leq D \iff D' \leq D_1. \tag{3.5.14}$$

Now, recalling that $d = r + t - 1$, it follows from the definition of $R_L(D)$ and $\mathbf{d}_L(x)$ that

$$\begin{aligned} R_L(D) &= \#\left\{(x_i) \in \mathbb{A}^d : \gamma \sup_{\substack{i\in I_{t-1}\\ j\in I_{r-1}}} \left((x_d)_\infty, (x_{t-1+j})_\infty + a_j(x_i)_\infty, a_r(x_i)_\infty\right) + \xi \sup_{i\in I_{t-1}} \left((x_i)_\infty\right) \leq D\right\} \\ &= \sum_{0\leq D'} \bar{N}_{t-1}(D')\#\left\{(x_i)\in \mathbb{A}^r : \gamma \sup_{j\in I_{r-1}}((x_r)_\infty, (x_j)_\infty + a_j D', a_r D') + \xi D' \leq D\right\}. \end{aligned}$$

Since

$$c_L D' \leq \gamma \sup_{j\in I_{r-1}}((x_r)_\infty, (x_j)_\infty + a_j D', a_r D') + \xi D',$$

writing D as in (3.5.13) we get (using (3.5.14))

$$R_L(D) = \sum_{0\leq D'\leq D_1} \bar{N}_{t-1}(D')\#\left\{(x_i)\in \mathbb{A}^r : \gamma \sup_{j\in I_{r-1}}((x_r)_\infty, (x_j)_\infty + a_j D', a_r D') + \xi D' \leq D\right\}.$$

Furthermore, using (3.5.14) we see that for D' satisfying $0 \leq D' \leq D_1$, the condition

$$\gamma \sup_{j\in I_{r-1}}((x_r)_\infty, (x_j)_\infty + a_j D', a_r D') + \xi D' \leq D,$$

is equivalent to the system of inequalities

$$\begin{cases} \gamma(x_r)_\infty \leq D - \xi D', \\ \gamma(x_j)_\infty \leq D - (\xi + \gamma a_j)D', \quad \text{for all } j \in I_{r-1}. \end{cases}$$

Therefore, we obtain

$$R_L(D) = \sum_{0 \le D' \le D_1} \overline{N}_{t-1}(D') N_1(D - \xi D') \prod_{j=1}^{r-1} N_1\left(D - (\xi + \gamma a_j) D'\right).$$

Putting $a_0 := 0$, we rewrite this identity as

$$R_L(D) = \sum_{0 \le D' \le D_1} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1\left(D - (\xi + \gamma a_j) D'\right). \tag{3.5.15}$$

Step 2. Analysis of the formal series $\sum_{D \ge 0} R_L(D) T^{\deg(D)}$. Let us define

$$Z_1(T) := \sum_{D \ge 0} R_L(D) T^{\deg(D)}.$$

Using (3.5.13) we have

$$Z_1(T) = \sum_{D_1 \ge 0; D_2 \in \mathrm{Div}_{c_L - 1}} R_L\left(c_L D_1 + D_2\right) T^{\deg(c_L D_1 + D_2)}.$$

Fixing now $D_1 \ge 0, D_2 \in \mathrm{Div}_{c_L - 1}$ and using (3.5.15) we have

$$\begin{aligned} & R_L\left(c_L D_1 + D_2\right) T^{\deg(c_L D_1 + D_2)} \\ = & \sum_{0 \le D' \le D_1} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1\left(c_L D_1 + D_2 - (\xi + \gamma a_j) D'\right) T^{\deg(c_L D_1 + D_2)} \\ = & \sum_{0 \le D' \le D_1} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1\left(c_L (D_1 - D') + D_2 + \gamma(a_r - a_j) D'\right) T^{\deg(c_L (D_1 - D') + D_2 + c_L D')}. \end{aligned}$$

