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K-polystability of higher dimensional Fano varieties

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Resumen

Esta tesis explora aspectos fundamentales y contemporáneos de la geometría algebraica, con un enfoque particular en las variedades de Fano y la noción de K-estabilidad. El trabajo se estructura en torno a la filosofía del Programa de Modelos Minimales (MMP), cuyo objetivo es clasificar variedades algebraicas seleccionando representantes canónicos dentro de cada clase de equivalencia birracional. El Capítulo I introduce los resultados fundamentales del MMP, enfatizando las generalizaciones en dimensiones superiores del Teorema del Cono de Mori y el rol de las singularidades en el programa.

El Capítulo II se centra en las variedades de Fano, presentando sus propiedades generales, la técnica del Δ-género de T. Fujita y la geometría birracional de familias específicas de variedades 4-dimensionales, como las variedades de del Pezzo. Estas discusiones destacan cómo los invariantes numéricos pueden contribuir a clasificar las variedades de Fano y analizar sus estructuras geométricas.

El Capítulo III ofrece una introducción a la K-estabilidad, un concepto utilizado para determinar la existencia de métricas de Kähler-Einstein en variedades algebraicas, con un énfasis particular en las variedades de Fano. Cubre avances teóricos y criterios numéricos prácticos, incluyendo vínculos recientes entre la K-estabilidad y el MMP. Se discuten invariantes computables, como y , junto con ejemplos que ilustran sus aplicaciones.

Finalmente, el Capítulo IV se enfoca en un trabajo en progreso sobre la K-estabilidad de las variedades de Fano-Mukai 4-dimensionales de género 9, extendiendo la clasificación de variedades K-estables a dimensiones superiores. A través de cálculos explícitos y conexiones con otras variedades, este capítulo proporciona evidencia inicial que respalda la K-estabilidad de estas 4-variedades y examina sus grupos de automorfismos.

Abstract

This thesis explores foundational and contemporary aspects of algebraic geometry, with a particular focus on Fano varieties and the notion of K-stability. The work is structured around the philosophy of the Minimal Model Program (MMP), which aims to classify algebraic varieties by selecting canonical representatives within each birational equivalence class. Chapter I introduces the foundational results of MMP, emphasizing the higher-dimensional generalizations of Mori's Cone Theorem and the role of singularities in the program.

Chapter II centers on Fano varieties, presenting their general properties, T. Fujita's Δ-genus technique, and the birational geometry of specific families of fourfolds, such as del Pezzo varieties. These discussions highlight how numerical invariants can aid in classifying Fano varieties and analyzing their geometric structures.

Chapter III provides an introduction to K-stability, a concept used to determine the existence of Kähler-Einstein metrics on algebraic varieties, with a particular emphasis on Fano varieties. It covers theoretical advancements and practical numerical criteria, including recent links between K-stability and the MMP. Computable invariants, such as and , are discussed alongside examples illustrating their applications.

Finally, Chapter IV focuses on a work in progress concerning the K-stability of Fano-Mukai fourfolds of genus 9, extending the classification of K-stable varieties to higher dimensions. Through explicit computations and connections to other varieties, this chapter provides initial evidence supporting the K-stability of these fourfolds and examines their automorphism groups.

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Contents

Introduction

Algebraic geometry, often described as the study of solutions to polynomial equations, has among its primary goals the classification of all projective varieties $X \subseteq \mathbb{P}^n$ over $\mathbb{C}$ up to isomorphism. A common approach to tackle this classification problem can be outlined as follows:

1. Classify varieties up to birational equivalence.
2. Identify suitable "canonical" representatives within each birational equivalence class.
3. Study how these canonical models relate to one another and to what extent an arbitrary variety deviates from these representatives.

This philosophy, known as the *Mori Program*, has proven highly effective in the study of complex algebraic surfaces, leading to the celebrated Enriques-Kodaira Classification. In simple words, the Minimal Model Program (MMP) in dimension 2 operates as an algorithm that iteratively contracts specific curves, ultimately yielding one of two outcomes: either a unique minimal surface or a ruled surface (where uniqueness is not guaranteed). These foundational ideas culminated in an elegant and complete classification of surfaces, details of which can be found in [Bea83].

Chapter I of this work is devoted to foundational aspects of the Mori Theory/MMP, now regarded as a cornerstone of modern geometry. It aims to present the philosophy that S. Mori employed to generalize the MMP from surfaces to higher dimensions, along with the key results that underpin this extension. In this context, Mori's celebrated Cone Theorem (presented in [Mor82]) laid the groundwork for the higher-dimensional MMP. The concept of (-1)-curves from the theory of surfaces transitions in higher dimensions to the notion of extremal contractions, and Mori demonstrated both their existence and uniqueness, and that these contractions serve to "hide" K_X-negative curves.

Sections I.1, I.2, and I.3 largely follow O. Debarre's book [Deb01], as well as his lecture notes on Mori Theory, which provide a more concise treatment of the same material. Additionally, these results are supported by key insights from [KM98, Mat02]. §I.4, which addresses the necessity of studying singularities to carry out the goals of the Minimal Model Program, is primarily based on the classical reference [KM98]. Finally, section §I.5, focusing on divisor volumes, draws heavily from [Laz04, LM09].

One of the principal families of varieties in algebraic geometry is that of Fano varieties, i.e., varieties X such that K_X is an ample divisor. These varieties generalize, in certain respects, specific complete intersections in projective space. Chapter II of this thesis has the following objectives:

1. Discuss general properties of Fano varieties.
2. Provide a modern exposition of T. Fujita's classical works on Fano varieties, focusing on the Δ-genus technique.
3. Present examples of Fano fourfolds and study their birational geometry in detail using techniques from intersection theory.

Each section in Chapter II corresponds to one of these objectives. The primary reference for this chapter is [IP99], with §II.1 following closely the exposition in §2.1 of this work. Section §II.2 builds upon the foundational papers [Fuj75, Fuj77, Fuj80, Fuj81] by T. Fujita, where he introduced the Δ-genus, a numerical invariant that enabled a clean classification of certain varieties. Notably, these works provide characterizations of Fano varieties with high index, including del Pezzo varieties. A comprehensive (though occasionally less detailed) presentation of these results can also be found in [Fuj90, §I].

Finally, in §II.3, the birational geometry of del Pezzo varieties of degree 4 and 5 is examined in depth, primarily following the works [PZ16, PZ17]. Additionally, various intersection theory computations are presented to illustrate different techniques that employ more advanced methods than those introduced in §I.2. Most of these techniques are rooted in the classical reference [Ful84], while a more modern treatment can be found in [EH16].

Chapter III aims to serve as an introduction to K-stability, a notion originally introduced by G. Tian to detect the existence of certain metrics on algebraic varieties. Its origins trace back to the Calabi problem, which concerns the existence of Kähler-Einstein metrics on a differentiable manifold. More precisely, given a smooth projective variety X, the question is whether there exists a Kähler metric ω on X such that

$$\mathrm{Ric}(\omega) = \lambda\omega \quad \text{for some} \quad \lambda \in \{-1, 0, 1\} \quad \text{(Kähler-Einstein Equation)}.$$

In terms of the trichotomy that dominates algebraic geometry (Calabi-Yau / canonically polarized / Fano), classical results by Aubin and Yau in complex geometry demonstrate that such metrics always exist in the first two cases. However, not every Fano variety admits such a metric. The works of Chen-Donaldson-Sun (2012) and Tian (2015) revealed that this phenomenon is intrinsically algebraic in nature, and the algebro-geometric concept underlying this behavior is known as *K-stability*. Sections §III.2, §III.3 aim to provide an initial approach to K-stability theory.

One of the main challenges of K-stability lies in its computational complexity: verifying it requires examining an entire family of degenerations of a variety, rendering manual verification impractical. This has spurred significant efforts to develop numerical criteria for determining K-stability. Notable advancements in this direction include the works [LX14, Fuj18, Fuj19b, Fuj19a, Li17, FO18, BHJ17, BJ20]. This is mainly discussed in §III.5.

Moreover, one of the most remarkable discoveries of the last decade has been the close connection between the Minimal Model Program and K-stability, primarily due to the works of Yuji Odaka and Chenyang Xu. A brief overview of this relationship is presented in §III.4, drawing mainly from [Oda13a, Oda12, OX12, Oda13b].

To conclude Chapter III, §III.6 addresses the numerical invariants and , which provide computable criteria for K-stability. Notably, [AZ22] introduces a highly fruitful technique in K-stability, a form of adjunction for the -invariant that reduces the dimensionality of the computations. Examples illustrating these techniques are provided at the end of the chapter.

In general, the preparation of this chapter relied heavily on the lecture notes by Harold Blum and Kristin DeVleming, as well as the book [ACC+23], which compiles much of the theory developed in recent years.

I am grateful to Professor Giancarlo Urzúa for inviting me to participate in the "Algebraic Geometry Seminar" organized at Pontificia Universidad Católica de Chile, as well as to the seminar attendees. Chapter III originated as lecture notes for a series of talks I co-presented with Professor Pedro Montero during this seminar. Undoubtedly, participating in this seminar allowed me to gain valuable insights into the topic.

Chapter IV presents the initial steps of a work in progress aimed at studying the K-stability properties of Fano-Mukai fourfolds of genus 9, characterized by S. Mukai. This research is motivated by the ongoing exploration of K-stable Fano varieties. As detailed in [ACC⁺23], the K-stability of the general members of all families of Fano 3-folds has been established, making the next natural challenge the extension of these results to dimension 4.

In Section IV.1, the K-stability of del Pezzo fourfolds is addressed. Building on the works [ST24, AGP06, Fuj17, Liu22], this section concludes that the problem has already been resolved.

Given this context, §IV.2 focuses on characterizing the Fano-Mukai fourfolds of genus 9 (denoted V_{16}), with primary references [Muk89, IR05]. Additionally, this section establishes a link between these varieties and del Pezzo fourfolds of degree 5, which were extensively studied in §II.3.

Subsequently, §IV.3 calculates the beta invariant associated with the divisor on V_{16} arising from the link established in §IV.2. This serves as initial evidence for the K-stability of these varieties. The methodology follows the ideas of [Fuj17], where it was shown that del Pezzo fourfolds of degree 5 are not K-stable using the link discussed in §II.3. The original computation presented corresponds to Proposition IV.3.6.

Finally, §IV.4 explores characterizations of the automorphism group of V_{16}, with the main source being [DM22]. In this section we prove that the action of $\mathrm{Aut}(V_{16})$ does not have fixed points. The details are presented in Proposition IV.4.6.

To conclude, I would like to express my gratitude to Professor Adrien Dubouloz for the opportunity to participate in the Workshop "K-stability, Geometry and Group Actions", organized at the University of Poitiers. This event provided me with the valuable chance to present the content of Chapter IV of this work. I am also deeply thankful to Professors Kento Fujita and Takashi Kishimoto, who attended the workshop and kindly guided me on the posed problem, answering several questions I had along the way.

Chapter I

Mori theory

The classification of complex algebraic surfaces, with foundational contributions stemming from the classical Italian school of algebraic geometry (notably through the works of M. Noether, G. Castelnuovo, and F. Enriques), and later profoundly developed by K. Kodaira, undoubtedly represents one of the most remarkable achievements in mathematics of the past century. The central philosophy of this endeavor can be summarized as follows:

1. classify varieties up to birational equivalence,
2. within each birational equivalence class, identify a model that is as simple as possible,
3. and recognize that the geometry of a variety is fundamentally governed by its canonical divisor.

The Minimal Model Program (MMP) for surfaces thus consists of an algorithm which, starting with a smooth surface, proceeds by contracting specific curves. The process yields one of two outcomes: either a unique minimal surface or a ruled surface, in which case uniqueness is not guaranteed. These foundational ideas culminated in an elegant and comprehensive classification of surfaces, which the interested reader may consult in [Bea83].

Following these breakthroughs, one of the central problems in algebraic geometry throughout the 20th century was the extension of this classification paradigm to higher dimensions, the pursuit of what is now known as the Minimal Model Program in arbitrary dimension. A key distinction between the surface case and higher-dimensional settings lies in the increased complexity of the morphisms involved (cf. Castelnuovo's Contractibility Criterion). In 1982, S. Mori, in his seminal work [Mor82], introduced the celebrated Cone Theorem, which provided the pivotal tools for generalizing the Minimal Model Program to higher dimensions. By employing the concept of extremal contractions, Mori established structural theorems about the associated morphisms, enabling the construction of models that are, in a precise sense, as simple as possible.

This chapter aims, first, to establish the notation and theoretical framework that will be employed throughout this work, and second, to provide a comprehensive account of the fundamental tools and results in birational geometry and the Minimal Model Program. For the most of this chapter X will be a projective variety over an algebraically closed field k.

I.1 Birational geometry and Intersection theory

I.1.1 Divisors

Let X be an algebraic variety. For us, a divisor will be a Weil divisor, and the abelian group of divisors will be denoted $\mathrm{WDiv}(X)$. The class group of X, denoted $\mathrm{Cl}(X)$, is the group of divisors modulo linear equivalence, and the Picard group $\mathrm{Pic}(X)$ of X is the group of Cartier divisors on X modulo linear equivalence (or equivalently, isomorphisms classes of line bundles). To allow greater flexibility in working with divisors, we introduce the following definitions.

Definition I.1.1 ($\mathbb{Q}$-divisors and $\mathbb{R}$-divisors)**.** A $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) on X is an element of the $\mathbb{Q}$-vector space $\mathrm{WDiv}(X) \otimes \mathbb{Q}$ (resp. $\mathrm{WDiv}(X) \otimes \mathbb{R}$) where $\mathrm{WDiv}(X)$ denotes the group of Weil divisors on X. Explicitly, a $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) D is a linear combination $D = \sum_i a_i D_i$ where $a_i \in \mathbb{Q}$ (resp. $a_i \in \mathbb{R}$) and $D_i \in \mathrm{WDiv}(X)$. A $\mathbb{Q}$-divisor (resp. a $\mathbb{R}$-divisor) is $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) if some multiple is a Cartier divisor. We say that an $\mathbb{R}$-divisor $D = \sum_i a_i D_i$ is effective if $a_i > 0$ for all i.

Definition I.1.2. Let $f : Y \to X$ be a proper birational morphism, and let $U = \mathrm{Dom}(f)$ be the domain of the rational map f^{-1}, i.e., the largest open subset of X in which f is an isomorphism. The exceptional locus of f is the closed subset $\mathrm{Exc}(f) := Y \setminus f^{-1}(U)$. We say that a Weil divisor on Y is f-exceptional (or simply exceptional if f is understood in the context) if its support is contained in $\mathrm{Exc}(f)$.

Remark I.1.3. Note that if X is a normal variety, Zariski's Main Theorem implies that $\operatorname{codim}_X(X \setminus U) \geq 2$, and then a prime divisor E on Y is exceptional if and only if $\dim(f(E)) < \dim(E)$.

Definition I.1.4. Let $f : X \dashrightarrow Y$ be a rational map defined in an open set $U \subseteq X$. Given a subvariety $Z \subseteq X$, the **birational** or **strict transform** of Z is $f_*(Z) := \overline{f(Z \cap U)}$. If $g : Y \to X$ is a birational map we use $g^{-1}(Z)$ to denote the birational transform of Z by g^{-1}.

If $f : Y \to X$ be a proper, birational morphism with X being a normal variety, the condition $\operatorname{codim}_X(X \setminus U) \geq 2$ implies that every prime divisor D on X intersects U. Hence, its birational transform is well-defined, and we adopt the notation $\widetilde{D}$ for it. In a more general setting, if $D = \sum_i a_i D_i$ is an $\mathbb{R}$-divisor we denote $\widetilde{D} = \sum_i a_i \widetilde{D_i}$.

I.1.2 Intersection theory and Nakai-Moishezon criterion for ampleness

A longstanding and fundamental problem in algebraic geometry has been the development of a robust and coherent intersection theory—a theoretical framework capable of rigorously formalizing the intersection of subvarieties and accurately defining intersection multiplicities. Over the years, numerous attempts have been made to tackle this challenge, ultimately leading to the definition that will be presented in this section. While this definition may lack an immediately intuitive geometric interpretation, it encapsulates the essential properties and behaviors one expects from an intersection product, ensuring both mathematical rigor and applicability. For the exposition in this section, we largely follow the framework laid out in [Deb01].

One of the most celebrated results in this context is Bézout's theorem, which asserts that the intersection of two plane curves in $\mathbb{P}^2$, counted with multiplicity, equals the product of their degrees. This elegant theorem serves as a cornerstone in intersection theory, illustrating the interplay between geometry and algebra. More generally, the intersection product on a surface can be defined as follows. The precise formulation is provided in [Bea83, Definition I.3].

Definition I.1.5. Let C and D be two curves on a projective surface X with no common components, let x be a point of $C \cap D$, and let f and g be respective generators of the ideals of C and D at x. We define the intersection multiplicity of C and D at x to be

$$m_x(C \cap D) = \dim_{\mathbf{k}} \mathcal{O}_{X,x}/(f, g)$$

Then we set the intersection number of C and D as the integer

$$(C \cdot D) = \sum_{x \in C \cap D} m_x(C \cap D).$$

Building upon this definition, we arrive at the following fundamental property, which further elucidates its significance.

Lemma I.1.6. *For any smooth curve C on X and any Cartier divisor D on X, we have*

$$(D \cdot C) = \deg\left(\mathcal{O}_X(D)|_C\right)$$

Naturally, the next question arises: for which objects does it make sense to define an intersection product? Intuitively, imposing an equation reduces the dimension of a variety by at most 1, as suggested by Krull's Principal Ideal Theorem. Consequently, if X is a variety of dimension n and we consider divisors $D_1, \ldots, D_n$, it follows that $\dim(D_1 \cap \cdots \cap D_m) > 0$ whenever $m < n$. This observation naturally leads to the definition of an intersection product of n divisors on an n-dimensional variety.