Writing $D_1 - D' = D''$ with $D'' \ge 0$ we get (using (3.5.12))

$$\begin{aligned} Z_1(T) = & \sum_{\substack{D' \ge 0; D'' \ge 0 \\ D_2 \in \mathrm{Div}_{c_L - 1}}} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1\left(c_L D'' + D_2 + \gamma(a_r - a_j) D'\right) T^{\deg(c_L D'' + D_2 + c_L D')} \\ = & \sum_{D \ge 0; D' \ge 0} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1\left(D + \gamma(a_r - a_j) D'\right) T^{\deg(D + c_L D')} \\ = & \sum_{\substack{D \ge 0; D' \ge 0 \\ D_1 \in \mathrm{Div}_{\gamma - 1}}} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1\left(\gamma D + D_1 + \gamma(a_r - a_j) D'\right) T^{\deg(\gamma D + D_1 + c_L D')} \\ = & \sum_{\substack{D \ge 0; D' \ge 0 \\ D_1 \in \mathrm{Div}_{\gamma - 1}}} \overline{N}_{t-1}(D') \prod_{j=0}^{r-1} N_1^{1}\left(D + (a_r - a_j) D'\right) T^{\deg(\gamma D + D_1 + c_L D')}. \end{aligned}$$

Recall that $N_X = \#\{j \in \{1, \ldots, r\} : a_j = a_r\}$ and note that $j = r - N_X$ is the largest non-negative index satisfying $a_j < a_r$. Hence, we have

$$Z_1(T) = \sum_{\substack{D \geq 0; D' \geq 0 \\ D_1 \in \mathrm{Div}_{\geq 1}}} \tilde{N}_{t-1}(D') N_1^1(D)^{N_X - 1} \prod_{j=0}^{r-N_X} N_1^1(D + (a_r - a_j)D')\, T^{\deg(-D + D_1 + c_L D')}.$$

Now, define

$$Z_2(T) := \sum_{\substack{D \geq 0; D' \geq 0 \\ D_1 \in \mathrm{Div}_{\geq 1}}} \tilde{N}_{t-1}(D') N_1^1(D)^{N_X - 1} \prod_{j=0}^{r-N_X} q^{1-g} q^{\deg(D) + (a_r - a_j)\deg(D')} T^{\deg(-D + D_1 + c_L D')}.$$

Putting $A := \min\{a_r - a_j : j \in \{0, \ldots, r - N_X\}\} = a_r - a_{r-N_X}$, we note by the Riemann–Roch theorem (see Section 2.1) that

$$N_1^1(D + (a_r - a_j)D') = q^{1-g} q^{\deg(D) + (a_r - a_j)\deg(D')} \quad \text{for all } j \in \{0, \ldots, r - N_X\},$$

provided $\deg(D + AD') > 2g - 2$. This implies

$$\begin{aligned} Z_1(T) - Z_2(T) &= \sum_{\substack{D \geq 0; D' \geq 0 \\ \deg(D) + A\deg(D') \leq 2g-2 \\ D_1 \in \mathrm{Div}_{\geq 1}}} \tilde{N}_{t-1}(D') N_1^1(D)^{N_X - 1} \left(\prod_{j=0}^{r-N_X} N_1^1(D + (a_r - a_j)D') \right. \\ &\qquad \left. - q^{(r+1-N_X)(1-g)} \prod_{j=0}^{r-N_X} q^{\deg(D) + (a_r - a_j)\deg(D')} \right) T^{\deg(-D + D_1 + c_L D')} \\ &= P_1(T) \sum_{D_1 \in \mathrm{Div}_{\geq 1}} T^{\deg(D_1)}, \end{aligned} \tag{3.5.16}$$

where $P_1(T) \in \mathbb{Q}[T]$, since $A > 0$ and there are only finitely many pairs of divisors $D \geq 0, D' \geq 0$ satisfying $\deg(D + AD') \leq 2g - 2$. Next, we further compute

$$\begin{aligned} Z_2(T) = & q^{(r+1-N_X)(1-g)} \left(\sum_{D' \geq 0} \tilde{N}_{t-1}(D') q^{((r+1)a_r - |a|)\deg(D')} T^{c_L \deg(D')} \right) \\ & \left(\sum_{D \geq 0} N_1^1(D)^{N_X - 1} \left(q^{r+1-N_X} T^{-1} \right)^{\deg(D)} \right) \left(\sum_{D_1 \in \mathrm{Div}_{\geq 1}} T^{\deg(D_1)} \right). \end{aligned}$$

Putting

$$
\begin{aligned}
Z_3(T) :=& q^{(r+1-N_X)(1-g)} \left(\sum_{D'\geq 0} \overline{N}_{t-1}(D') q^{((r+1)a_r-|a|)\deg(D')} T^{c_L \deg(D')}\right)\\
&\left(\sum_{D\geq 0} \left(q^{1-g+\deg(D)}\right)^{N_X-1} \left(q^{r+1-N_X} T\right)^{\deg(D)}\right)\left(\sum_{D_1\in \mathrm{Div}_{\geq 1}} T^{\deg(D_1)}\right)\\
=& q^{r(1-g)} \left(\sum_{D'\geq 0} \overline{N}_{t-1}(D') q^{((r+1)a_r-|a|)\deg(D')} T^{c_L \deg(D')}\right)\\
&\left(\sum_{D\geq 0} (q^r T)^{\deg(D)}\right)\left(\sum_{D_1\in \mathrm{Div}_{\geq 1}} T^{\deg(D_1)}\right)
\end{aligned}
$$

we have, using the Riemann–Roch theorem once more, that

$$
\begin{aligned}
Z_2(T) - Z_3(T) =& q^{(r+1-N_X)(1-g)} \left(\sum_{D'\geq 0} \overline{N}_{t-1}(D') q^{((r+1)a_r-|a|)\deg(D')} T^{c_L \deg(D')}\right)\\
&P_2(T)\left(\sum_{D_1\in \mathrm{Div}_{\geq 1}} T^{\deg(D_1)}\right),
\end{aligned}
\tag{3.5.17}
$$

with

$$
P_2(T) := \sum_{\substack{D\geq 0\\ \deg(D)\leq 2g-2}} \left(N_1^1(D)^{N_X-1} - \left(q^{1-g+\deg(D)}\right)^{N_X-1}\right)\left(q^{r+1-N_X} T\right)^{\deg(D)} \in \mathbb{Q}[T].
$$

Step 3. Rewriting the formal series $Z_3(T)$. Taking into account that $(N_{t-1}^1, \overline{N}_{t-1})$ is a μ-couple, and using Lemma 97, we have

$$
\begin{aligned}
&\sum_{D'\geq 0} \overline{N}_{t-1}(D') q^{((r+1)a_r-|a|)\deg(D')} T^{c_L \deg(D')}\\
=& \sum_{D\geq 0}\sum_{0\leq D'\leq D} \mu(D-D') N_{t-1}^1(D') q^{((r+1)a_r-|a|)\deg(D)} T^{c_L \deg(D)}\\
=& \sum_{D\geq 0}\sum_{D'\geq 0} \mu(D) N_{t-1}^1(D') q^{((r+1)a_r-|a|)\deg(D+D')+} T^{c_L \deg(D+D')}\\
=& \left(\sum_{D'\geq 0} N_1^1(D')^{t-1} q^{((r+1)a_r-|a|)\deg(D')} T^{c_L \deg(D')}\right)\left(\sum_{D\geq 0} \mu(D) q^{((r+1)a_r-|a|)\deg(D)} T^{c_L \deg(D)}\right)\\
=& \left(\sum_{D'\geq 0} N_1^1(D')^{t-1} q^{((r+1)a_r-|a|)\deg(D')} T^{c_L \deg(D')}\right)\frac{1}{Z_K\left(q^{(r+1)a_r-|a|} T^{c_L}\right)}.
\end{aligned}
$$

Using again the Riemann–Roch theorem, we obtain

$$\sum_{D' \geq 0} \mathcal{N}_{t-1}(D') q^{((r+1)a_r - |a|)\deg(D')} T^{c_L \deg(D')}$$
$$= \left(\sum_{D' \geq 0} q^{(t-1)(1-g)} q^{(t-1)\deg(D')} q^{((r+1)a_r - |a|)\deg(D')} T^{c_L \deg(D')} + P_3(T) \right) \frac{1}{Z_K\left(q^{(r+1)a_r - |a|} T^{c_L}\right)}$$
$$= \left(q^{(t-1)(1-g)} Z_K\left(q^{(r+1)a_r - |a| + t - 1} T^{c_L}\right) + P_3(T) \right) \frac{1}{Z_K\left(q^{(r+1)a_r - |a|} T^{c_L}\right)},$$

where

$$P_3(T) := \sum_{\substack{D' \geq 0 \\ \deg(D') \leq 2g-2}} \left(N_1^1(D')^{t-1} - q^{(t-1)(1-g+\deg(D'))} \right) q^{((r+1)a_r - |a|)\deg(D')} T^{c_L \deg(D')} \in \mathbb{Q}[T].$$