To extend this framework, suppose we are working within a subvariety $Y \subset X$ of dimension $\dim(Y) = m$. In this case, we can restrict the divisors to Y and compute the product $D_1|_Y \cdots D_m|_Y$. Thus, the objective is to define an intersection product, denoted $(D_1 \cdots D_n)$, that satisfies the following essential properties:

1. The integer $(D_1 \cdot \ldots \cdot D_n)$ is symmetric and multilinear as a function of its arguments;

2. $(D_1 \cdot \ldots \cdot D_n)$ depends only on the linear equivalence classes of the D_i;

3. If $D_1, \ldots, D_n$ are effective divisors that meet transversely at smooth points of X, then
$$(D_1 \cdot \ldots \cdot D_n) = \#\{D_1 \cap \ldots \cap D_n\}.$$

The justification for point 2 is as follows: when calculating the intersection $(D_1 \cdot \ldots \cdot D_k \cdot Y)$ of divisors $D_1, \ldots, D_k$ within an irreducible subvariety $Y \subset X$ of dimension k, the product is always well-defined. This is because we can replace each divisor by a linearly equivalent divisor whose support does not intersect Y, ensuring the intersection is properly defined.

Due to the above discussion, the intersection product is defined for divisors, and it is defined by Hilbert polynomials as follows [Deb01, Definition 1.7].

Definition I.1.7. Let $D_1, \ldots, D_r$ be Cartier divisors on a projective variety X over a field, with $r \leq \dim(X)$. We define the intersection number

$$(D_1 \cdot \ldots \cdot D_r)$$

as the coefficient of $m_1 \cdots m_r$ in the rational polynomial

$$\chi(X, m_1 D_1 + \cdots + m_r D_r)$$

If $Y \subseteq X$ is a subvariety of $\dim(Y) = k$ we denote

$$(D_1 \cdot \ldots \cdot D_k \cdot Y) := (D_1|_Y \cdot \ldots D_k|_Y)$$

Remark I.1.8. Note that this definition works as we expect in the cases of curves, because by Riemann-Roch theorem for algebraic curves we have

$$\chi(C, mD) = m\deg(D) + \chi(C, \mathcal{O}_C) \quad \text{then} \quad D \cdot C = \deg(\mathcal{O}_C(D)),$$

and by I.1.6, this definition generalizes the intersection product on surfaces. Furthermore, [Kol96, Theorem VI.2.8] shows that, indeed, this intersection number counts the points in $D_1 \cap \cdots \cap D_n$ with multiplicity.

Remark I.1.9. The independence of linear classes of divisors in the previous definition means that we can think of the intersection product only in terms of line bundles, because the expression $(\mathcal{O}_X(D_1) \cdot \ldots \cdot \mathcal{O}_X(D_n))$ makes sense. Note that, if $D_1, \ldots, D_k$ are divisors in X and Y is a k-dimensional subvariety:

$$(D_1 \cdot \ldots \cdot D_k \cdot Y) = (\mathcal{O}_X(D_1)|_Y \cdot \ldots \cdot \mathcal{O}_X(D_k)|_Y)$$

Example I.1.10. If D is a Cartier divisor on the n-dimensional projective variety X, the function $m \mapsto \chi(X, mD)$ is a polynomial $P(T) = \sum_{i=0}^{n} a_i T^i$ such that

$$\chi(X, m_1 D + \ldots + m_n D) = \sum_{i=0}^{n} a_i (m_1 + \ldots + m_n)^i$$

and then the coefficient of $m_1 \cdots m_n$ is $a_n n!$, hence

$$\chi(X, mD) = m^n \frac{(D^n)}{n!} + O(m^{n-1})$$

This definition is highly flexible, because it also works for $\mathbb{Q}$-divisors by linearity. Moreover, it possesses the desired properties.

Proposition I.1.11. *Let $D_1, \ldots, D_n$ be Cartier divisors on a projective variety X of* $\dim(X) = n$*.*

1. *The map*
$$(D_1, \ldots, D_n) \mapsto (D_1 \cdot \cdots \cdot D_n)$$
is $\mathbb{Z}$-multilinear, symmetric and takes integral values.

2. *If D_n is effective,*
$$(D_1 \cdot \cdots \cdot D_n) = (D_1|_{D_n} \cdot \cdots \cdot D_{n-1}|_{D_n})$$

A crucial result to carry out several calculations is the projection formula.

Definition I.1.12. Let $\varphi : X \to Y$ be a morphism between varieties and let $C \subset X$ be a curve. We define the 1-cycle (i.e., a formal linear combination of irreducible curves) as

$$\varphi_* C := \begin{cases} 0 & \text{if} \quad \dim \varphi(C) = 0 \\ \deg(C \to \varphi(C))\, \varphi(C) & \text{if} \quad \dim \varphi(C) = 1 \end{cases}$$

Theorem I.1.13 (projection formula, [Deb01, Proposition 1])**.** *Let $\varphi : X \to Y$ be a surjective morphism between projective varieties. Let $D_1, \ldots, D_r$ be Cartier divisors on Y such that $r \geq \dim(X)$. Then*

$$(\varphi^* D_1 \cdot \cdots \cdot \varphi^* D_r) = \deg(\varphi)(D_1 \cdot \cdots \cdot D_r)$$

In particular, if $C \subset X$ is a curve and $D \in \mathrm{Div}(Y)$ is a Cartier divisor, then:

$$(\varphi^* D \cdot C) = (D \cdot \varphi_* C)$$

The following example shows an explicit calculation using the previous properties.

Example I.1.14. Let X be a n-dimensional smooth projective variety, $\widetilde{X}$ the blow-up on a point and E the exceptional divisor of $\widetilde{X}$, i.e., $E = \mathbb{P}^{n-1}$. We will calculate (E^n). We have that

$$\omega_E = \omega_{\mathbb{P}^{n-1}} = \mathcal{O}_{\mathbb{P}^{n-1}}(-n)$$

Now, the fact that

$$\mathrm{Pic}(\tilde{X}) = \pi^{*}\,\mathrm{Pic}(X) \oplus \mathbb{Z}[\mathcal{O}_X(E)]$$

and a classical argument using adjunction formula gives us:

$$\begin{aligned}\omega_E = (\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E))|_E &= \omega_{\tilde{X}}|_E \otimes \mathcal{O}_{\tilde{X}}(E)|_E \\ &= \mathcal{O}_{\tilde{X}}((n-1)E)|_E \otimes \mathcal{O}_{\tilde{X}}(E)|_E \\ &= \mathcal{O}_{\tilde{X}}(nE)|_E\end{aligned}$$

and then, since $\mathrm{Pic}(\mathbb{P}^{n-1}) = \mathbb{Z}$ is torsion-free, we have

$$\mathcal{O}_E(E) := \mathcal{O}_{\tilde{X}}(E)|_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$$

By Proposition I.1.11 we have

$$(E^n) = (E|_E^{n-1}) = ((-H)^{n-1}) = (-1)^{n-2}(H^{n-1})$$

where H is an hyperplane in $\mathbb{P}^{n-1}$, and this can be calculated explicitly. We have

$$\chi(\mathbb{P}^{n-1}; mH) = \chi(\mathbb{P}^{n-1}; \mathcal{O}_{\mathbb{P}^{n-1}}(m)) = h^0(\mathbb{P}^{n-1}; \mathcal{O}_{\mathbb{P}^{n-1}}(m)) - h^{n-1}(\mathbb{P}^{n-1}; \mathcal{O}_{\mathbb{P}^{n-1}}(m))$$

and by the Example I.1.10, it follows that $(H^{n-1}) = 1$ and then $(E^n) = (-1)^{n-1}$.

Now, the main theorem of this section is presented. It corresponds to Nakai-Moishezon ampleness criterion, which gives a numerical characterization of the fundamental concept of an ample divisor.

Theorem I.1.15 (Nakai-Moishezon)**.** *A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a projective variety X is ample if and only if, for every prime divisor Y of X, of dimension r,*

$$((D|_Y)^r) = (D^r \cdot Y) > 0$$

This theorem also motivates the definition of another widely used class of divisors, these are the *nef divisors*.

Definition I.1.16 (nef divisor)**.** Let X be a projective variety. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X)$ is *nef* if for every subvariety $Y \subset X$ of dimension r satisfies

$$(D^r \cdot Y) \geq 0$$

We have the following well-known properties of nef divisors (see e.g. [Laz04, §1.4]).

Theorem I.1.17. *Let X be a projective variety. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is nef if it has nonnegative intersection with every curve on X.*

Lemma I.1.18. *Let X be a projective variety. If D, E are nef divisors on X and H is an ample divisor on X, then $D + H$ is ample and $D + E$ is nef.*

I.1.3 The cone of curves and the effective cone

The fundamental definition introduced by Mori pertains to the cone of curves, a vector space that captures much of the geometry of the variety by employing the intersection product studied in the previous section. In the following, we introduce the key concepts of Mori's theory. Remarkably, the Kleiman's criterion is presented, alongside the notion of an extremal ray and its characterization in the case of surfaces.

Definition I.1.19. Let X be a smooth projective variety over a field $\mathbf{k}$. Two Cartier divisors $D, D' \in \mathrm{Div}(X)$ are *numerically equivalent*, denoted $D \equiv_{\mathrm{num}} D'$, if

$$(D \cdot C) = (D' \cdot C)$$

for all irreducible curves $C \subset X$. The quotient of $\mathrm{Div}(X)$ by this equivalence relation is denoted $N^1(X)_{\mathbb{Z}}$, and we define

$$N^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes \mathbb{Q} \quad ; \quad N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

Similarly, we consider the set of formal linear combinations of irreducible curves on X modulo numerical equivalence, i.e., $C \equiv_{\mathrm{num}} C'$ are equivalent if:

$$(D \cdot C) = (D \cdot C') \quad \forall D \in \mathrm{Div}(X)$$

denoted $N_1(X)_{\mathbb{Z}}$ and define:

$$N_1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes \mathbb{Q} \quad ; \quad N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

Elements of $N_1(X)_{\mathbb{R}}$ are called 1-cycles.

Theorem I.1.20. *The real vector spaces $N^1(X)_{\mathbb{R}}, N_1(X)_{\mathbb{R}}$ are finite dimensional and the intersection pairing*

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$$

is non-degenerate. The dimension of these spaces, denoted by ρ_X, is called the Picard number of X.

Definition I.1.21 (cone of curves, effective/nef/ample cone)**.** The cone of curves $\mathrm{NE}(X)$ is the set of classes of effective 1-cycles in $N_1(X)_{\mathbb{R}}$. By duality we define the effective cone $\mathrm{Eff}(X)$ as the set of classes of effective divisors in $N^1(X)_{\mathbb{R}}$. The nef (resp. ample) cone is the convex cone of all nef (resp. ample) $\mathbb{R}$-divisor classes on X.

Remark I.1.22. It is worth noting that these cones are convex, but not necessarily closed (see e.g. [Laz04, Example 1.5.1]). For this reason, we consider their closures in the metric topology, denoted by $\overline{\mathrm{NE}}(X)$ and $\overline{\mathrm{Eff}}(X)$, which are referred to as the *closed cone of curves* and the *pseudoeffective cone*, respectively.

In terms of this new framework, we arrive at Kleiman's criterion for ampleness.

Theorem I.1.23 (Kleiman's criterion)**.** *Let X be a projective variety.*

1. *A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is ample if and only if $D \cdot z > 0$ for all nonzero z in $\overline{\mathrm{NE}}(X)$.*
2. *For any ample divisor H and any $k \in \mathbb{Z}$, the set $\{z \in \overline{\mathrm{NE}}(X) \mid H \cdot z \leq k\}$ is compact and therefore contains only finitely many classes of curves.*

A particularly useful and practical perspective on this theorem, in the context of the new terminology, is through the lens of the duality of cones.

Corollary I.1.24 ([Laz04, Proposition 1.4.23])**.** *Let X be a projective variety over a field. Then*

1. *The dual of the closed cone of curves $\overline{\mathrm{NE}(X)}$ on X is the nef cone $\mathrm{Nef}(X)$.*
2. *The interior of the nef cone is the ample cone, i.e., $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$, and the nef cone is the closure of the ample cone, i.e., $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$.*

In addition to the definition of a nef divisor, another significant numerical concept is that of a big divisor, inspired by the asymptotic version of the Riemann-Roch theorem.

Theorem I.1.25 ([Deb01, Proposition 1.31])**.** *Let D be a Cartier divisor on a projective variety X of $\dim(X) = n$.*

1. *For all i, it is verified that*
$$h^i(X, mD) = O\left(m^n\right).$$
2. *If D is nef, we have*
$$h^i(X, mD) = O\left(m^{n-1}\right)$$
for all $i > 0$, hence
$$h^0(X, mD) = m^n \frac{(D^n)}{n!} + O\left(m^{n-1}\right)$$

Definition I.1.26. Let X be a projective variety. A Cartier divisor is big if[1]

$$\limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^n} > 0$$

Remark I.1.27. Another usual definition of *big divisors* is by means of its associated line bundle. A Cartier divisor D on X is big if and only if $\mathcal{O}_X(D)$ has maximal Iitaka dimension.

The motivation behind the definition of a big divisor lies in the fact that the class of big and nef divisors exhibits particularly good behavior for numerous purposes. The following result serves as an illustrative example.

Proposition I.1.28. *Let D be a nef and big $\mathbb{Q}$-divisor on a projective variety X. There exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor E on X such that $D - tE$ is ample for all rationals t in $(0, 1]$.*

Below, we introduce the concept of *extremal ray*, one of the central objects in Mori's work. Its profound significance will become evident later in this chapter.

Definition I.1.29 (extremal ray)**.** Let V be a cone in $\mathbb{R}^m$. A subcone $W \subset V$ is called extremal if it is closed and convex and if any two elements of V whose sum is in W are both in W. An extremal ray of V is an extremal subcone of dimension 1.

In the case of smooth surfaces we have good characterizations of extremal rays.

Proposition I.1.30. *Let X be a smooth projective surface.*

1. *The class of an irreducible curve C with $(C^2) \leq 0$ is in $\partial\overline{\mathrm{NE}}(X)$.*

2. *The class of an irreducible curve C with $(C^2) < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$.*

3. *If the class of an irreducible curve C with $(C^2) = 0$ and $(K_X \cdot C) < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$, the surface X is ruled over a smooth curve, C is a fiber and X has Picard number 2.*

4. *If r spans an extremal ray of $\overline{\mathrm{NE}}(X)$, either $r^2 \leq 0$ or X has Picard number 1.*

[1] This notion will be revisited in Section I.5, where the number involved in this definition will be called the volume.

5. *If r spans an extremal ray of $\overline{\mathrm{NE}}(X)$ and $r^2 < 0$, the extremal ray is spanned by the class of an irreducible curve.*

We end the section analyzing some explicit examples.

Example I.1.31.

1. We will describe the cone of curves $\overline{\mathrm{NE}}(X)$ of the blow-up $X = \mathrm{Bl}_p(\mathbb{P}^n)$ of projective space at a point $p \in \mathbb{P}^n$. We know that $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}[H]$ is generated by the class of a hyperplane, and by duality we have that $N_1(X)_{\mathbb{R}} = \mathbb{R}[\ell]$ is generated by the class of a line. In particular, the cone of curves is simply $\overline{\mathrm{NE}}(\mathbb{P}^n) = \mathbb{R}^+[\ell]$. If $\pi : X \to \mathbb{P}^n$ denotes the blow-up and E denotes the exceptional divisor, a local computation shows that $D := \pi^* H - E$ is basepoint-free (in particular is nef). Since $\rho_X = 2$, we know $N^1(X)_{\mathbb{R}} = \mathbb{R}^2$, and we can calculate the cone of curves in terms of a pair of non-numerically equivalent classes. Consider $\ell_p \subset \mathbb{P}^n$ a line through p, and $e \subset E = \mathbb{P}^{n-1}$ a line inside the exceptional divisor. Since $[\ell_p], [e] \in N_1(X)_{\mathbb{R}}$ are two different effective classes and $\pi^* H - E$ is nef, the computation
$$(x\ell_p + ye) \cdot (\pi^* H - E) = x + y$$
shows that $\overline{\mathrm{NE}}(X)$ is contained in the cone spanned by the classes $[\ell_p], [e]$. The converse inclusion is obvious since ℓ_p and e are effective curves. We conclude
$$\overline{\mathrm{NE}}(X) = \mathbb{R}^+[\ell_p] + \mathbb{R}^+[e].$$

2. Consider an abelian surface X. The homogeneity of X implies that every curve has non-negative self-intersection, because for a given curve $C \subset X$, we can move it by an element $g \in X$, so $C^2 = C \cdot (g + C) \geq 0$. If H is an ample divisor on X, the cone of curves is given by
$$\overline{\mathrm{NE}}(X) = \{z \in N_1(X)_{\mathbb{R}} : z^2 \geq 0, H \cdot z \geq 0\}.$$
Indeed, inclusion of RHS in $\overline{\mathrm{NE}}(X)$ is clear, and the other inclusion is by 4. in Proposition I.1.30. Note for example that if $N_1(X)_{\mathbb{R}}$ is of dimension 3, $\overline{\mathrm{NE}}(X)$ is not finitely generated.

3. Let C be a smooth curve of genus $g(C)$ and $X := C \times C$ with projections $p_1, p_2 : X \to C$. We denote by f_1, f_2, Δ the numerical classes in $N_1(X)_{\mathbb{R}}$ of $\{pt\}$

$C; fptg \quad C$ and diagonal respectively. The relations

$$(\Delta \quad f_1) = (\Delta \quad f_2) = (f_1 \quad f_2) = 1$$
$$(f_1)^2 = (f_2)^2 = 0$$

are clear. We will calculate $(\Delta)^2$. It is known that

$$K_X \ {}_{\text{lin}}\ p_1 K_C + p_2 K_C$$

By Riemann-Roch theorem for curves

$$\deg(K_C) = 2g(C) \quad 2$$

and since all vertical (and horizontal) fibers of X is numerically equivalents we have

$$p_1 K_C \ {}_{\text{num}}\ (2g(C) \quad 2) f_1; \qquad p_2 K_C \ {}_{\text{num}}\ (2g(C) \quad 2) f_2$$

Furthermore, $\Delta = C$ and by Riemann-Roch theorem for surfaces

$$g(C) = 1 + \frac{1}{2}(\Delta^2 + (K_X \quad C))$$

and then

$$2g(C) \quad 2 = \Delta^2 + (p_1 K_C \quad \Delta) + (p_2 K_C \quad \Delta) = (\Delta)^2 + (2g(C) \quad 2)((f_1 + f_2) \quad \Delta)$$

resulting $\Delta^2 = 2 \quad 2g(C)$.

Let us now assume that C is an elliptic curve, i.e., $g(C) = 1$, and then X is an abelian surface. In this case $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)$, i.e., any effective curve in X is nef. This is because

$$\overline{\mathrm{NE}}(X) = \left\{ z \ 2 \ N_1(X)_{\mathbf{R}} \ j \ z^2 \quad 0; H \quad z \quad 0 \right\}$$

for an ample divisor H on X, and in an abelian variety for any curve $C^0 \quad X$ we can consider $g^0 \ 2 \ X$ such that $(C^0)^2 = (C^0 \quad gC^0) \quad 0$. If we write

$$= x f_1 + y f_2 + z\Delta \tag{I.1}$$

then $\quad 2$ Nef(X) if and only if

$$xy + xz + yz \quad 0$$
$$x + y + z \quad 0$$

When $\quad (X) = 3$ these equations defines exactly the nef and effective cones.

I.2 "Bend-and-break lemmas" and The cone theorem

I.2.1 Parametrization of morphisms

In this section, we present basic results related to Hilbert schemes that parametrize morphisms. Many of these results make use of techniques from Deformation Theory in their proofs. Throughout this section, we will work with schemes and primarily follow [Deb01, §2].

In geometry, many objects vary algebraically in terms of parameters that define families with certain common characteristics. A basic example is the case of conics in $\mathbb{P}^2$, which are parametrized by points in $\mathbb{P}^5$. This is an example of a moduli problem, which, roughly speaking, involves classifying geometric objects (eventually modulo a certain equivalence relation).

We start the section with a definition that arises from category theory.

Definition I.2.1 (representable functor)**.** Let $\mathcal{C}$ be a locally small category and Set the category of sets. A contravariant functor $F : \mathcal{C}^{opp} \to$ Set is said to be *representable* if it exists an object $A \in \mathcal{C}$ such that F is naturally isomorphic to the functor of points $\mathrm{Hom}_{\mathcal{C}}(-, A)$.

We consider projective S-schemes X and Y, with S locally noetherian, and the goal of this section is to construct a scheme $\mathrm{Mor}_S(Y, X)$ that, in a certain sense, parametrizes S-morphisms $Y \to X$. The idea is to define the functorial properties that this scheme should satisfy and to formulate the problem of its existence in terms of the representability of a functor.

The property we desire for the scheme $\mathrm{Mor}_S(Y, X)$ is the following: we require the existence of a universal morphism

$$f^{\mathrm{univ}} : Y \times \mathrm{Mor}_S(Y, X) \to X$$

such that for any S-scheme T, the correspondence between

morphisms $\varphi : T \to \mathrm{Mor}_S(Y, X)$ and

morphisms $f : Y \times T \to X$

obtained by sending φ to

$$f(y,t) = f^{\text{univ}}(y,\varphi(t))$$

is one-to-one.

All these properties translate into the fact that, given an S-scheme T, a T-point of $\mathrm{Mor}_S(Y,X)$ is the same as a morphism $Y \times_S T \to X \times_S T$ (by using the universal property of the fibered product). This motivates the following definition.

Definition I.2.2. Let Sch/S the category of schemes over a scheme S, and let X and Y be schemes over S. Define $\underline{\mathrm{Mor}}_S(Y,X)$ as the functor

$$\underline{\mathrm{Mor}}_S(Y,X) : (\mathrm{Sch}/S)^{opp} \to \mathrm{Set}, \quad T \mapsto \{T\text{-morphisms } Y \times_S T \to X \times_S T\}$$

The goal is then to represent the functor $\mathrm{Mor}_S(Y,X)$ by a scheme. In order to prove this, we introduce Hilbert's functor and scheme.

Definition I.2.3. Let X be a projective scheme over a scheme S. The *Hilbert functor* defined on the category $(\mathrm{Sch}/S)^{opp}$ of schemes over S is defined as follows

$$Hilb_{X/S} : (\mathrm{Sch}/S)^{opp} \to \mathrm{Set}, \quad T \mapsto \{Z \subset X_T := X \times_S T \big| Z \to T \text{ is flat}\}$$

Furthermore, if $P \in \mathbb{Q}[z]$ is a polynomial and we fix an ample Cartier divisor H on X, we define the functor

$$Hilb^P_{X/S} : (\mathrm{Sch}/S)^{opp} \to \mathrm{Set}$$
$$T \mapsto \{Z \subset X_T := X \times_S T \big| Z \to T \text{ is flat}, \ \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(nH)) = P(n) \ \forall n \in \mathbb{Z}\}$$

associating the set of all flat families of subschemes of X with Hilbert polynomial P.

Hilbert's functor then associates to each scheme T the set of flat families of subschemes of X parameterized by T, and then the problem of parameterizing subschemes translates into the representability of the Hilbert functor. This is a classic result by Grothendieck.

Theorem I.2.4 (Grothendieck). *Let X be a projective scheme over S and let $P \in \mathbb{Q}[z]$ be a polynomial. Then $Hilb^P_{X/S}$ is representable. We denote by $\mathrm{Hilb}^P_{X/S}$ the scheme representing this functor.*

Definition I.2.5. Let X be a projective scheme over S. We define the *Hilbert scheme* of X as

$$\mathrm{Hilb}_{X/S} := \bigsqcup_{P \in \mathbb{Q}[z]} \mathrm{Hilb}^P_{X/S}$$

Now, the following result (see [Kol96, Theorem 1.10]) allows us to assert the representability of the functor $\underline{\mathrm{Mor}}_S(Y, X)$.

Theorem I.2.6. *Let X and Y be projective schemes over S. Assume that X is flat over S. Then $\underline{\mathrm{Mor}}_S(Y, X)$ is represented by an open subscheme*[2]

$$\mathrm{Mor}_S(Y, X) \subset \mathrm{Hilb}_{(X \times_S Y/S)}$$

As the notation suggests, the scheme given by the previous theorem satisfy the desired property. The fact that $\mathrm{Mor}_S(Y, X)$ represents the functor $\underline{\mathrm{Mor}}_S(Y, X)$ means that there exists a natural isomorphism

$$\underline{\mathrm{Mor}}_S(Y, X) = Hom(-, \mathrm{Mor}_S(Y, X))$$

and by Yoneda Lemma this the same as a morphism $u \in \underline{\mathrm{Mor}}_S(Y, X)(\mathrm{Mor}_S(Y, X))$, satisfying the following universal property

$$\forall T/S, \forall v \in \underline{\mathrm{Mor}}_S(Y, X)(T), \exists! f : T \to \mathrm{Mor}_S(Y, X), F(f)(u) = v.$$

Identifying the S-morphisms $Y \to X$ with morphisms $Y \times_S T \to X \times_S T$, we see that u plays the role of the required universal morphism.

From now on, we fix a field k and we adopt the notation $\mathrm{Mor}(Y, X)$ for $\mathrm{Mor}_k(Y, X)$. An incredible result is that we can understand the tangent space of $\mathrm{Mor}(Y, X)$ using only its functorial properties. The next proposition is proven in [Deb01, Proposition 2.4].

Proposition I.2.7. *Let X and Y be varieties over k, with X quasi-projective and Y projective, let $f : Y \to X$ be a k-morphism, and let $[f]$ be the corresponding $\mathbf{k}$-point of $\mathrm{Mor}(Y, X)$. One has*

$$T_{\mathrm{Mor}(Y,X),[f]} = H^0(Y, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y))$$

Proof. Suppose $Y = \mathrm{Spec}(B), X = \mathrm{Spec}(A)$ are affine schemes (i.e., A, B are k-algebras). The morphism $f : Y \to X$ comes from a ring morphism $f^\# : A \to B$ and then B is an A-algebra. Remember that a tangent vector on $T_{\mathrm{Mor}(Y,X),[f]}$ is the same as a morphism $\mathrm{Spec}(k[\varepsilon]/\langle\varepsilon^2\rangle) \to \mathrm{Mor}(Y, X)$ with image $[f]$. By the universal property of $\mathrm{Mor}(Y, X)$ this is the same as a morphism:

$$f_\varepsilon : Y \times_k \mathrm{Spec}(k[\varepsilon]/\langle\varepsilon^2\rangle) = \mathrm{Spec}(B \otimes_k k[\varepsilon]/\langle\varepsilon^2\rangle) \to \mathrm{Spec}(A)$$

[2] This inclusion is simply to note that a morphism can be identified with its image.

and again this is the same as a ring morphism $f_\varepsilon^\# : A \to B[\varepsilon]/\langle\varepsilon^2\rangle$. Moreover, this morphism $f_\varepsilon^\#$ must extend $f^\#$, and then it must be of the form

$$f_\varepsilon^\#(a) = f(a) + \varepsilon g(a)$$

for a certain ring homomorphism $g : A \to B$. Note that the condition $f_\varepsilon^\#(aa') = f_\varepsilon^\#(a) f_\varepsilon^\#(a')$ gives that

$$g(aa') = f^\#(a) g(a') + f^\#(a') g(a) \quad \forall a, a' \in A$$

i.e., g is a k derivation of the A-module B. Thus, the tangent space of $\mathrm{Mor}(Y, X)$ identifies with $Hom_A(\Omega_A, B)$ where Ω_A is the Kähler differentials module of A (section II.8 in [Har77]). By adjunction property of scalar extension we have

$$\mathrm{Hom}_A(\Omega_A, B) = \mathrm{Hom}_B(\Omega_A \otimes_A B, B)$$

and recalling that the coherent sheaf associated to Ω_A is Ω_X we have

$$\begin{aligned} T_{\mathrm{Mor}(Y,X),[f]} = \mathrm{Hom}_B(\Omega_A \otimes_A B, B) &= H^0(Y, \widetilde{Hom_B(\Omega_A \otimes_A B, B)}) \\ &= H^0(Y, \mathrm{Hom}(\widetilde{\Omega_A \otimes_A B}, \widetilde{B})) \\ &= H^0(Y, \mathrm{Hom}(f^*\Omega_X, \mathcal{O}_Y)) \end{aligned}$$

In the general case we cover X by affine open sets $U_i = \mathrm{Spec}(A_i)$ and Y by $V_i = \mathrm{Spec}(B_i)$ such that $f(V_i) \subset U_i$. The previous arguments give that on each open set extensions of $f|_{V_i} : V_i \to U_i$ are parameterized by

$$\mathrm{Hom}_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i) = H^0(V_i, \mathrm{Hom}(f^*\Omega_X, \mathcal{O}_Y))$$

and gluing these extensions on a global extension of f is the same that asking

$$g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j} \quad \forall i, j$$

which is exactly the gluing condition for sections of a sheaf. □

Remark I.2.8. Note that when X is smooth along $f(Y)$ we have

$$T_{\mathrm{Mor}(Y,X),[f]} = H^0(Y, f^* T_X)$$

The following theorem gives a lower bound for the dimension of the space of morphisms, and the proof uses Deformation Theory techniques.

Theorem I.2.9. *Let X and Y be projective varieties over a field* $\mathbf{k}$ *and let $f: Y \to X$ be a* $\mathbf{k}$*-morphism such that X is smooth along $f(Y)$. Locally around $[f]$, the scheme* $\mathrm{Mor}(Y, X)$ *can be defined by $h^1(Y, f^*T_X)$ equations in a smooth scheme of dimension $h^0(Y, f^*T_X)$. In particular, any (geometric) irreducible component of* $\mathrm{Mor}(Y, X)$ *through $[f]$ has dimension at least*

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

The bend-and-break lemmas, fundamental results by Mori that will be presented in the next section, are based on the assumption that the subscheme of $\mathrm{Mor}(Y, X)$ fixing a finite set of points has at least dimension 1. To understand this we have the following proposition.

Proposition I.2.10. *Let X, Y be varieties with Y projective and X quasi-projective, and $\{y_1, \dots, y_r\} \subset Y$, $\{x_1, \dots, x_r\} \subset X$ finite sets of points. Then if $f(Y)$ is smooth on X, the tangent space of* $\mathrm{Mor}(Y, X; y_i \mapsto x_i)$ *(subscheme of morphisms mapping $y_i \mapsto x_i$) in the point $[f] \in$* $\mathrm{Mor}(Y, X; y_i \mapsto x_i)$ *is*

$$T_{\mathrm{Mor}(Y,X;y_i \mapsto x_i),[f]} = H^0(Y, f^*T_X \otimes I_{y_1,\dots,y_r})$$

In particular, the dimension of its irreducible components through $[f]$ is at least

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X) - r\dim(X)$$

Remark I.2.11. In Proposition I.2.10, if $Y = C$ is a curve, Riemann-Roch theorem gives the bound

$$\begin{aligned} \dim_{[f]} \mathrm{Mor}\,(C, X; c_i \mapsto f(c_i)) &\geq \chi(C, f^*T_X) - r\dim(X) \\ &= -K_X \cdot f_*C + (1 - g(C) - r)\dim(X) \end{aligned}$$

I.2.2 "Bend-and-break" lemmas

The main ingredient to prove The cone theorem is a version of its famous "bend-and-break" lemmas. We present two different versions and some geometric consequences. We present only sketches of the proofs.

Theorem I.2.12 ([Deb01, Proposition 3.5])**.** *Let X be a projective variety and let H be an ample Cartier divisor on X. Take a smooth curve $f: C \to X$ on X and B a finite non-empty subset of C such that*

$$\dim_{[f]} \mathrm{Mor}(C, X; B \mapsto f(B)) \geq 1.$$

There exists a rational curve Γ *on* X *which meets* $f(B)$ *and such that*

$$(H \cdot \Gamma) \le \frac{2\,(H \cdot f_* C)}{\mathrm{Card}(B)}$$

Sketch of the proof. We list below the steps of the proof.

1. Let $b = \mathrm{Card}(B)$ and C' the normalization of C. If C' is rational and f has degree $\le b/2$ take $\Gamma = C'$.

2. Note that the dimension of the space of morphisms from C to $f(C)$ sending B to $f(B)$ is 0 in the case that C' is irrational or C' rational and degree of f is $< b/2$.

3. Take T a 1-dimensional subvariety of $\dim_{[f]} \mathrm{Mor}(C, X; B \mapsto f(B))$ through $[f]$. By previous step the images are not all the same. Take $\overline{T}$ a smooth compactification of T and solve the indeterminacies of the evaluation map $e : S \to C \times \overline{T} \overset{ev}{\dashrightarrow} X$, and note that its image is a surface.

4. For every $i = 1, \ldots, b$, consider the inverse images on S of the (-1)-curves that appear blowing-up points on the strict transform of $\{b_i\} \times \overline{T}$. Denote this curves by $E_{i,1}, \ldots, E_{i,n_i}$. Note that

$$(E_{i,j} \cdot E_{i',j'}) = -\delta_{i,j\;i',j'}.$$

5. Prove that

$$((e^* H)^2) \le \sum_{i,j} a_{i,j} \left(\frac{2d}{b} - a_{i,j} \right)$$

   where

$$a = \left(e^* H \cdot \varepsilon \bar{T}\right) \ge 0 \quad ; \quad a_{i,j} = (e^* H \cdot E_{i,j}) \ge 0 \quad ; \quad ba = \sum_{i,j} a_{i,j}$$

   and conclude that exists i_0, j_0 such that $0 < a_{i_0,j_0} < \frac{2d}{b}$. Conclude that exists a rational component of $e_* E_{i_0,j_0}$ which passes through $f(c_{i_0})$. This curve is the desired Γ.

$\square$

Theorem I.2.13 (Miyaoka-Mori)**.** *Let X be a projective variety, let H be an ample divisor on X, and let $f : C \to X$ be a smooth curve such that X is smooth along $f(C)$ and $(K_X \cdot f_*C) < 0$. Given any point x on $f(C)$, there exists a rational curve Γ on X through x with*

$$(H \cdot \Gamma) \leq 2\dim(X)\frac{(H \cdot f_*C)}{(-K_X \cdot f_*C)}$$

If X is smooth, it is also verified that $(-K_X \cdot \Gamma) \leq \dim(X) + 1$.

Sketch of the proof. The main idea of the proof is suppose in first place $\mathbf{k}$ has characteristic $p > 0$ and use Frobenius morphism to raise the degree of f. Compose f with m Frobenius morphism to obtain $f_m : C_m \to X$ with degree $p^m \deg(f)$ and consider $B_m \subset C_m$ with b_m points. The bound of Observation I.2.11 gives

$$\dim_{[f_m]} \mathrm{Mor}\,(C_m; X; B_m \mapsto f_m(B_m)) \geq p^m(-K_X \cdot f_*C) + (1 - g(C) - b_m)\dim(X);$$

Take m sufficientely large and b_m such that the previous bound is positive. Use Theorem I.2.12 to produce a rational curve through some point of $f_m(b_m)$. Conclude the proof in positive characteristic.

The proof in characteristic 0 is done considering a finitely generated domain where $X; C; f; H$ are defined and taking a quotient that is a finite field. The conclusion follows applying the previous result. $\square$

The previous theorem allows us to derived an interesting characterization of varieties whose anti-canonical divisor is nef.