We conclude

$$Z_3(T) = \left(q^{(t-1)(1-g)} Z_K\left(q^{(r+1)a_r - |a| + t - 1} T^{c_L}\right) + P_3(T) \right)$$
$$\cdot \frac{q^{r(1-g)} Z_K\left(q^r T\right)}{Z_K\left(q^{(r+1)a_r - |a|} T^{c_L}\right)} \left(\sum_{D_1 \in \mathrm{Div}_{1}} T^{\deg(D_1)} \right). \tag{3.5.18}$$

Note that, by (3.5.17), we also get

$$Z_2(T) - Z_3(T) = q^{(r+1-N_X)(1-g)} \left(q^{(t-1)(1-g)} Z_K\left(q^{(r+1)a_r - |a| + t - 1} T^{c_L}\right) + P_3(T) \right)$$
$$\cdot \frac{P_2(T)}{Z_K\left(q^{(r+1)a_r - |a|} T^{c_L}\right)} \left(\sum_{D_1 \in \mathrm{Div}_{1}} T^{\deg(D_1)} \right). \tag{3.5.19}$$

Step 4. Analytic behaviour of $\zeta_{U;D}(s)$. Since all $D \geq 0$ can be written as $D = \gamma D_1 + D_2$, with $D_1 \geq 0$ and $D_2 \in \mathrm{Div}_{1}$, we have

$$Z_K(T) = \left(\sum_{D \geq 0} T^{\gamma \deg(D)} \right) \left(\sum_{D \in \mathrm{Div}_{1}} T^{\deg(D)} \right).$$

This implies

$$\sum_{D \in \mathrm{Div}_{1}} T^{\deg(D)} = \frac{Z_K(T)}{Z_K(T^{\gamma})},$$

and by Lemma 97 together with (3.5.16) we get

$$Z_{U;L}(T) = \frac{Z_1(T)}{Z_K(T)} = \frac{Z_2(T)}{Z_K(T)} + \frac{P_1(T)}{Z_K(T^{\gamma})}.$$

Then, using (3.5.19) we have

$$Z_{U;L}(T) = \frac{Z_3(T)}{Z_K(T)} + \frac{q^{(d+1-N_X)(1-g)} P_2(T) Z_K\left(q^{(r+1)a_r - |a| + t - 1} T^{c_L}\right)}{Z_K\left(q^{(r+1)a_r - |a|} T^{c_L}\right) Z_K(T^{\gamma})}$$
$$+ \frac{q^{(r+1-N_X)(1-g)} P_3(T) P_2(T)}{Z_K\left(q^{(r+1)a_r - |a|} T^{c_L}\right) Z_K(T^{\gamma})} + \frac{P_1(T)}{Z_K(T^{\gamma})}$$

Finally, using (3.5.18) we get

$$\begin{aligned}
Z_{U,L}(T) = {}& \frac{q^{d(1-g)}\, Z_K\left(q^{(r+1)a_r-|a|+t-1}T^{c_L}\right) Z_K\left(q^r T^{\gamma}\right)}{Z_K\left(q^{(r+1)a_r-|a|}T^{c_L}\right) Z_K\left(T^{\gamma}\right)} + \frac{q^{r(1-g)} P_3(T)\, Z_K\left(q^r T^{\gamma}\right)}{Z_K\left(q^{(r+1)a_r-|a|}T^{c_L}\right) Z_K\left(T^{\gamma}\right)} \\
&+ \frac{q^{(d+1-N_X)(1-g)} P_2(T)\, Z_K\left(q^{(r+1)a_r-|a|+t-1}T^{c_L}\right)}{Z_K\left(q^{(r+1)a_r-|a|}T^{c_L}\right) Z_K\left(T^{\gamma}\right)} + \frac{q^{(r+1-N_X)(1-g)} P_3(T) P_2(T)}{Z_K\left(q^{(r+1)a_r-|a|}T^{c_L}\right) Z_K\left(T^{\gamma}\right)} \\
&+ \frac{P_1(T)}{Z_K\left(T^{\gamma}\right)}.
\end{aligned}$$