Theorem I.2.14 ([Deb01, Theorem 3.10])**.** *If X is a smooth projective variety with $-K_X$ nef,*

- *either K_X is numerically trivial,*
- *or there is a rational curve through any point of X.*

Proof. Consider an embedding $X \hookrightarrow \mathbb{P}^N_{\mathbf{k}}$ given by an hyperplane section H on X, i.e., a very ample divisor, and suppose $(K_X \cdot H^{n-1}) = 0$. Let $C \subset X$ be a curve and take hypersurfaces $H_1, \dots, H_{n-1} \in \mathbb{P}^N_{\mathbf{k}}$ of degrees $d_1, \dots, d_{n-1}$ such that $Z :=$ $X \cap H_1 \cap \cdots \cap H_{n-1}$ contains C and has pure dimension 1 (this is always possible by making succesive cuts). Now, every hypersurface in $\mathbb{P}^N_{\mathbf{k}}$ is linearly equivalent to a multiple of H, more especifically, $H_i \sim d_i H$ for every $i = 1, \dots, n-1$, and then

$$0 \leq (-K_X \cdot C) \leq (-K_X \cdot Z) = d_1 \cdots d_{n-1}\left(-K_X \cdot H^{n-1}\right) = 0$$

so K_X is numerically trivial. Since $-K_X$ is nef, we assume now $(K_X \cdot H^{n-1}) < 0$. Take a point $x \in X$. By Bertini theorem we can take hyperplane sections of X by $n-1$ hyperplanes such that their intersection is an irreducible curve containing x. Let C be the normalization of this curve. Then $(K_X \cdot C) = (K_X \cdot H^{n-1}) < 0$ and by Theorem I.2.13 there is a rational curve on X which passes through x. □

Example I.2.15. An abelian variety A has trivial canonical divisor, and by the above Theorem doesn't contain rational curves. Alternatively, by the universal property of the Albanese map, we have that every morphism $\mathbb{P}^1 \to A$ is necessarily constant (see [Bea83, Theorem V.13]).

I.2.3 The cone theorem

Using the bend-and break techniques presented in the previous section we will be able to prove the cone theorem. For a subset $S \subset N_1(X)_{\mathbb{R}}$ and a Cartier divisor D on X, we use the notation

$$S_{D \geq 0} := \{z \in S : D \cdot z \geq 0\}$$

and similarly for $S_{D \leq 0}, S_{D>0}$ and $S_{D<0}$.

Theorem I.2.16 (Mori's Cone Theorem, [Deb01, Kol96])**.** *Let X be a smooth projective variety. There exists a countable family $(\Gamma_i)_{i \in I}$ of rational curves on X such that*

$$0 < (-K_X \cdot \Gamma_i) \leq \dim(X) + 1$$

and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i];$$

where the $\mathbb{R}^+[\Gamma_i]$ are all the extremal rays of $\overline{\mathrm{NE}}(X)$ that meet $N_1(X)_{K_X<0}$; these rays are locally discrete in that half-space, i.e., these rays can only accumulate in the hyperplane $K_X = 0$.

Proof. First, we know that the set of classes of irreducible curves $[C]$ of X is discrete, and in particular numerical classes of rational curves satisfying $0 < (-K_X \cdot z_i) \leq \dim(X)+1$ are countable, and we pick a representative Γ_i for each class z_i of this form.

We define

$$V := \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i]$$

Now the proof is divided in 3 steps. First, we prove that negative extremal rays are locally discrete, secondly, we prove $\overline{\mathrm{NE}}(X) = \overline{V}$ and finally we prove that V is closed.

Step 1: the rays $\mathbb{R}^+ z_i$ *are locally discrete in* $N_1(X)_{K_X<0}$. Let H be an ample divisor on X. By a limit argument we note that:

$$N_1(X)_{K_X<0} = \bigcup_{\epsilon>0} N_1(X)_{K_X+\epsilon H<0}$$

and then is sufficient to prove that there are only finite classes z_i in $N_1(X)_{K_X+\epsilon H<0}$ for each $\epsilon > 0$ (because we can isolate each class) sufficiently separated from $\{K_X = 0\}$. We note that if $((K_X + \epsilon H) \cdot \Gamma_i) < 0$ then

$$(H \cdot \Gamma_i) < \frac{1}{\epsilon}(-K_X \cdot \Gamma_i) \leq \frac{1}{\epsilon}(\dim(X) + 1)$$

and Kleiman's criterion allows us to conclude.

Step 2: $\overline{\mathrm{NE}}(X)$ *is equal to the closure of* V. Suppose by contradiction that $\overline{\mathrm{NE}}(X) \neq \overline{V}$, in which case $\overline{V}$ is a proper closed subcone of $\overline{\mathrm{NE}}(X)$. By definition the effective cone $\overline{\mathrm{NE}}(X)$ contains no lines, and this implies that exists a linear form in $N_1(X)_{\mathbb{R}}$ which is positive on $\overline{V} \setminus \{0\}$ and vanishes on some extremal ray of $\overline{\mathrm{NE}}(X)$. As the intersection product is a perfect pairing we can identify the dual of $N_1(X)_{\mathbb{R}}$ with $N^1(X)_{\mathbb{R}}$ and this translates into the existence of an $\mathbb{R}$-divisor M such that is nonnegative on $\overline{\mathrm{NE}}(X)$, positive on $\overline{V} \setminus \{0\}$ and $(M \cdot z) = 0$ for some nonzero $z \in \overline{\mathrm{NE}}(X)$. By construction $\overline{V}$ contains all K_X- positive classes, and then $K_X \cdot z < 0$. It is important to note M is a nef divisor.

Now choose a norm on $N_1(X)_{\mathbb{R}}$ such that $\|[C]\| \geq 1$ for each irreducible curve $C \subseteq X$. Note that this is possible because numerical classes of curves are a discrete set. In addition, we can suppose $(M \cdot v) \geq \|v\|$ for all $v \in \overline{V}$ because we can replace M by a positive multiple without change his properties. For the $z \in \overline{\mathrm{NE}}(X)$ taken above we see that

$$2\dim(X)(M \cdot z) = 0 < (-K_X \cdot z)$$

As M is nef, we can approximate it by ample $\mathbb{Q}$-divisor, and since $z \in \overline{\mathrm{NE}}(X)$ we can approximate by effective 1-cycles. these facts allow us to consider H ample $\mathbb{Q}-$ divisor and Z effective 1-cycle verifying

$$2\dim(X)(H \cdot Z) < (-K_X \cdot Z)$$

and such that H preserves the norm condition

$$(H \cdot v) \geq \|v\| \quad \forall v \in \overline{V}$$

A short argument allows us to assume that every component C of Z verify $(-K_X \cdot C) > 0$, because if $(-K_X \cdot C) \leq 0$, the number $(H \cdot Z)$ goes down by removing C (H is ample) and $(-K_X \cdot Z)$ increases.

By construction every rational curve Γ such that $(-K_X;\Gamma) \leq \dim(X)+1$ is in $\overline{V}$, so for each curve of this type we have $(H \cdot \Gamma) \geq \|\Gamma\| \geq 1$. Then by Mori's bend-and-break lemma, since X is smooth, for every component C of Z exists a rational curve Γ such that

$$2\dim(X)\frac{(H \cdot C)}{(-K_X \cdot C)} \geq (H \cdot \Gamma)$$

and even more, we can assume $(-K_X;\Gamma) \leq \dim(X)+1$. This implies $[\Gamma] \in \overline{V}$, and then

$$2\dim(X)(H \cdot C) \geq (-K_X \cdot C)$$

Adding the inequality above over every component of Z we reach a contradiction with previous inequality.

Step 3: for any set of indices J the cone

$$\overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{j \in J} \mathbb{R}^+ z_j$$

is closed. Let V_J be this cone. By definition V_J contains no lines, and then $\overline{V_J}$ is equal to the convex hull of its extremal rays. Then, to prove $V_J = \overline{V_J}$ it is sufficient to prove that any extremal ray $\mathbb{R}^+ r \in \overline{V_J}$ satisfying $(K_X \cdot e)$ is in V_J.

To prove this we use the first step. Take H an ample divisor on X and $\varepsilon > 0$ such that $((K_X + \varepsilon H) \cdot r) < 0$. The first step gives us that there are only finitely many classes $z_{j_1}, \dots, z_{j_n}$ such that $((K_X + \varepsilon H) \cdot z_j) < 0$. Sinces $r \in \overline{V_J}$ we can write it as the limit of a sequence $(r_m + s_m)_{m \in \mathbb{N}}$ with $r_m \in \overline{\mathrm{NE}}(X)_{K_X + \varepsilon H \geq 0}$ and

$$s_m = \sum \lambda_{j,m} z_j$$

By construction we have

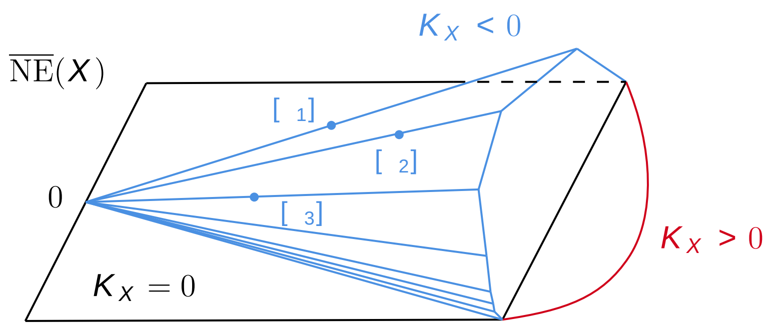
$$(H \cdot r_m) > 0; \qquad (H \cdot z_j) > 0; \qquad \lambda_{i,m} > 0 \quad \forall i,m$$

This implies that the sequences $(H \cdot r_m), (\lambda_{i,m})$ are bounded because we have $(H \cdot r_m + s_m) \to (H \cdot r)$. Now, Kleiman criterion gives that the set $\{z \in \overline{\mathrm{NE}}(X) : (H \cdot z) \leq k\}$ is compact, and therefore taking subsequences we may assume $(r_m)_{m \in \mathbb{N}}$ and $(\lambda_{i,m})_{m \in \mathbb{N}}$ are convergent. Write the limit of $(r_m + s_m)_{m \in \mathbb{N}}$ as $\overline{r} + \overline{z}$. Because r is an extremal ray and $r = \overline{r} + \overline{z}$, the classes $\overline{r}, \overline{z}$ are positive multiples of r. However, we know that $(K_X + \varepsilon H \cdot r) < 0$, but $(r_m)_{m \in \mathbb{N}} \subset \overline{\mathrm{NE}}(X)_{K_X + \varepsilon H \geq 0}$ and then

$$(K_X + \varepsilon H \cdot \overline{r}) \geq 0 \quad \text{and hence} \quad \overline{r} = 0$$

Thus, we conclude that r is a multiple of one of the z_j, and therefore is in V_J.

Clearly, steps 2 and 3 finish the proof. $\square$



A representation of the cone of curves.

We finish this section with the *supporting divisor* corollary.

Corollary I.2.17. *Let X be a smooth projective variety and let R be a K_X-negative extremal ray. There exists a nef divisor M_R on X such that*

$$R = \left\{z \in \overline{\mathrm{NE}}(X) \mid M_R \cdot z = 0\right\}.$$

For any such divisor, $mM_R - K_X$ is ample for all $m \gg 0$. M_R is called a supporting divisor for R.

I.3 Contractions of extremal rays

In the classification of surfaces, Castelnuovo's contractibility criterion is a crucial statement because it says that the key question to construct minimal surfaces is: **is there**

a (-1)-curve on S?. In this sense, the first step to generalize this process to higher dimension is to replace this question for a new one. Then, Mori formulated the question

"Is K_S nef?"

With this perspective, the cone theorem states:

"If it is not, then there exists a K_S-negative extremal ray".

And then, Mori asserts:

"If a K_S-negative extremal ray exists, we can contract it".

This framework forms the foundation of the *Minimal Model Program* (MMP). In this section we adress the existence of extremal contractions and we classify them into three distinct types.

I.3.1 Existence of extremal contractions

The fundamental result that makes it possible to carry out the Minimal Model Program is the existence of K_X negative extremal rays contractions. By a contraction of an extremal ray R of a variety X (i.e., R is an extremal ray of $\overline{\mathrm{NE}}(X)$) we refer to a surjective morphism $c_R : X \to Y$ such that

1. if $\Gamma \subset X$ is an irreducible curve, $\dim(c_R(\Gamma)) = 0 \iff [\Gamma] \in R$.
2. $c_{R*} \mathcal{O}_X = \mathcal{O}_Y$.

To prove the existence of extremal contractions we will need the following fundamental result (see [KMM87, Theorem 3-1-1]).

Theorem I.3.1 (Kawamata)**.** *Let X be a smooth complex projective variety and let D be a nef divisor on X such that $aD - K_X$ is big and nef for some $a \in \mathbb{Q}^{>0}$. The divisor mD is generated by its global sections for all $m \gg 0$.*

The next theorem is the fundamental result about the existence of contractions.

Theorem I.3.2 ([Deb01, Theorem 7.39], [KM98, Corollary 3.17])**.** *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray.*

1. *The contraction $c_R : X \to Y$ of R exists, where Y is a normal projective variety. It is given by the Stein factorization of the morphism defined by any sufficiently high multiple of any supporting divisor of R. This morphism is unique up to isomorphism.*

2. *Let C be any integral curve on X with class in R. There is an exact sequence*

$$
\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Pic}(Y) & \xrightarrow{c_R^*} & \operatorname{Pic}(X) & \longrightarrow & \mathbb{Z} \\
 & & & [D] & \longmapsto & (C \cdot D)
\end{array}
$$

   and $\rho_Y = \rho_X - 1$.

Proof.

1. Let M_R be a supporting divisor for R. The hypothesis of Kawamata theorem are fullfilled so mM_R is generated by global sections for all $m \gg 0$. Thus, we obtain a regular morphism $\varphi : X \to \mathbb{P}^N$ which has a Stein factorization

$$X \xrightarrow{c_R} Y \xrightarrow{\varphi_R} \mathbb{P}^N$$

   and the one we search is c_R, which has connected fibers.

   Let's see that c_R contracts R. The morphism φ_R is finite and then the divisor $D_m := \varphi_R^* \mathcal{O}_{\mathbb{P}^n}(1)$ is ample on Y. On the one hand, projection formula shows

$$(c_R^* D_m \cdot \Gamma) = (D_m \cdot (c_R)_*(\Gamma)) = 0 \quad \iff \quad \dim((c_R)_*(\Gamma)) = 0$$

   and on the other hand

$$(c_R^* D_m \cdot \Gamma) = (c_R^* \varphi_R^* \mathcal{O}_{\mathbb{P}^n}(1) \cdot \Gamma) = (mM_R \cdot \Gamma) = 0 \quad \iff \quad [\Gamma] \in R$$

   This shows that c_R is the desired morphism.

2. We prove first injectivity of c_R^*. Since $c_{R*}\mathcal{O}_X = \mathcal{O}_Y$, if $L \in \operatorname{Pic}(Y)$ by projection formula

$$(c_R)_*(c_R^* L) = (c_R)_*(c_R^* L \otimes \mathcal{O}_Y) = L \otimes (c_R)_* \mathcal{O}_X = L$$

   and c_R^* is injective, because we can recover L aplying $(c_R)_*$. Let $C \subset X$ be an irreducible curve with class in R, and $D \in \operatorname{Div}(X)$ a divisor such that $(D \cdot C) = 0$. The supporting divisor M_R has the property that is positive on the cone

$$V = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{R' \in \mathcal{R} \setminus \{R\}} R'$$

where R denotes the set of extremal rays, and we can take m sufficiently large such that mM_R+D is also positive on $V \setminus \{0\}$, i.e., mM_R+D is also a supporting divisor for R. Now, by uniqueness of the contraction c_R, a multiple $m'(mM_R+D)$ defines the same morphism. Using the same idea as before of using the finite morphism of Stein factorization to pulling back a divisor, there exists a divisor $E_{m,m'}$ on X such that $c_R E_{m,m'} \sim_{\text{lin}} m'(mM_R+D)$. We get

$$c_R(E_{m,m'+1} - E_{m,m'} - D_m) \sim_{\text{lin}} (mM_R+D) - mM_R \sim_{\text{lin}} D$$

which ends the proof.

□

I.3.2 Types of contractions

In the previous section we solved the problem of the existence of contractions in the smooth case, so we can always contract a negative extremal ray. Now, the problem is to determine how singular the resulting variety is, in order to can continuate this process.

Definition I.3.3 (relative cone)**.** Let X, Y be projective varieties and $\varphi : X \to Y$ a morphism. The relative cone of curves of φ is the convex subcone $\mathrm{NE}(\varphi)$ of $\mathrm{NE}(X)$ generated by the classes of curves contracted by φ. If H is an ample divisor on Y, $\varphi_*(C) = 0$ if and only if $(\varphi^*H \cdot C) = 0$. Thus, $\mathrm{NE}(\varphi) = \mathrm{NE}(X) \cap (\varphi^*H)^\perp$ is closed in $\mathrm{NE}(X)$ and

$$\overline{\mathrm{NE}}(\varphi) \subseteq \overline{\mathrm{NE}}(X) \cap (\varphi^*H)^\perp$$

Example I.3.4. Consider $X = \mathbb{P}^n \times \mathbb{P}^m$ the product of two projective spaces. Since $\mathrm{Pic}(X) = \mathbb{Z}^2$ we have $N_1(X)_{\mathbb{R}} = \mathbb{R}^2$, and a basis is given by the classes of a line $\ell_1 \subset \mathbb{P}^n$ and a line $\ell_2 \subset \mathbb{P}^m$. The cone of curves of X is clearly given by $\overline{\mathrm{NE}}(X) = \mathbb{R}^+[\ell_1]+\mathbb{R}^+[\ell_2]$. If we consider the coordinate projections $\mathrm{pr}_1 : X \to \mathbb{P}^n, \mathrm{pr}_2 : X \to \mathbb{P}^m$, its relative cones are

$$\mathrm{NE}(\mathrm{pr}_1) = \mathbb{R}^+[\ell_1] \text{ and } \mathrm{NE}(\mathrm{pr}_2) = \mathbb{R}^+[\ell_2]$$

In terms of the relative cone of curves, we can state a relative version of Kleiman's ampleness criterion, whose proof follows immediately.