By Lemma 95 we conclude

$$\begin{aligned}
\zeta_{U,L}(s) = {}& \frac{q^{d(1-g)}\zeta_K\left((\gamma a_r+\xi)s-((r+1)a_r-|a|+t-1)\right)\zeta_K(\gamma s-r)}{\zeta_K\left((\gamma a_r+\xi)s-((r+1)a_r-|a|)\right)\zeta_K(\gamma s)} \\
&+ \frac{q^{r(1-g)}P_3(q^{-s})\zeta_K(\gamma s-r)}{\zeta_K\left((\gamma a_r+\xi)s-((r+1)a_r-|a|)\right)\zeta_K(\gamma s)} \\
&+ \frac{q^{(d+1-N_X)(1-g)}P_2(q^{-s})\zeta_K\left((\gamma a_r+\xi)s-((r+1)a_r-|a|+t-1)\right)}{\zeta_K\left((\gamma a_r+\xi)s-((r+1)a_r-|a|)\right)\zeta_K(\gamma s)} \\
&+ \frac{q^{(r+1-N_X)(1-g)}P_3(q^{-s})P_2(q^{-s})}{\zeta_K\left((\gamma a_r+\xi)s-((r+1)a_r-|a|)\right)\zeta_K(\gamma s)} + \frac{P_1(q^{-s})}{\zeta_K(\gamma s)}.
\end{aligned}$$

It follows from the properties of the zeta function $\zeta_K(s)$ (see Section 3.2) that $\zeta_{U,L}(s)$ is a rational function in q^{-s}. Moreover, using Corollary 92 and (2.10.4) we see that $\zeta_{U,L}(s)$ converges absolutely for

$$\Re(s) > \max\{A_L, B_L\} = a(L),$$

with A_L, B_L defined in (3.5.9). Finally, recalling that $r \ge 1, t \ge 2$, we get the following properties:

1. If $A_L = B_L$, then $\zeta_{U,L}(s)$ has a pole of order 2 at $s = a(L)$ with

$$\begin{aligned}
\lim_{s\to a(L)}(s-a(L))^2\zeta_{U,L}(s) &= \frac{q^{d(1-g)}}{\zeta_K(t)\zeta_K(r+1)}\frac{\left(\mathrm{Res}_{s=1}\zeta_K(s)\right)^2}{(\gamma+a_r\xi)\gamma} \\
&= \frac{q^{(d+2)(1-g)}h_K^2}{\zeta_K(t)\zeta_K(r+1)(\gamma+a_r\xi)\gamma(q-1)^2\log(q)^2}.
\end{aligned}$$

2. If $A_L < B_L$, then $\zeta_{U,L}(s)$ satisfies

$$\begin{aligned}
\lim_{s\to a(L)}(s-a(L))\zeta_{U,L}(s) &= \left(q^{d(1-g)}\zeta_K(\gamma B_L-r)+q^{(d+1-N_X)(1-g)}P_2(q^{-B_L})\right) \\
&\qquad \frac{\mathrm{Res}_{s=1}\zeta_K(s)}{\zeta_K(t)\,\zeta_K(\gamma B_L)(\gamma a_r+\xi)} \\
&= \left(q^{d(1-g)}\zeta_K(\gamma B_L-r)+q^{(d+1-N_X)(1-g)}P_2(q^{-B_L})\right) \\
&\qquad \frac{h_K q^{1-g}}{\zeta_K(t)\,\zeta_K(\gamma B_L)(\gamma a_r+\xi)(q-1)\log(q)}.
\end{aligned}$$

From the definition of $P_2(T)$ and (3.5.11) we have

$$
\begin{aligned}
&q^{d(1-g)}\zeta_K(\gamma B_L - r) + q^{(d+1-N_X)(1-g)}P_2(q^{-B_L})\\
&= q^{(d+1-N_X)(1-g)} \sum_{\substack{D\geq 0\\ \deg(D)\leq 2g-2}} q^{(N_X-1)\ell(D)+(r+1-N_X-B_L)\deg(D)} + q^{d(1-g)} \sum_{\substack{D\geq 0\\ \deg(D)>2g-2}} q^{-(B_L-r)\deg(D)}\\
&= q^{(d+1-N_X)(1-g)}\mathbf{R}_K(1-N_X, \gamma B_L - r + N_X - 1).
\end{aligned}
$$

Since this value is positive, we conclude that $\zeta_{U;L}(s)$ has a simple pole at $s = a(L)$ in this case.

3. If $A_L > B_L$, then $\zeta_{U;L}(s)$ satisfies

$$
\begin{aligned}
\lim_{s\to a(L)}(s-a(L))\zeta_{U;L}(s) &= \left(q^{d(1-g)}\zeta_K\left((\gamma a_r+\xi)A_L - ((r+1)a_r - |a| + t - 1)\right) + q^{r(1-g)}P_3(q^{-A_L})\right)\\
&\qquad \frac{\mathrm{Res}_{s=1}\zeta_K(s)}{\zeta_K\left((\gamma a_r+\xi)A_L - ((r+1)a_r - |a|)\right)\zeta_K(r+1)\gamma}\\
&= \left(q^{d(1-g)}\zeta_K(A_L\xi + |a| - t + 1) + q^{r(1-g)}P_3(q^{-A_L})\right)\\
&\qquad \frac{h_K q^{1-g}}{\zeta_K(A_L\xi+|a|)\zeta_K(r+1)\gamma(q-1)\log(q)}.
\end{aligned}
$$

Using the definition of $P_3(T)$ and (3.5.11), we get

$$
\begin{aligned}
&q^{d(1-g)}\zeta_K(A_L\xi + |a| - t + 1) + q^{r(1-g)}P_3(q^{-A_L})\\
&= q^{r(1-g)} \sum_{\substack{D\geq 0\\ \deg(D)\leq 2g-2}} q^{\ell(D)(t-1)-\deg(D)(A_L+|a|)} + q^{d(1-g)} \sum_{\substack{D\geq 0\\ \deg(D)>2g-2}} q^{-(A_L+|a|-t+1)\deg(D)}\\
&= q^{r(1-g)}\mathbf{R}_K(1-t, A_L\xi + |a|).
\end{aligned}
$$

As in the previous case, we conclude that $\zeta_{U;L}(s)$ has a simple pole at $s = a(L)$.

Since the condition $A_L = B_L$ (resp. $A_L < B_L$, $A_L > B_L$) is equivalent to $\xi = \left(\frac{t-|a|}{r+1}\right)\gamma$ (resp. $\xi < \left(\frac{t-|a|}{r+1}\right)\gamma$, $\xi > \left(\frac{t-|a|}{r+1}\right)\gamma$), the desired result follows from the properties stated above. This completes the proof of the theorem. □

The much simpler case when $a_r = 0$ is given by the following theorem, where we see that there is no need to remove a closed subvariety of $X = X_d(a_1,\dots,a_r)$ and we can directly give the analytic properties of the height zeta function

$$
\zeta_{X;L}(s) := \sum_{P\in X(K)} H_L(P)^{-s}.
$$

Note that, in this case, we have

$$
A_L = \frac{r+1}{\gamma}, \quad B_L = \frac{t}{\xi}.
$$

Theorem 100. *Let $X \cong \mathbb{P}^r \times \mathbb{P}^{t-1}$ be a Hirzebruch–Kleinschmidt variety over the global function field $K = \mathbb{F}_q(C)$ with $a_r = 0$ and let $L = \gamma h + \xi f \in \mathrm{Pic}(X)$ big. Then, the height zeta function $\zeta_{X;L}(s)$ is a rational function in q^{-s}. Moreover, $\zeta_{X;L}(s)$ converges absolutely for $\Re(s) > a(L)$ and it has a pole of order $b(L)$ at $s = a(L)$ with*