Definition I.3.5. Let $f : X \to Y$ be a morphism between normal varieties, and let D be a Cartier divisor on X. We say D is f-ample (or relatively ample respect to f) if the restriction of D to every fiber of f is ample.

Theorem I.3.6 (Kleiman's relative criterion, [KM98, Theorem 1.44])**.** *Let $\ : X \to Y$ be a projective morphism, and let D be a Cartier divisor on X. Then D is $\ $-ample if and only if*

$$D_{>0} \quad \overline{\mathrm{NE}}(\ ) \setminus \{0\};$$

i.e., if D is positive inside the relative cone.

Consider X as a smooth projective variety, R a K_X-negative extremal ray and c_R : $X \quad Y$ its contraction. According to the definition above, the relative cone of c_R is R and since $(c_R)\ O_X = O_Y$ we have two possbilities: either $\dim(Y) < \dim(X)$ or c_R is birational[3].

Definition I.3.7 (exceptional locus)**.** Let $\ : X \to Y$ be a proper birational morphism. The exceptional locus of $\ $ is the closed set of points of X such that $\ $ is not a local isomorphism, or equivalently, it is the domain of the inverse map. This locus is denoted Exc( ). If c_R is the contraction of a K_X-negative extremal ray R of X, the exceptional locus of c_R is called the locus of R, denoted as locus(R).

From this discussion we conclude that we can divide contractions into 3 different classes, listed in the following definition.

Definition I.3.8. Let X be a smooth complex variety, let R be a K_X-negative extremal ray R of X, and let $c_R : X \quad Y$ be its contraction. We say that c_R is a

1. *fiber contraction* if locus(R) $= X$.
2. *divisorial contraction* if $\mathrm{codim}_X(\mathrm{locus}(R)) = 1$.
3. *small contraction* if $\mathrm{codim}_X(\mathrm{locus}(R)) \quad 2$.

Remark I.3.9. By virtue of the very construction of the contraction morphism via Stein factorization, the resulting variety of a contraction is necessarily normal.

[3] This is because of the upper semicontinuity of the dimension of the fibers. As c_R has connected fibers, these are either a point or a positive dimension subvariety of X, and the set of points where $\dim_x(c_R^{-1}(c_R(x))) = 0$ is an open set.

Proposition I.3.10 ([Deb01, Proposition 6.10])**.** *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray of X. The locus of R is closed and if Z is an irreducible component of* locus(R)*,*

1. *Z is covered by rational curves contracted by c_R;*
2. *if Z has codimension 1, then it is equal to locus (R);*
3. *the following inequality holds*

$$\dim(Z) \geq \frac{1}{2}\left(\dim(X) + \dim\left(c_R(Z)\right)\right)$$

The previous Proposition motivates the following definition.

Definition I.3.11 (length of an extremal ray)**.** Let X be a smooth complex projective variety and R a K_X-negative extremal ray of X. We define the length of R as

$$\ell(R) = \min\{(-K_X \cdot \Gamma) \mid \Gamma \text{ rational curve on } X \text{ with class in } R\}$$

The next theorem is presented in [Wis91].

Theorem I.3.12. *Let X be an smooth complex projective variety and R a K_X-negative extremal ray of X. Let F be an irreducible component of a non-trivial fiber of the contraction c_R. Then*

$$\dim F + \dim(\text{locus}(R)) \geq \dim X + \ell(R) - 1.$$

In particular, F is covered by rational curves of $(-K_X)$-degree at most $\dim(F) + 1 - \operatorname{codim}(\text{locus}(R))$.

Building upon the structural results of contractions established thus far, we proceed to examine each type of contraction individually.

Fiber contractions

First and foremost, we introduce the notion of a Fano variety, a significant class of varieties that will constitute the primary focus of Chapter 2.

Definition I.3.13. A Fano variety is a smooth complex projective variety X such that its anticanonical divisor $-K_X$ is ample. A Fano surface is called a del Pezzo surface[4]

[4] See Proposition II.1.5 for a complete description of these surfaces.

Here is the reason to introduce Fano varieties. Consider X as a smooth complex variety and R a K_X-negative extremal ray with $c_R : X \to Y$ a fiber type contraction. In this case $\dim(Y) < \dim(X)$ and by Proposition I.3.10 $X = \text{locus}(R)$ is covered by rational curves, and moreover a general fiber F of c_R is smooth and verifies $-K_F = -(K_X)|_F$ is ample because by Kleiman criterion $-K_X$ is c_R-ample. Thus, fiber contractions are fibrations on Fano varieties.

Proposition I.3.14. *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray. If the contraction $c_R : X \to Y$ is of fiber type, Y is locally factorial.*

The main example of a fiber type contraction is a projective bundle. This is explained in the following example.

Example I.3.15. Let E be a vector bundle (or a locally free sheaf) of rank r over a smoth projective variety Y, and $X = \mathbb{P}(E)$ the associated projective bundle with projection $\pi : X \to Y$. Denote by ξ the class of $\mathcal{O}_X(1)$. In this case we have the formula[5]:

$$\omega_X = \pi^*(\omega_Y \otimes \det(E)) \otimes \mathcal{O}_X(-r)$$

In terms of divisors this formular is written as

$$K_X = -r\xi + \pi^*(K_Y + \det(E)).$$

Let ℓ be a line contained in a fiber of π. This fiber is simply $\mathbb{P}^{r-1}$, and then by projection formula

$$(K_X \cdot \ell) = -r(\xi \cdot \ell) + (K_Y + \det(E) \cdot \pi_*(\ell)) = -r(\xi \cdot \ell) = -r$$

Observe that the numerical class $[\ell]$ spans a K_X-negative extremal ray whose contraction is π. Indeed, every curve contracted by π is a curve on $\mathbb{P}^r$ whose intersection product is governed by the degree, and therefore every contracted curve is numerically equivalent to a multiple of $[\ell]$. Thus, $\overline{\text{NE}}(\pi) = \mathbb{R}^+[\ell]$ and then this ray is extremal.

In [Mor82], Mori classified all possible fiber type contractions of threefolds in terms of the description of the fibers.

[5] This formula follows the Grothendieck convention of projective bundle as the space of hyperplanes on each fiber.

Theorem I.3.16. *Let X be a smooth complex projective threefold, R be a K_X-negative extremal ray with contraction $c_R : X \to Y$ of fiber type.*

1. *If $\dim(Y) = 2$ we have the following possibilities*
   (a) *$c_R : X \to Y$ is a conic bundle with a singular fiber.*
   (b) *$c_R : X \to Y$ is a $\mathbb{P}^1$-bundle.*
2. *If $\dim(Y) = 1$ the possibilities are*
   (a) *the general fiber of $c_R : X \to Y$ is a del Pezzo surface S of degree $d := (-K_S)^2 \in \{1, \ldots, 6\}$.*
   (b) *c_R is a quadric bundle.*
   (c) *c_R is a $\mathbb{P}^2$-bundle.*

Divisorial contractions

Analogous to the case of fiber type contractions, the outcome of a divisorial contraction may be singular. Nevertheless, it is guaranteed to be at least locally $\mathbb{Q}$-factorial (i.e., every Weil divisor has a multiple which is Cartier), allowing us to apply intersection theory without any problem. Indeed, if X is locally $\mathbb{Q}$-factorial, C is a curve and D is a Weil divisor which is not Cartier, we can take m such that mD is Cartier and define:

$$(D \cdot C) = \frac{1}{m} \deg \mathcal{O}_C(mD)$$

The only point to bear in mind is that the result may be a rational number.

Proposition I.3.17. *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray whose contraction $c_R : X \to Y$ is divisorial. Then Y is locally $\mathbb{Q}$-factorial.*

Example I.3.18. A smooth blow-up is the prototypical example of a divisorial contraction. Let Y be a smooth complex projective variety and $Z \subset Y$ and smooth subvariety of codimensión c. The blow-up $\pi : X \to Y$ along Z satisfies:

$$K_X = \pi^* K_Y + (c - 1)E,$$

where E denotes the exceptional divisor. Moreover, if ℓ is a line contained in a fiber F of the projection $E \to Z$ we have $(K_X \cdot \ell) = -(c - 1)$. A argument analogous to that in Example I.3.15 shows that $[\ell]$ generates an K_X-negative extremal ray whose contraction is π. This is a divisorial contraction because $\operatorname{locus}(R) = E$.

In a similar fashion to the case of fiber contractions, Mori classified all divisorial contractions of a smooth projective threefold.

Theorem I.3.19 ([IP99, Theorem 1.4.3])**.** *Let X be a smooth complex projective threefold, R be a K_X-negative extremal ray with contraction $\mathrm{c}_R : X \to Y$ of divisorial type. Then the situation is one of the following:*

(E1) Y is smooth and X is the blow-up of Y along a smooth curve.

(E2) Y is smooth and X is the blow-up of Y at a point.

(E3) X is the blow-up of Y on a singular point analytically isomorphic to

$$0 \in \operatorname{Spec} \mathbb{C}[X,Y,Z,W]/\langle X^2+Y^2+Z^2+W^2\rangle$$

where all the curves on the exceptional divisor $E = \mathbb{P}^1 \times \mathbb{P}^1$ are numerically proportional.

(E4) X is the blow-up of Y on a singular point analytically isomorphic to

$$0 \in \operatorname{Spec} \mathbb{C}[X,Y,Z,W]/\langle X^2+Y^2+Z^2+W^3\rangle$$

(E5) X is the blow-up of Y on a singular point analytically isomorphic to

$$0 \in \operatorname{Spec} \mathbb{C}[X,Y,Z]^{\langle i\rangle}$$

where $\mathbb{C}[X,Y,Z]^{\langle i\rangle}$ denotes the ring of invariants by the action of the involution $i : (X,Y,Z) \to (-X,-Y,-Z)$.

Example I.3.20 (divisorial contraction with a singular image)**.** As an example we comment the case 3. of Theorem I.3.19. Let Z be a smooth complex projective threefold and let C be an irreducible curve in Z with a unique singularity being a node, i.e., an ordinary double point. In this case, locally analytically, the ideal of C is generated by xy, z where x, y, z are local parameters. The situation is then

$$C = \{xy = z = 0\} \subset Z \overset{\text{analytically}}{=} \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x,y,z]$$

and the blow up Y of Z along C is given by

$$\left\{((x,y,z),[u,v]) \in \mathbb{A}^3_k \times \mathbf{P}^1_k \mid xyv = zu\right\}$$

which is smooth except at $q = ((0,0,0),[0,1])$. Now let X be the blow-up of Y at q. A local computation shows that X is smooth and the exceptional divisor of $X \to Z$ is a quadric surface $E = \mathbb{P}^1 \times \mathbb{P}^1$.

Small contractions

In this section we consider X as a smooth complex projective variety and R being a K_X-negative extremal ray with $c_R : X \to Y$ a small contraction, i.e., $\operatorname{codim} \operatorname{locus}(R) \geq 2$. This is the worst case of contraction, because Y is always very singular, as is indicated by the following result.

Proposition I.3.21. *Let Y be a normal and locally $\mathbb{Q}$-factorial variety and let $\varphi : X \to Y$ be a birational proper morphism. Every irreducible component of the exceptional locus of φ has codimension 1 in X.*

Proof. Denote by $E = \operatorname{Exc}(\varphi)$ the exceptional locus of φ, and let $x \in E, y = \varphi(x)$. We will prove that there is a codimension 1 component of E through x, which will give us the conclusion. We have an isomorphism of fields $\varphi^* : k(Y) \to k(X)$, and $\varphi^*(\mathcal{O}_{Y,y}) \subsetneq \mathcal{O}_{X,x}$ is a proper subring (because φ is not an isomorphism at x). In particular, there exists an element $t \in \mathfrak{m}_{X,x} \setminus \varphi^*(\mathcal{O}_{Y,y})$. We write $\operatorname{div}(t) = D - D'$, i.e., D is the divisor of zeros and D' the divisor of poles, and we assume that D, D' has no common components. Since Y is locally $\mathbb{Q}$-factorial, there exists $m \in \mathbb{N}_{\geq 1}$ such that mD and mD' are both effective Cartier divisors, so there exists $u, v \in \mathcal{O}_{Y,y}$ such that $\varphi_*(t^m) = \frac{u}{v}$. We claim that $u, v \in \mathfrak{m}_{Y,y}$. Indeed, if $v \notin \mathfrak{m}_{Y,y}$ then $(\varphi_*(t))^m \in \mathcal{O}_{Y,y}$, and as Y is normal we obtain $\varphi_*(t) \in \mathcal{O}_{Y,y}$, and it follows that $u = v\varphi_*(t^m) \in \mathfrak{m}_{Y,y}$. The subvariety $Z = \{u = v = 0\} \subset Y$ has codimension 2 and it contains $y = \varphi(x)$, but $x \in \varphi^{-1}(Z) = \{v = 0\}$ and $\varphi^{-1}(Z)$ has codimension 1, i.e., $\varphi^{-1}(Z) \subset E$. □

By the previous Proposition we have that the result of a small contraction is not even locally $\mathbb{Q}$-factorial, and therefore we cannot use intersection theory to perform calculations.

Remark I.3.22. An interesting fact is that there are no small contractions on smooth threefolds. In effect, by Theorem I.3.12 we have that any positive-dimensional irreducible component F of a fiber of c_R verifies

$$\dim(F) \geq \operatorname{codim}(\operatorname{locus}(R)) + \ell(R) - 1 = 1 + \ell(R) \geq 2$$

and then by Proposition I.3.10

$$\dim(X) \geq \dim(c_R(\operatorname{locus}(R))) + 4$$

Thus, by dimensional reasons small contractions cannot happen in this case.

In dimension 4 small contractions can be appear, and we must solve this problem. Faced with this situation, Mori proposed the idea that there should exist another projective variety X^+ with a small contraction $c^+ : X \to Y$ such that K_{X^+} is c^+-ample, i.e., we can *flip* the extremal ray of the contration in such a way that it becomes a K_X-positive ray. In other words, we are looking for a way to avoid the small contractions. This motivates the following fundamental definition.

Definition I.3.23 (flip)**.** Let

$$c : X \to Y$$

be a small contraction of an extremal ray with respect to K_X between normal projective varieties such that K_X is $\mathbb{Q}$-Cartier and K_X is c-ample. A flip is a small contraction $c^+ : X^+ \to Y$ such that

1. X^+ is a normal projective variety.

2. K_{X^+} is $\mathbb{Q}$-Cartier and c^+-ample.

Remark I.3.24. We finish this section summarizing the ideas behind MMP and the problems that there is solved in order to run the MMP. Given a variety X, you can succesively contract its K_X-negative extremal rays, until you end up with a variety which either has a fiber type contraction, or has nef canonical divisor. Thus, when we run this process three problems arise:

1. Because the result of a contraction can be singular, we have to expand our methods to allow singularities. This problem will be adressed briefly in the next section.

2. Since small contractions could be appear, a problem we have to surpass is the *existence of flips*. The existence of flips was proven in [BCHM10].

3. Another problem concerning flips is the *termination of flips*. Whereas fiber type and divisorial contractions decreases Picard number by 1, flips do not. The question that arises naturally is if it can exist an infinite sequence of flips. This is still an open problem in full generality.

I.3.3 An example of a flip

The goal of this section is to provide a detailed analysis of a specific example of a flip in dimension 3. This example is drawn from [Deb01]. To make the computations explicit, we will first introduce some preliminary results, primarily concerning projective bundles.

First, we discuss the computation of the normal bundle of a section in a projective bundle. Next, we establish a proposition that describes how to compute the normal bundle of the strict transform of a section within the blow-up of a threefold. Finally, we examine an explicit example of a flip.

Now, we turn to the first objective. The setup is as follows: let X be a complex projective variety of dimension $\dim(X) = n$, and let $E \xrightarrow{\pi} X$ be a vector bundle of rank r. We consider the associated projective bundle $P := \mathbb{P}_X(E)$, which is a projective variety of dimension $\dim(P) = \dim(X) + r - 1$, equipped with the natural projection $\pi : P \to X$.

Below, we recall the construction of the tautological bundle.

Construction I.3.25. In P we have the pullback bundle π^*E whose fiber at a point $p \in P$ is by definition $(\pi^*E)_p := E_{\pi(p)}$. This vector bundle has a natural sub-bundle N of rank $n-1$. A point $p \in P$ represents a point in the projectivization $\mathbb{P}(E_{\pi(p)})$, i.e., is an hyperplane H_p on the fiber $E_{\pi(p)}$. We define the line bundle N as $N_p := H_p$ at each point $p \in P$. Then the vector bundle $\mathcal{O}_P(1)$ is defined by the following exact sequence of vector bundles on P:

$$0 \to N \to \pi^*E \xrightarrow{u} \mathcal{O}_P(1) \to 0$$

Then, in a point $p \in P$ the fiber of $\mathcal{O}_P(1)$ corresponds to $(\mathcal{O}_P(1))_p := E_{\pi(p))}/H_p$, and the natural map u corresponds to the quotient projection:

$$u_p : (\pi^*E)_p \twoheadrightarrow (\mathcal{O}_P(1))_p; \quad x \mapsto x + H_p$$

Proposition I.3.26. *Let Y be a complex projective variety and let $f : Y \to X$ be a regular morphism. There is a correspondence:*

$$\begin{aligned} \{g : Y \to P \mid \pi \circ g = f\} \quad &\longleftrightarrow \quad \{L \in \mathrm{Pic}(Y) \text{ and } v : f^*E \to L \text{ surjective}\} \\ g \quad &\longmapsto \quad L := g^*\mathcal{O}_P(1) \quad \text{and} \quad v := g^*u \\ (g := y \mapsto \mathbb{P}(\ker(v_y))) \quad &\longmapsfrom \quad (L; v) \end{aligned}$$

Proof. Given a morphism $g : Y \to P$, we simply take the pullback $L := g^*\mathcal{O}_S(1)$ and the pullback morphism $v := g^*u$ is simply $v_y := u_{g(y)}$ at a point $y \in Y$. Conversely, if we have a line bundle $L \in \mathrm{Pic}(Y)$ and a surjective morphism $v : f^*E \twoheadrightarrow L$, by definition

$\ker(v_y)$ corresponds to an hyperplane on the fiber $E_{f(y)}$, i.e., it is on the fiber of $f(y)$ along and then $g = f$.