$$\lim_{s \to a(L)} (s - a(L))^{b(L)} \zeta_{X;L}(s) = \begin{cases} \frac{q^{(d+2)(1-g)} h_K^2}{\zeta_K(t)\zeta_K(r+1)(q-1)^2 \log(q)^2} & \text{if } \xi = \left(\frac{t}{r+1}\right)\gamma, \\ \frac{\zeta_{\mathbb{P}^r}(\gamma B_L) h_K q^{t(1-g)}}{\zeta_K(t)(q-1)\log(q)} & \text{if } \xi < \left(\frac{t}{r+1}\right)\gamma, \\ \frac{\zeta_{\mathbb{P}^{t-1}}(\xi A_L) h_K q^{(r+1)(1-g)}}{\zeta_K(r+1)(q-1)\log(q)} & \text{if } \xi > \left(\frac{t}{r+1}\right)\gamma. \end{cases}$$

Proof. By (3.5.1) we have

$$\zeta_{X;L}(s) = \zeta_{\mathbb{P}^r}(\gamma s)\zeta_{\mathbb{P}^{t-1}}(\xi s).$$

Then, using Theorem 84, we see that $\zeta_{X;L}(s)$ is a rational function in q^{-s} and it converges absolutely on $\Re(s) > \max\{A_L, B_L\} = a(L)$. Moreover, we have the following properties:

1. If $A_L = B_L$, then $\zeta_{X;L}(s)$ has a double pole at $s = a(L)$ with

$$\begin{aligned} \lim_{s \to a(L)} (s - a(L))^2 \zeta_{X;L}(s) &= \frac{\mathrm{Res}_{s=r+1}\zeta_{\mathbb{P}^r}(s)\,\mathrm{Res}_{s=t}\zeta_{\mathbb{P}^{t-1}}(s)}{\gamma\xi} \\ &= \frac{h_K^2 q^{(d+2)(1-g)}}{\gamma\xi\zeta_K(r+1)\zeta_K(t)(q-1)^2\log(q)^2}. \end{aligned}$$

2. If $A_L < B_L$, then $\zeta_{X;L}(s)$ has a simple pole at $s = a(L)$ with

$$\begin{aligned} \lim_{s \to a(L)} (s - a(L)) \zeta_{X;L}(s) &= \frac{\zeta_{\mathbb{P}^r}(\gamma B_L)\,\mathrm{Res}_{s=t}\zeta_{\mathbb{P}^{t-1}}(s)}{\xi} \\ &= \frac{\zeta_{\mathbb{P}^r}(\gamma B_L) h_K q^{t(1-g)}}{\xi\zeta_K(t)(q-1)\log(q)}. \end{aligned}$$

3. If $A_L > B_L$, then $\zeta_{X;L}(s)$ has a simple pole at $s = a(L)$ with

$$\begin{aligned} \lim_{s \to a(L)} (s - a(L)) \zeta_{X;L}(s) &= \frac{\mathrm{Res}_{s=r+1}\zeta_{\mathbb{P}^r}(s)\zeta_{\mathbb{P}^{t-1}}(\xi A_L)}{\gamma} \\ &= \frac{\zeta_{\mathbb{P}^{t-1}}(\xi A_L) h_K q^{(r+1)(1-g)}}{\gamma\zeta_K(r+1)(q-1)\log(q)}. \end{aligned}$$

These cases correspond exactly to ξ equal, less than, or greater than $\left(\frac{t}{r+1}\right)$, respectively. This proves the theorem. □

3.5.4 The anticanonical height

We now prove Theorem 7 from the Introduction. We choose $L = -K_X = \gamma f + \xi f$ with $\gamma = r + 1$ and $\xi = t - |a|$. Hence, $A_L = B_L = 1$, $a(L) = 1$ and $b(L) = 2$. The absolute convergence of $\zeta_{U,-K_X}(s)$ on $\Re(s) > 1$ and the fact $\zeta_{U,-K_X}(s)$ is a rational function on q^{-s} follows directly from Theorems 98 and 100. Now, assuming $a_r > 0$, we get by Theorem 98 the equality

$$\lim_{s\to 1}(s-1)^2\zeta_{U,-K_X}(s) = C$$

with C given by (0.0.5). This proves the result in the case $a_r > 0$. Assume $a_r = 0$. Then, Theorem 100 gives