These constructions are inverses one to the other because L_y is the cokernel of the embedding $N_{g(y)} \quad E_{f(y)}$. □

Remark I.3.27. The previous proposition with $Y = X$ and $f = \mathrm{Id}$ gives us a 1 1 correspondence between sections of the natural projection and rank 1 quotients of E.

The question that we want to solve here is how to calculate the normal bundle of a section $s : X \to P$ of the projection , which by the Remark is the same as a rank 1 quotient of E, which we call L.

Then we have a surjection $E \quad L$ and therefore an exact sequence of vector bundles on X:

$$0 \to F \to E \xrightarrow{q} L \to 0$$

Taking pullbacks is an exact operation, so:

$$0 \to \quad F \to \quad E \to \quad L \to 0$$

is an exact sequence on P. Composing maps we obtain a morphism $\quad F \to O_P(1)$.

Twisting the above morphism by $O_P(\ 1)$ we have $\quad F \quad O_P(\ 1) \to O_P$, whose image defines an ideal sheaf on P. Now we prove the subvariety defined by the ideal sheaf is exactly the section $D := s(X)$. To prove this we analyze the image of $\quad F \to O_P(1)$. By construction, we have

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & O_P(1) \\ F_{(p)} & \longrightarrow & \ker(q_{(p)}) & \longrightarrow & E_{(p)}/\ker(q_{(p)}) \end{array}$$

where the diagram above shows the image on each fiber. Then, by definition of the section s we have that:

$$p \in D := s(X) \qquad (\ ) \qquad \mathrm{Im}((\quad F \to O_P(1))_p) = 0$$

Thus, the support of the ideal sheaf defined by the morphism $\quad F \quad O_P(\ 1) \to O_P$ is exactly the section D. The rank of F is $r \quad 1$, which is exactly $\mathrm{codim}_P(D)$, and then D is locally a complete intersection because his ideal sheaf I_D is locally generated by $r \quad 1$ elements. To conclude we prove the following statement.

Proposition I.3.28. *Let E be a vector bundle of rank r which maps onto an ideal sheaf I_D defining a local complete intersection of codimension r, denoted D. Then the normal bundle of D is $E^\vee$.*

Proof. We have a surjection $E \to I_D \to 0$. If $\iota : D \hookrightarrow X$ we can restrict to have

$$E \to i^* I_D \to 0$$

on D. We note that

$$\iota^* I_D = I_D \otimes O_D = I_D \otimes (O_X/I_D) = \iota^*(I_D/I_D^2)$$

and this is the conormal sheaf of D. Note that $\iota^* E; i^*(I_D/I_D^2)$ are vector bundles of the same rank r and then the surjection $\iota^* E \to \iota^*(I_D/I_D^2)$ is an isomorphism. Thus, the normal bundle of D is $(\iota^* E)^\vee$. □

Then, using Proposition I.3.28 we have a surjective map

$$s^*(\pi^* F \otimes O_P(-1)) = F \otimes L^\vee \to s^*(I_D) = N^\vee_{D/X}$$

because by Proposition I.3.26 we have $L = s^* O_P(1)$.

Proposition I.3.29. *Let X be a smooth complex projective variety, $E \xrightarrow{\pi} X$ be a vector bundle and $P := \mathbb{P}_X(E)$ its projective bundle with projection $\pi : P \to X$. If $s : X \to P$ is a section of π with associated exact sequence $0 \to F \to E \to L \to 0$, we have*

$$N_{D/P} = F^\vee \otimes L$$

The following result allows to calculate normal bundles of some curves inside the blow-up of a smooth threefold.

Proposition I.3.30. *Let $C \subset Y$ be a smooth rational curve on a smooth threefold Y and $\varepsilon : X \to Y$ the blow-up along C. Let $\widetilde{C}$ be a smooth section of the projection $E = \mathbb{P}(N^\vee_{C/Y}) \to C$ associated to the exceptional divisor E. Then*

$$\deg(N_{C/Y}) = \widetilde{C}^2 + 2E \cdot \widetilde{C};$$

and if $\deg(N_{C/Y}) \geq 3\widetilde{C}^2 + 2$, the normal bundle of $\widetilde{C}$ in X is

$$N_{\widetilde{C}/X} = N_{\widetilde{C}/E} \oplus (N_{E/X})|_{\widetilde{C}}.$$

Proof. We have the exact sequence of normal bundles

$$0 \to \underbrace{N_{\tilde{C}/E}}_{F=} \to N_{\tilde{C}/X} \to \underbrace{\left(N_{E/X}\right)\big|_{\tilde{C}}}_{L=} \to 0$$

and we want to prove this sequence splits. Because L is a line bundle, a well-known criterion says that the obstruction to the splitting of the exact sequence is an element lying in $H^1(X; E \quad L^{-})$, so it is enough to prove the vanishing of this cohomology group. We calculate the degree of these vector bundles. We have

$$\deg(N_{\tilde{C}/X}) = \deg(T_X|_{\tilde{C}}/T_{\tilde{C}}) = \deg(K_{\tilde{C}}) \quad \deg(K_X|_{\tilde{C}}) = \deg(K_{\tilde{C}}) \quad K_X \quad \tilde{C}$$
$$\deg(N_{\tilde{C}/E}) = \tilde{C}^2$$
$$\deg(N_{E/X}|_{\tilde{C}}) = \deg(\mathcal{O}_X(E)|_E) = E \quad \tilde{C}$$

Because of the additivity of degree in exact sequences, we have

$$\deg(K_{\tilde{C}}) \quad K_X \quad \tilde{C}\,\deg(N_{\tilde{C}/X} = \deg(N_{\tilde{C}/E}) + \deg(N_{E/X}|_{\tilde{C}}) = \tilde{C}^2 + E \quad \tilde{C}$$

Now, by adjunction we have $K_X = \varepsilon^* K_Y + E$, and we obtain

$$\begin{aligned}\deg(N_{C/Y}) &= \deg(K_{\tilde{C}}) \quad K_Y \quad C\\ &= \deg(K_{\tilde{C}}) + E \quad \tilde{C} \quad K_X \quad \tilde{C}\\ &= \tilde{C}^2 + 2E \quad \tilde{C}\end{aligned}$$

Then the exact sequence corresponds to

$$0 \to \mathcal{O}_{\tilde{C}}\left(\tilde{C}^2\right) \to N_{\tilde{C}/X} \to \mathcal{O}_{\tilde{C}}\left(\frac{1}{2}\left(\deg\left(N_{C/Y}\right) \quad \tilde{C}^2\right)\right) \to 0$$

Note then that $H^1(\tilde{C}; F \quad L^{-}) = 0$ if $\deg(F \quad L^{-}) \quad 2g(C) \quad 1 = \quad 1$, and thus we obtain the condition

$$\deg(F \quad L^{-}) = \tilde{C}^2 \quad \frac{1}{2}\deg(\left(N_{C/Y}\right) \quad \tilde{C}^2) \qquad 1 \quad (\ ) \quad \deg(N_{C/Y}) \quad 3\tilde{C}^2 + 2.$$

□

Now, we present an example of a flip, which will be presented in the inverse direction, i.e., we begin with a variety and a contraction of a positive curve, and performing different types of birational transformations we arrive to a variety in which the positive curve is replaced by a negative curve.

Example I.3.31. We will consider a threefold X and a small contraction $X \to Y$. We suppose that the resulting variety of the associated flip, denoted by X^+, is such that it contains a rational curve $\Gamma^+ = \mathbb{P}^1 \subset X^+$ whose normal bundle corresponds to $N_{\Gamma^+/X^+} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. This situation in fact exists. Consider the rank 3 vector bundle $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1$, and consider $\pi : X^+ := \mathbb{P}_{\mathbb{P}^1}(E) \to \mathbb{P}^1$ the associated projective bundle. Proposition I.3.29 implies that the section $s : \mathbb{P}^1 \to \Gamma^+ := s(\mathbb{P}^1) \subset \mathbb{P}(E)$ associated to the trivial quotient $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1}$ is as we want, because its corresponding exact sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

We know the canonical divisor of X^+ is given by

$$K_{X^+} = -3\xi + \pi^*(K_{\mathbb{P}^1} + \det(E)) = -3\xi + \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$$

where ξ denote the class of $\mathcal{O}_{X^+}(1)$ and we calculate

$$\det(E) = \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(2) = \mathcal{O}_{\mathbb{P}^1}(3).$$

By projection formula we observe

$$K_{X^+} \cdot \Gamma^+ = \mathcal{O}_{\mathbb{P}^1}(1) \cdot \pi_*\Gamma^+ = 1,$$

so $\mathbb{R}^+[\Gamma^+]$ is a K_{X^+}-positive ray. Moreover, since $\mathcal{O}_{X^+}(1) \cdot \Gamma^+ = 0$ have zero intersection, the morphism associated to the linear system $\mathcal{O}_{X^+}(1)$ is the contraction of the ray $\mathbb{R}^+[\Gamma^+]$. We will apply some geometrical operations in order to discover what is the original X.

Step 1. Consider $\varepsilon_1 : X_1^+ \to X^+$ the blow-up along Γ^+. The exceptional divisor of this morphism is

$$S_1^+ = \mathbb{P}(N^\vee_{\Gamma^+/X^+}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right) \overset{\text{def}}{=} \mathbb{F}_1,$$

which is known that contains a section $E_1^+ \subset S_1^+$ such that $(E_1^+)^2 = -1$. Note that in this case $\deg(N_{\Gamma^+/X^+}) = \deg(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = -3$, and the condition $\deg(N_{\Gamma^+/X^+}) \le 3(E_1^+)^2 + 2$ is verified and Proposition I.3.30 implies[6]

[6]Remember $\mathbb{F}_1$ is isomorphic to the blow-up $\mathrm{Bl}_p(\mathbb{P}^2)$ at a point with E_1^+ as the exceptional divisor, so $N_{E_1^+/S_1^+} = \mathcal{O}_{E_1^+}(-1)$.

$$N_{E_1^+/X_1^+} = N_{E_1^+/S_1^+} \oplus (N_{S_1^+/X_1^+})|_{E_1^+} = \mathcal{O}_{E_1^+}(-1) \oplus \mathcal{O}_{E_1^+}(-1)$$

Step 2. Consider now $\varepsilon_2 : X_0 = \mathrm{Bl}_{E_1^+}(X_1^+) \to X_1^+$ the blow-up of X_1^+ along the curve E_1^+. In this case the exceptional divisor corresponds to

$$S_0 = \mathbb{P}(N^{\vee}_{E_1^+/X_1^+}) = \mathbb{P}(\mathcal{O}_{E_1^+}(1) \oplus \mathcal{O}_{E_1^+}(1)) = \mathbb{P}^1 \times \mathbb{P}^1$$

with normal bundle $N_{S_0/X_0} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. Define the following curves

1. Γ_0 will be a fiber of the natural projection $S_0 \to E_1^+$.
2. F_0 will be the strict transform under ε_2 of a fiber of the projection $S_1^+ \to \Gamma^+$
3. E_0^+ will be the intersection of the strict transform of S_1^+, which we will denote $\widetilde{S_1^+}$, and the exceptional divisor S_0.

Consider the entire composition

$$\varphi : X_0 \xrightarrow{\varepsilon_2} X_1^+ \xrightarrow{\varepsilon_1} X^+ \to Y$$

The key observation is to note that the relative cone of curves $\overline{\mathrm{NE}}(\varphi) = \langle [\Gamma_0], [F_0], [E_0^+] \rangle$ is generated by the previously defined classes, because all of them are contracted by φ, and they are generators because $\rho_{X_0} = 4, \rho_Y = 1$. Now, we want to prove $\mathbb{R}^+[E_0^+]$ is a K_{X_0}-negative extremal ray of X_0. Indeed, by adjunction

$$K_{X_0} \cdot E_0^+ = K_{X^+} \cdot \Gamma^+ + S_1^+ \cdot E_1^+ + S_0 \cdot E_0^+ = 1 + (-1) + (-1) = -1.$$

In order to prove $\mathbb{R}^+[E_0^+]$ is extremal, we use that $[\Gamma_0], [F_0], [E_0^+]$ generates $\overline{\mathrm{NE}}(\varphi)$ and the fact that the relative cone of curves is an extremal subcone of $\overline{\mathrm{NE}}(X_0)$ (see [Deb01, Proposition 1.14]). If $\mathbb{R}^+[E_0^+]$ were not extremal, there exists $a, b > 0$ such that $[E_0^+] = a[F_0] + b[\Gamma_0]$. On one hand, we can intersect with S_0 to obtain the relation $-1 = a - b$, and on the other hand if we intersect with the strict transform of S_1^+ we have $-1 = -a + b$, which is a contradiction.

Step 3. Since $\mathbb{R}^+[E_0^+]$ is an extremal ray, we can consider its contraction $\psi : X_0 \to X_1$, which we can describe using Mori's characterization of extremal contractions on threefolds. In fact, note that E_0^+, Γ_0 are the rulings of the exceptional divisor $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and we already know E_0^+, Γ_0 are not numerically equivalent in X_0 and the contracted ray $\mathbb{R}^+[E_0^+]$ is not numerically effective because $S_0 \cdot E_0^+ = -1$. By [IP99, Theorem 1.4.3], this contraction is divisorial, so the exceptional locus corresponds to the divisor S_0, but the ruling Γ_0 is not contracted so it has to be an $(E1)$ type contraction. The previous fact implies that X_1 is a smooth threefold, $\varphi(S_0)$ is a smooth curve and φ is nothing more than the blow-up of X_1 along $\varphi(S_0) =: \Gamma_1$.

We can also calculate the normal bundle of Γ_1 into X_1. The proof of Proposition I.3.30 gives us a formula for the degree of N_{Γ_1/X_1}, and we obtain

$$\deg(N_{\Gamma_1/X_1}) = \underbrace{\Gamma_0^2}_{=0} + 2\underbrace{S_0 \cdot \Gamma_0}_{=-1} = -2$$

and because we have characterized the contraction φ as the blow-up of Γ_1, we know that

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = S_0 = \mathbb{P}(N_{\Gamma_1/X_1});$$

so N_{Γ_1/X_1} corresponds to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ twisted by a certain line bundle over $\mathbb{P}^1$, but since its degree is exactly -2, we conclude

$$N_{\Gamma_1/X_1} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

In the following we calculate the normal bundle N_{S_1/X_1}. Note that the ruled surface $\widetilde{S_1^+} = \mathbb{F}_1$ gets blow-down onto a projective plane under φ, because it is just the contraction of E_0^+, which corresponds to the exceptional curve of $\widetilde{S_1^+}$. More precisely, $S_1 := \varphi(\widetilde{S_1^+}) = \mathbb{P}^2$.

Now we calculate N_{S_1/X_1}. Since $S_1 = \mathbb{P}^1$, it is enough to calculate the degree of the restriction $N_{S_1/X_1}|_\ell$, where $\ell = \mathbb{P}^1 \subset S_1$ is a line that doesn't pass through the point $\varphi(E_0^+)$. Note first that

$$\deg(N_{S_1/X_1}|_\ell) = S_1 \cdot \ell = \widetilde{S_1^+} \cdot \ell = \deg(N_{\widetilde{S_1^+}/X_0}|_\ell);$$

where we also denote by ℓ a line in $\widetilde{S_1^+}$ that it doesn't intersect E_1^+. This implies that he calculation of $\deg(N_{S_1/X_1}|_\ell)$ can be carried out in X_0, and moreover, since $\widetilde{S_1^+}$ is nothing more than the strict transform of $S_1^+ \subset X_1$, the calculation can be done in X_1^+.

In light of the above, we consider a line $\ell \subset S_1^+$ such that does not intersect E_1^+. We claim that ℓ is a section the $\mathbb{P}^1$-bundle $S_1^+ \to \Gamma^+$. In order to prove this, we recall that $S_1^+ = \mathbb{F}_1$, and we know the intersection theory on this surface. We have $\mathrm{Pic}(\mathbb{F}_1) = \mathbb{Z}[\sigma] \oplus \mathbb{Z}[f]$, where σ is the class of a section and f is the class of a fiber (see

[Bea83, Proposition IV.1]), and $E_1^+ = \quad f$. If we write $\ell = a \quad + bf$ or some $a, b \in \mathbb{Z}$, we compute $0 = E_1^+ \quad \ell = b$, i.e., ℓ is in fact a section. By Proposition I.3.30, we obtain that

$$\deg(N_{S_1/X_1}) = \deg(N_{S_1^+/X_1}|_\ell) = \frac{1}{2}(\deg(N_{\quad^+/X^+}) - \ell^2) = \frac{1}{2}(-3 - 1) = -2.$$

From the previous calculation we obtain $N_{S_1/X_1} = \mathcal{O}_{S_1}(-2)$. Via adjunction formula we can use the previous computation to derive

$$K_{X_1}|_{S_1} = K_{S_1} \otimes \det(N_{S_1/X_1}^{-}) = \mathcal{O}_{S_1}(-3) \otimes \mathcal{O}_{S_1}(2) = \mathcal{O}_{S_1}(-1).$$

Step 4. From the last calculation follows that, if $\ell_1 = \mathbb{P}^1 \subset S_1$ is a line disjoint from Γ_1, we have $K_{X_1} \cdot \ell_1 = \deg(K_{X_1}|_{\ell_1}) = -1$ and we consider its contraction $c : X_1 \to X$. By [IP99, Theorem 1.4.3] its contraction is of type $(E5)$, and this means that S_1 is contracted to a quotient singularity, which is locally analytically isomorphic to the quotient of $\mathbb{A}^3$ by the involution $(x, y, z) \mapsto (-x, -y, -z)$.