$$\lim_{s\to 1}(s-1)^2\zeta_{X,-K_X}(s) = C$$

with the same C as before. Now, by Lemma 87 we have

$$X \setminus U \simeq \begin{cases} X_{d-1}(a_1,\dots,a_r) \sqcup (\mathbb{P}^r_1 \setminus \{P_0\}) \sqcup \left(\bigsqcup_{2\le t'<t} U_{t'+r-1}(a_1,\dots,a_r)\right) & \text{if } r > 1,\\ \mathbb{P}^{t-1}_2 \sqcup (\mathbb{P}^1_1 \setminus \{P_0\}) \sqcup \left(\bigsqcup_{2\le t'<t} U_{t'+r-1}(a_1)\right) & \text{if } r = 1,\end{cases}$$

with $X_{d-1}(a_1,\dots,a_r) \simeq \mathbb{P}^{t-1}\times\mathbb{P}^{r-1}$ in the case $r > 1$. Using Theorems 100 and 84, we see that

$$\zeta_{X\setminus U,-K_X}(s) := \sum_{P\in X(K)\setminus U(K)} H(P)^{-s}$$

has a simple pole at $s = 1$. We conclude that

$$\lim_{s\to 1}(s-1)^2\zeta_{U,-K_X}(s) = \lim_{s\to 1}(s-1)^2\zeta_{X,-K_X}(s) = C.$$

This completes the proof of Theorem 7.

3.5.5 Example: Hirzebruch surfaces

Given $a > 0$ let $X := X_1(a)$ be a Hirzebruch surface and $L = \gamma h + \xi f \in \mathrm{Pic}(X)$ big. By Lemma 87 we have

$$X \simeq \mathbb{P}^1 \sqcup \mathbb{A}^1 \sqcup U_1(a), \tag{3.5.20}$$

with associated height zeta functions $\zeta_{\mathbb{P}^1}(\xi s), \zeta_{\mathbb{A}^1}(\gamma s) = \zeta_{\mathbb{P}^1}(\gamma s) - 1$ and $\zeta_{U,L}(s)$ where $U := U_1(a)$. In this case we have

$$A_L = \frac{2}{\gamma}, \quad B_L = \frac{a+2}{\gamma a + \xi},$$

and the analytic properties of $\zeta_{U,L}(s)$ are given in Theorem 98. In particular, it converges absolutely in $\Re(s) > \max\{A_L, B_L\}$, is has a pole at $s = \max\{A_L, B_L\}$, and this pole is of order two if $A_L = B_L$, and of order one otherwise. For the zeta function $\zeta_{\mathbb{A}^1}(\gamma s)$, we see that it converges absolutely in $\Re(s) > A_L$, it has a pole at $s = A_L$, and this is a simple pole. Finally, the zeta function $\zeta_{\mathbb{P}^1}(\xi s)$ has no finite abscissa of absolute convergence if $\xi \le 0$, and it converges absolutely in $\Re(s) > \frac{2}{\xi}$ with a simple pole at $s = \frac{2}{\xi}$ if $\xi > 0$. This allows for a complete analysis of the contribution of each component in (3.5.20) to the number of rational points of bounded height H_L in X.

In order to illustrate this, we choose $a = 1$ for simplicity. In the following table we denote by σ_1, σ_2 and σ_3 the abscissas of absolute convergence of $\zeta_{\mathbb{P}^1}(\xi s), \zeta_{\mathbb{A}^1}(\gamma s)$ and $\zeta_{U,L}(s)$, respectively, for the first possible choices of $\gamma > 0$ and $\mu > \quad a\gamma$ (so that L is big). We also record the order $b(L)$ of the pole of $\zeta_{U;L}(s)$ at $s = \sigma_3$.

γ	ξ	σ_1	σ_2	σ_3	$b(L)$
1	0	1	2	3	1
1	1	2	2	2	1
1	2	1	2	2	1
2	1	1	1	3	1
2	0	1	1	3/2	1
2	1	2	1	1	2
3	2	1	2/3	3	1
3	1	1	2/3	3/2	1
3	0	1	2/3	1	1

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