Now we can prove K_X is not Cartier. Write $K_{X_1} = c^* K_X + aS_1$ for some $a \in \mathbb{Q}$. If we restrict to S_1 we obtain

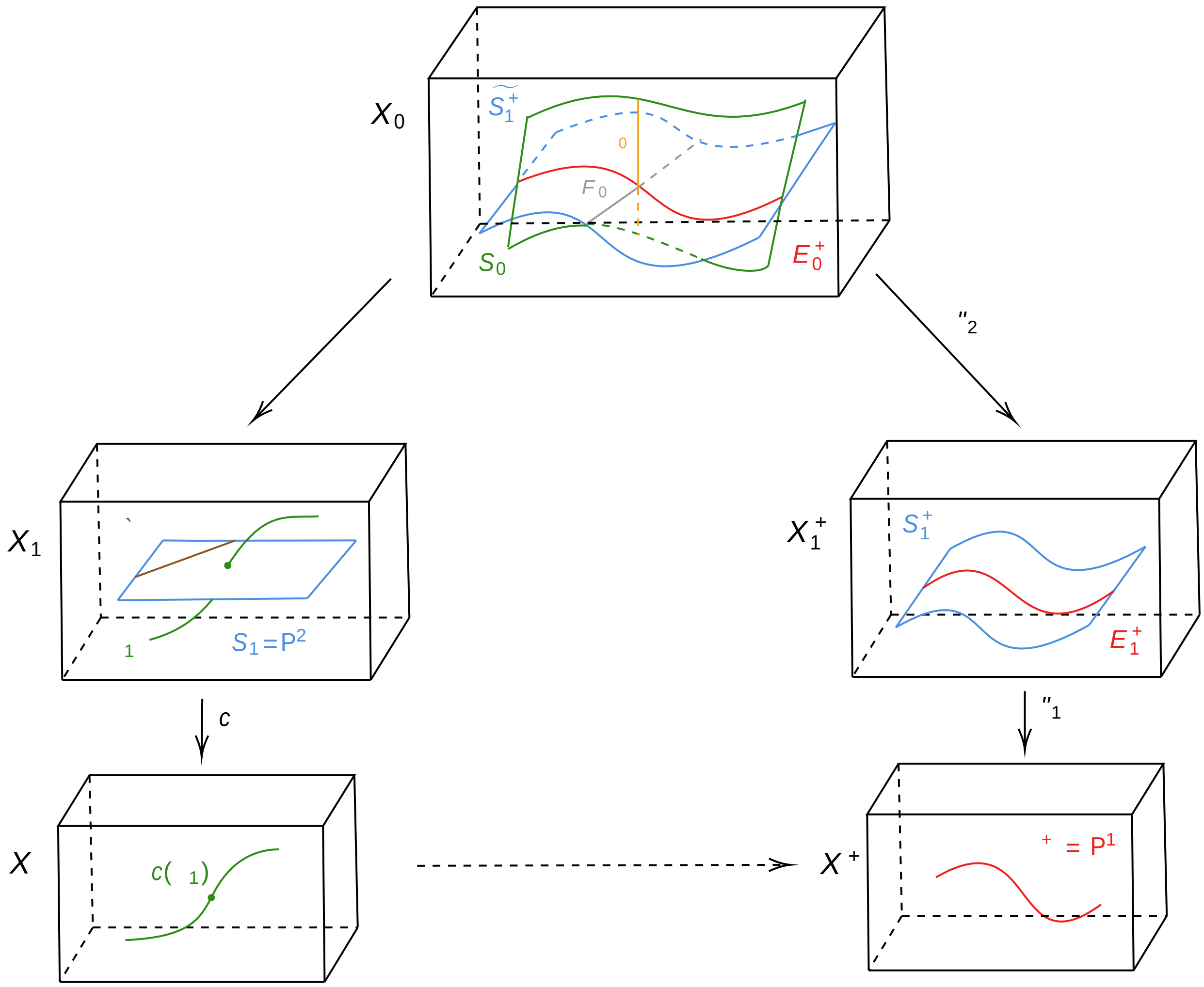
$$-H = K_{X_1}|_{S_1} = \underbrace{c^* K_X|_{S_1}}_{=0} + aS_1|_{S_1} = -2aH \Rightarrow a = \frac{1}{2}$$

where H denotes the ample generator of $\mathrm{Pic}(S_1)$, so

$$K_X \cdot c(\Gamma_1) = c^* K_X \cdot \Gamma_1 = \underbrace{K_{X_1} \cdot \Gamma_1}_{=0} - \frac{1}{2}\underbrace{S_1 \cdot \Gamma_1}_{=1} = -\frac{1}{2}.$$

This calculation also proves $\mathbb{R}^+[c(\Gamma_1)]$ is a K_X-negative ray, and in fact it is extremal. Then we can perform the contraction of the ray $\mathbb{R}^+[c(\Gamma_1)]$, which results to be a small contraction. The map $X^+ \dashrightarrow Y$ correspond to a flip of this contraction, and its effect is to replace the K_X-negative curve $c(\Gamma_1)$ by the K_{X^+}-positive curve Γ^+.

The following picture summarizes the complete procedure.



I.4 Singularities of the MMP

In the last section we discovered that study singularities is inevitable to the MMP. The aim of this section is to present some general definitions and facts about the singularities that can be appear when we run the MMP.

I.4.1 Log pairs and log discrepancies

Definition I.4.1. Let X be a variety and let $f : Y \to X$ be a proper, birational morphism with Y normal. In particular we have an isomorphism $k(Y) \cong k(X)$ of k-extensions given by the pullback of rational functions, and for a prime divisor $E \subset Y$ the local ring $\mathcal{O}_{Y,E}$ (which is the stalk at the generic point of E) is a DVR (discrete valuation ring) of $k(Y)$, and by the identification above it is also a DVR of $k(X)$. If $Y_1 \to X, Y_2 \to X$ are two proper, birational morphisms to X, we identify prime divisors $E_1 \subset Y_1, E_2 \subset Y_2$ if they give the same DVR of $k(X)$, and an equivalence class of such divisors is called a divisor over X.[7]

The following definition contains fundamental concepts related to resolutions of singularities.

Definition I.4.2. Let X be a normal variety. A divisor D on X is **simple normal crossing** (abbreviated snc) if X is smooth and at each $p \in \mathrm{Supp}(D)$ there exists local coordinates $x_1, \dots, x_n$ such that D is locally defined at p by $x_1 \cdots x_r$ for some $1 \leq r \leq n$.

A *log resolution* of (X, D) where D is a $\mathbb{Q}$-divisor on X is a proper birational morphism $f : Y \to X$ such that

1. Y is smooth.
2. $\mathrm{Exc}(f)$ is pure codimension 1.
3. $\mathrm{Exc}(f) \cup \mathrm{Supp}(f^{-1}(D))$ is snc.

The existence of log resolutions in characteristic 0 is a celebrated result due to Hironaka.

Now, since our focus lies on the study of singular varieties, it becomes essential to extend the notion of a canonical divisor to this broader context.

Definition I.4.3. Let X be a normal variety. A **canonical divisor** on X is a divisor K_X such that

$$\omega_U = \omega_U(K_X|_U)$$

where $U = X_{\mathrm{reg}}$ denotes the smooth locus of X. This notion is well-defined because, due to the normality of X, $\mathrm{codim}_X(X \setminus U) \geq 2$, ensuring that any two canonical divisors on X are linearly equivalent.

Lemma I.4.4. *Let $f : Y \to X$ be a proper birational morphism between normal varieties, and let K_Y be a canonical divisor on Y. Then $f_* K_Y$ is a canonical divisor on X.*

[7] If $f : Y \to X$ is a proper, birational morphism and $E \subset Y$ is a prime divisor, we say that f *extracts* the divisor E.

Proof. Let $\mathrm{Exc}(f) \subset Y$ be the exceptional locus of f, and consider the restriction $V := Y \setminus \mathrm{Exc}(f) \to U := X \setminus \mathrm{Exc}(f)$ of the morphism f. This is an isomorphism and then $f_*(K_Y)|_U$ is a canonical divisor on X. Because f is birational, $\mathrm{codim}_X(U) \geq 2$, and by normality of X, we have that $f_* K_Y$ is a canonical divisor. □

Warning. Whenever we are in the setting of the previous lemma, we will assume that K_X, K_Y are canonical divisors such that $K_X = f_* K_Y$.

Definition I.4.5. Let $f : Y \to X$ be a proper birational morphism of normal varieties, and assume that K_X is $\mathbb{Q}$-Cartier. The relative canonical divisor of f is

$$K_{Y/X} := K_Y - f^* K_X$$

Remark I.4.6. Because K_X is chosen such that $K_X = f_* K_Y$, we have $f_* K_{Y/X} = 0$, i.e., $K_{Y/X}$ is an exceptional divisor.

Example I.4.7. If X is a smooth projective variety and $Z \subset X$ is a smooth subvariety of codimension $r = \mathrm{codim}_X(Z) \geq 2$, the relative canonical divisor of the blow-up $\widetilde{X} = \mathrm{Bl}_Z(X)$ is

$$K_{\widetilde{X}/X} = (r - 1)E,$$

where E is the exceptional divisor.

To carry out the purposes of the MMP, we must be more flexible with the use of canonical divisor, and for this reason is introduced the notion of a log pair.

Definition I.4.8. Let X be a normal variety. We say that (X, Δ) is a *log pair* if Δ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Now, consider (X, Δ) a log pair, $f : Y \to X$ a birational proper morphism with Y normal and Δ_Y a divisor on Y such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

This divisor is independent of K_X because $f_*(\Delta_Y) = \Delta$. This means that for a prime divisor F on X, its coefficient on Δ is the same that the coefficient of its strict transform $\widetilde{F}$ on Δ_Y. Note that by definition this coefficient doesn't depend on f, it is a quantity associated to a *divisor over* X. This is the reason why the following definition makes sense.

Definition I.4.9 (log discrepancy)**.** Let (X,Δ) be a log pair. For every birational proper morphism $f : Y \to X$ with Y normal, we can consider a divisor Δ_Y on Y such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

The log discrepancy of a prime divisor $E \subset Y$ extracted by f, with respect to the pair (X,Δ) is

$$A_{(X,\Delta)}(E) := 1 - (\text{the coefficient of } E \text{ in } \Delta_Y).$$

Example I.4.10. Consider a log pair (X,Δ).

1. If $\Delta = 0$, by definition we have $\Delta_Y = -K_{Y/X}$. In this case log discrepancy is simply $A_X(E) = 1 + \mathrm{coeff}_E(K_{Y/X})$.

2. If $f : Y \to X, g : Z \to Y$ are proper birational morphisms, we can choose canonical divisors K_Y, K_Z such that $f_*(K_Y) = K_X$ and $(f \circ g)_*(K_Z) = K_Y$, and then follows $g_*(K_Z) = K_Y$ (because of the uniqueness of a divisor with this property). If we pullback the relation $K_Y = f^*K_X + K_{Y/X}$ we have

$$K_Z = g^*K_Y + K_{Z/Y} = (f \circ g)^*K_X + g^*K_{Y/X} + K_{Z/Y}$$

   and we obtain the formula

$$K_{Z/X} \overset{\text{def}}{=} K_Z - (f \circ g)^*K_X = g^*K_{Y/X} + K_{Z/Y}.$$

Definition I.4.11. Let (X,Δ) be a pair where X is a normal variety and $\Delta = \sum a_i D_i$ is a sum of distinct prime divisors with $a_i \in \mathbb{Q}$ such that $m(K_X + \Delta)$ is Cartier for some $m > 0$. We say that (X,Δ) is

1. *Kawamata log terminal* (klt) if $A_{(X,\Delta)}(E) > 0$ for all E divisor over X.

2. *purely log terminal* (plt) if $A_{(X,\Delta)}$

3. *log canonical* (lc) if $A_{(X,\Delta)}(E) \geq 0$ for all E divisor over X.

We say X is klt (resp. lc) if the log pair $(X,0)$ is klt (resp. lc).

Remark I.4.12. There is more classes of log pairs that are useful in the context of MMP, but here we will also consider the introduced ones. Between these classes, log canonical is the largest one. The interested reader can consult [KM98].

The following theorem simplifies the very definition of klt and lc.

Theorem I.4.13. *If $(X;\Delta)$ is a log pair and $f : Y \to X$ is a log resolution of $(X;\Delta)$, then $(X;\Delta)$ is lc (resp. klt) if and only if all coefficients of $\Delta_Y \leq 1$ (resp. < 1).*

I.4.2 Log canonical thresholds

In this section we introduce a notion that it will be very useful in Chapter III. We have already introduced the lc class of log pairs, which can be thought as a measure of how singular a variety is. With this sight, it is natural to consider the threshold of a pair for belonging to this class.

Definition I.4.14. Let X be a klt variety. The *log canonical threshold* (or simply lct) of an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor Δ on X is

$$\mathrm{lct}(X;\Delta) := \sup\{\lambda \in \mathbb{Q}_{\geq 0} : (X;\lambda\Delta) \text{ is lc}\}$$

Remark I.4.15. The lct of a pair $(X;\Delta)$ is always positive. In fact, if $f : Y \to X$ is a resolution of singularities of $(X;\Delta)$, since X is klt the divisor $K_{Y/X} \overset{\text{def}}{=} K_Y - f^*K_X$ has coefficients > -1. Hence, for $0 < c \ll 1$, the divisor $K_Y - f^*(K_X - c\Delta)$ has coefficients ≥ -1, and then $(X; c\Delta)$ is lc.

We present some worked examples involving classical resolutions of singularities.

Example I.4.16.

1. Consider $\Delta = V(P)$ where $P \in k[x_1;\dots;x_n]$ is a homogeneous polynomial of degree d such that D has an isolated singularity at the origin. Consider the log pair $(\mathbb{A}^n; \lambda\Delta)$ and let $f : Y \to \mathbb{A}^n$ be the blow-up of the origin. We have the exceptional divisor $E = \mathbb{P}^{n-1}$ and the strict transform $\widetilde{\Delta}$ of the hypersurface Δ. Note that

   $$\widetilde{\Delta} \cap E \subset E = \mathbb{P}^{n-1}$$

   corresponds to the hypersurface in $\mathbb{P}^{n-1}$ defined by P and then $\widetilde{\Delta}$ is smooth and intersects E transversely. Thus, f is a log resolution of $(\mathbb{A}^n; \lambda\Delta)$ and we calculate

   $$(\lambda\Delta)_Y = \lambda f^*(\Delta) - K_{Y/\mathbb{A}^n} = \lambda(\widetilde{\Delta} + dE) - (n-1)E = \lambda\widetilde{\Delta} + (\lambda d - n + 1)E$$

   and then we have $\mathrm{lct}(\mathbb{A}^n;\Delta) = \min\{1; n/d\}$.

2. Take the pair $(X, \Delta) = (\mathbb{A}^2, \Delta)$ with $\Delta = \{Y^2 - X^2(X+1) = 0\}$ the nodal curve. To solve the singularity at $(0,0)$ we do the blow-up of the origin $\pi : \widetilde{X} \stackrel{\text{def}}{=} \mathrm{Bl}_0(X) \to X$ and we calculate the pullback $\pi^*\Delta$.

   We have the blow-up coordinates

$$\widetilde{X} = \{((x,y),[u,v]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xv = yu\}$$

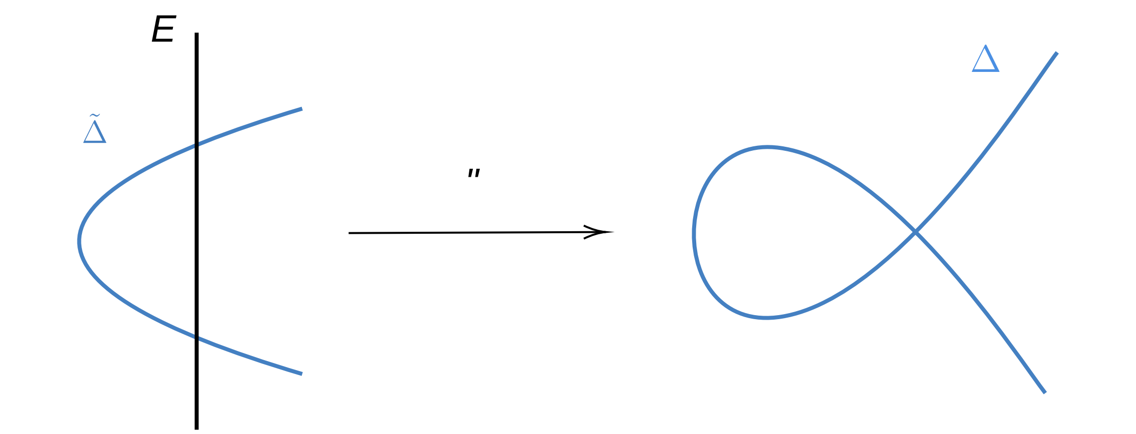
   ad in the affine chart $\{u \neq 0\}$ we have coordinates (x,v) with the relations $x = u, y = xv$ and the exceptional divisor corresponds to $E \stackrel{\text{loc}}{=} \{x = 0\}$. Here we compute

$$\pi^{-1}(\Delta) = \{(x,v) \in \mathbb{A}^2 : x^2v^2 - x^2(x+1) = x^2(v^2 - x - 1) = 0\} \text{ and then } \pi^*\Delta = \widetilde{\Delta} + 2E.$$

   Note that the strict transform $\widetilde{\Delta} \stackrel{\text{loc}}{=} \{(x,v) \in \mathbb{A}^2 : v^2 = 1 + x\}$ intersects transversely with E, so π is a resolution of singularities of the pair. Since $K_X = 0$, $K_{\widetilde{X}} = E$, we have $\Delta_Y = \widetilde{\Delta} + E$ and

$$(X, \Delta) \text{ is lc} \quad (\ ) \quad 1 \qquad 0$$

   and then $\mathrm{lct}(X;\Delta) = 1$. Geometrically this resolution looks like the following picture



3. We will consider the log pair $(X, \Delta) = (\mathbb{A}^2, \Delta)$ where $\Delta = \{Y^2 + X^3 = 0\}$ is the cuspidal cubic. In order to obtain a resolution we perform a blow-up $\pi_1 : X_1 \stackrel{\text{def}}{=} \mathrm{Bl}_0(X) \to X$ at the origin and we will calculate the pullback $\pi_1^*\Delta$.

   Explicitly, the blow-up is given by

$$X_1 = \{((x,y),[x_1,y_1]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xy_1 = x_1y\}$$

In the affine chart $\{y_1 \neq 0\}$, we have the set of coordinates (x_1, y) given by the relations $x = x_1 y, y = y_1$ and locally the exceptional divisor is given by $E_3 \overset{\text{loc}}{=} \{y = 0\}$, so

$$\pi_1^{-1}(\Delta) = \{(x_1, y) \in \mathbb{A}^2 : y^2(x_1^2 + y) = 0\} \text{ and then } \pi_1^*\Delta = \Delta_1 + 2E_3.$$

where Δ_1 denotes the strict transform of Δ, and in the chosen affine chart is given by the equation $\Delta_1 = \{(x_1, y) \in \mathbb{A}^2 : x_1^2 + y = 0\}$. Thus the strict transform is a parabola that intersects the exceptional divisor E_3, so this resolution is not simple normal crossings. We perform a second blow-up $\pi_2 : X_2 \to X_1$ at the intersection point of Δ_1 and E_3. Locally the blow-up looks like

$$X_2 \overset{\text{loc}}{=} \{((x_1, y), [x_2, y_2]) \in \mathbb{A}^2 \times \mathbb{P}^1 : x_1 y_2 = y x_2\},$$

and in the affine chart $\{x_2 \neq 0\}$ we have coordinates (x_1, y_2) such that $x_1 = x_2, y = x_1 y_2$. In this coordinate system the exceptional divisor is $F_2 = \{x_1 = 0\}$ and we calculate:

$$\pi_2^{-1}(\Delta_1) \overset{\text{loc}}{=} \{(x_1, y_2) \in \mathbb{A}^2 : x_1(x_1 + y_2) = 0\} \text{ and then } \pi_2^*(\Delta_1) = \Delta_2 + F_2$$

where $\Delta_2 = \{(x_1, y_2) \in \mathbb{A}^2 : x_1 + y_2 = 0\}$. We also have to calculate the pullback of E_3, which results

$$\pi_2^{-1}(E_3) \overset{\text{loc}}{=} \{(x_1, y_2) \in \mathbb{A}^2 : x_1 y_2 = 0\} \text{ and then } \pi_2^*(E_3) = F_3 + F_2$$

where $F_3 \overset{\text{loc}}{=} \{y_2 = 0\}$ is the strict transform of E_3 in X_2. In this case we obtain a triple intersection point of three lines, so we have to perform a third blow-up $\pi_3 : X_3 \to X_2$ at this point, and again the blow-up is locally given by

$$X_3 \overset{\text{loc}}{=} \{((x_1, y_2), [x_3, y_3]) \in \mathbb{A}^2 \times \mathbb{P}^1 : x_1 y_3 = y_2 x_3\}$$

Using the chart $\{x_3 \neq 0\}$ we have coordinates (x_1, y_3) such that $x_1 = x_3, y_2 = x_1 y_3$, and exceptional divisor $G_1 = \{x_1 = 0\}$. This description gives us

$$\pi_3^{-1}(\Delta_2) \overset{\text{loc}}{=} \{(x_1, y_3) \in \mathbb{A}^2 : x_1(y_3 + 1) = 0\} \text{ and then } \pi_3^*(\Delta_2) = \Delta_3 + G_1$$

where $\Delta_3 = \{(x_1, y_3) \in \mathbb{A}^2 : y_3 + 1 = 0\}$ corresponds to the strict transform of Δ_2 under π_3. In a similar fashion to what was done previously we calculate the pullbacks of F_3, F_2. It results

$$\pi_3^{-1}(F_3) \overset{\text{loc}}{=} \{(x_1, y_3) \in \mathbb{A}^2 : x_1 y_3 = 0\} \quad \text{then} \quad \pi_3^*(F_3) = G_3 + G_1$$
$$\pi_3^{-1}(F_2) \overset{\text{loc}}{=} \{(x_1, y_3) \in \mathbb{A}^2 : x_1 = 0\}$$

where $G_3 = \{y_3 = 0\}$ is the strict transform of F_3 in X_3 respectively.

Note that we didn't obtain equations for the strict transform of F_2 in $\pi_3(F_2)$, so we have to look for in the chart $\{y_3 \neq 0\}$ of X_3. In this case we have coordinates (x_3, y_2) with relations $x_1 = x_3 y_2, y_2 = y_3$ and exceptional divisor $G_1 \overset{\text{loc}}{=} \{y_2 = 0\}$. Now we have

$$\pi_3^{-1}(F_2) \overset{\text{loc}}{=} \{(x_3, y_2) \in \mathbb{A}^2 : x_3 y_2 = 0\} \text{ and then } \pi_3(F_2) = G_2 + G_1$$

where $G_2 = \{x_3 = 0\}$ is the strict transform of F_2. Finally we have arrived to a simple normal crossings configuration.

Denoting by $\pi = \pi_3 \circ \pi_2 \circ \pi_1$ the composition of the three blow-ups, we obtain

$$\begin{aligned} \pi^*(\Delta) &= \pi_3(\Delta_2) + 3\pi_3(F_2) + 2\pi_3 F_3 \\ &= \Delta_3 + 6G_1 + 3G_2 + 2G_3 \end{aligned}$$

Now using a previous example we calculate the relative canonical divisors

$$\begin{aligned} K_{X_1/X} &\overset{\text{def}}{=} K_{X_1} - \pi_1(K_X) = E_3 \\ K_{X_2/X_1} &\overset{\text{def}}{=} K_{X_2} - \pi_1(K_{X_1}) = F_2 \\ K_{X_3/X_2} &\overset{\text{def}}{=} K_{X_3} - \pi_1(K_{X_2}) = G_1 \\ K_{X_2/X} &= \pi_2(K_{X_1/X}) + K_{X_2/X_1} = 2F_2 + F_3 \\ K_{X_3/X} &= \pi_3(K_{X_2/X}) + K_{X_3/X_2} = G_3 + 2G_2 + 4G_1 \end{aligned}$$

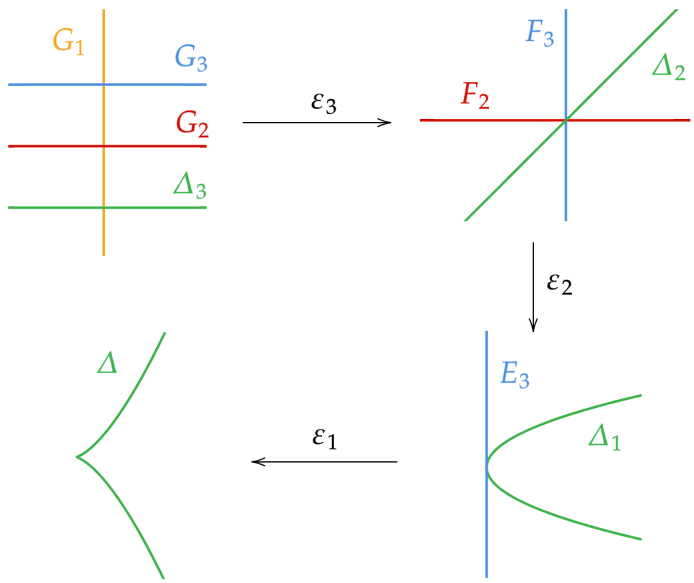
Thus we can compute the divisor Δ_{X_3} in the definition of log discrepancy, obtaining

$$\begin{aligned} \Delta_{X_3} &\overset{\text{def}}{=} \pi^* \Delta - K_{X_3/X} \\ &= (\Delta + 2G_3 + 3G_2 + 6G_1) - (G_3 - 2G_2 - 4G_3) \\ &= \Delta + (2c-1)G_3 + (3c-2)G_2 + (6c-4)G_3. \end{aligned}$$

This computation says that

$$(X, c\Delta) \text{ is lc} \iff 2c-1 \le 1, \quad 3c-2 \le 1, \quad 6c-4 \le 1$$

Thus we conclude $\operatorname{lct}(X, \Delta) = 5/6$. The big picture of this situation is illustrated in the following drawing.



I.5 Volumes of divisors

We now turn to study a notion used to measure the idea of *positivity* of line bundles or divisors. This idea, that comes from the differential geometry side and is highly related to the positivity of the Chern class $c_1(L)$ of a line bundle (hence the name), it is also related to the existence of sections of L, and therefore with classical algebraic geometry questions. The notion of volume measures the asymptotic growth of sections of a line bundle, and it will be turn relevant when we discuss about the valuative criterion of K-stability in Chapter III. In this section we introduce the notion of volume and a geometric way to think about it.

I.5.1 Definition of volume

In this section we define and summarize the principal properties of volume of a big divisor. This section is extracted mainly from [Laz04].

Definition I.5.1. Let X be an irreducible projective variety of dimension n and $L \in \operatorname{Pic}(X)$ a line bundle on X. The volume of the line bundle is defined as

$$\operatorname{vol}(L) = \operatorname{vol}_X(L) = \limsup_{m \to \infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}$$

The volume $\mathrm{vol}(D) = \mathrm{vol}_X(D)$ of a Cartier divisor is the volume of $\mathcal{O}_X(D)$.

Remark I.5.2.

1. Note that $\mathrm{vol}(L)$ is positive if and only if L is big. Moreover, if L is big and nef by an asymptotic version of Riemann-Roch (view [Laz04, Corollary 1.4.41]) we have that $\mathrm{vol}(L) = L^n$ is the top self-intersection.

2. For a positive integer $a > 0$,
$$\mathrm{vol}(aD) = a^n \,\mathrm{vol}(D)$$
where n is the dimension of the variety.

3. The definition of volume also makes sense for $\mathbb{Q}$-Cartier divisors. In fact, if D is $\mathbb{Q}$-Cartier we can take $a > 0$ such that aD is Cartier and define
$$\mathrm{vol}(D) = \frac{1}{a^n}\,\mathrm{vol}(aD)$$
which is independent of a by the previous observation.

The next proposition corresponds to [Laz04, Proposition 2.2.41].

Proposition I.5.3. *If D, D' are numerically equivalent Cartier divisors on X then $\mathrm{vol}(D) = \mathrm{vol}(D')$. Thus, it makes sense to take the volume of a class in $N^1(X)_{\mathbb{Q}}$, and the set of classes of big $\mathbb{Q}$-divisors in $N^1(X)_{\mathbb{R}}$ spans a convex cone.*

Theorem I.5.4 ([Laz04, Theorem 2.2.44])**.** *The function* $\mathrm{vol} : N^1(X)_{\mathbb{Q}} \to \mathbb{R}, \ \xi \mapsto \mathrm{vol}(\xi)$ *extends uniquely to a continuous function* $\mathrm{vol} : N^1(X)_{\mathbb{R}} \to \mathbb{R}$.

To finish this part we present a brief review of [Laz04, §2.3.E] in which is presented a method in surfaces used to extract the components of a divisor in which the volume is concentrated. It will be relevant in the next section and at the final of the Chapter III.

Theorem I.5.5 (Zariski decomposition, [Laz04, Theorem 2.3.19])**.** *Let X be a smooth projective surface and let D be a pseudo-effective integral divisor. Then, it exists a unique decomposition $D = P + N$ for P, N two $\mathbb{Q}$-divisors (called the positive and negative part of D, repectively) with the following properties:*

1. P is nef.

2. *$N = \sum a_i E_i$ is effective and if $N \neq 0$ its intersection matrix $A = (E_i \cdot E_j)_{i,j}$ is negative definite.*

3. *P is orthogonal to any component of N, i.e., $P \cdot E_i = 0$ for all i.*

In addition, the volume of D can be calculated as $\mathrm{vol}(D) = P^2$.

I.5.2 Newton-Okounkov bodies

To finish this chapter we comment a construction due to A. Okounkov following [LM09], which gives a geometric way to calculate volumes of divisors. The idea is to attach a convex body to a line bundle in such a way that the volume of the line bundle coincides with the volume of that convex body. We review this construction and as an example we will focus in the case of surfaces.

Construction I.5.6. Let X be a smooth projective variety and take an admissible flag in X, i.e., a sequence of closed subvarieties

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{p\}$$

with $\mathrm{codim}_X(Y_i) = i$. Let D be a divisor on X, $s = s_1 \in H^0(X, \mathcal{O}_X(D))$ a global section and $D_1 = \mathrm{div}(s_1)$ its effective divisor in the linear system $|D|$. If we define $\nu_1(s) = \mathrm{ord}_{Y_1}(D_1)$ as the multiplicity of Y_1 along D_1, $D_1 - \nu_1(s)Y_1$ is an effective divisor whose support does not contain Y_1, so we can define $D_2 = (D_1 - \nu_1(s)Y_1)|_{Y_1}$, and take $\nu_2(s) = \mathrm{ord}_{Y_2}(D_2)$. Inductively, we define $\nu_{Y_\bullet}(s) = (\nu_1(s), \dots, \nu_n(s))$ and moreover we have constructed a function

$$\nu_{Y_\bullet} : H^0(X, \mathcal{O}_X(D)) \to \mathbb{Z}^n, \quad s \mapsto \nu_{Y_\bullet}(s).$$

The *graded semigroup* of D is the sub-semigroup

$$\Gamma_{Y_\bullet}(D) = \left\{ (\nu_{Y_\bullet}(s), m) \mid 0 \neq s \in H^0\left(X, \mathcal{O}_X(mD)\right), m \geq 0 \right\}$$

of $\mathbb{N}^n \times \mathbb{N}$, and we define the *Newton-Okounkov body of D* with respect to $Y_\bullet$ as

$$\Delta_{Y_\bullet}(D) = \mathrm{cone}\left(\Gamma_{Y_\bullet}(D)\right) \cap \left(\mathbb{R}^n \times \{1\}\right),$$

where $\mathrm{cone}(\Gamma_{Y_\bullet}(D))$ is the closed convex cone in $\mathbb{R}^n \times \mathbb{R}$ spanned by $\Gamma_{Y_\bullet}(D)$.

Theorem I.5.7 ([LM09, Theorem A]). *If D is a big divisor on X, then*

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_Y(D)) = \frac{1}{n!}\ \mathrm{vol}_X(D).$$

Example I.5.8.

1. As an illustration we will consider the projective plane $S = \mathbb{P}^2$ and the flag $Y : S \quad Y_1 := V(X) \quad Y_2 = [0,0,1]$. We will construct the Newton-Okounkov body of each line bundle $\mathcal{O}_S(d)$.

   First of all we take the line bundle $L = \mathcal{O}_S(2)$. In this case we have the global sections $H^0(S, L) = \langle X^2, Y^2, Z^2, XY, XZ, YZ\rangle$ and we calculate explictly $\quad (Y^2)$ and $\quad (XZ)$. Taking $s_1 = Y^2$, we have

   $$D_1 = \mathrm{div}(Y^2) = 2V(Y) \text{ and by definition } {}_1(y^2) = \mathrm{ord}_{Y_1}(D_1) = 0.$$

   Then following the construction we directly compute

   $$D_2 = D_1|_{Y_1} = 2Y_2 \text{ so } {}_2(Y_2) = \mathrm{ord}_{Y_2}(D_2) = 2 \text{ and } {}_Y(y^2) = (0,2).$$

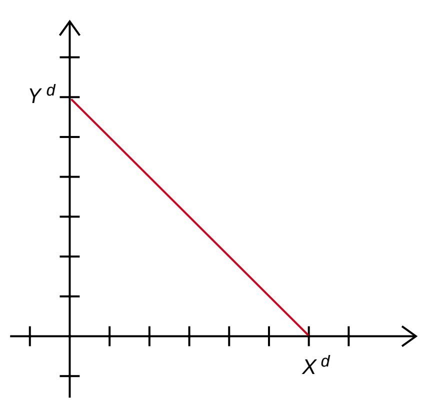
   Similarly, for $s_1 = XZ$ we have

   $$D_1 = \mathrm{div}(XZ) = Y_1 + V(Z) \text{ and } D_2 = V(Z)|_{Y_1} = [0,1,0].$$

   This means ${}_Y(XZ) = (1,0)$ and for the other sections we have

   $${}_Y(X^2) = (2,0), \quad {}_Y(Z^2) = (0,0), \quad {}_Y(XY) = (1,1), \quad {}_Y(YZ) = (0,1)$$

   Following this logic it follows directly that for any $d \in \mathbb{N}^1$ we have that the associated function to $L = \mathcal{O}_X(d)$ is the lexicographic valuation on monomials gave by ${}_Y(X^aY^bZ^c) = (a,b)$. The Newton-Okounkov body of L corresponds simply to



which in effect calculates the volume of L.

2. More generally, if we consider $X = \mathbb{P}^n$ with homogeneous coordinates $X_0, \dots, X_n$ and the flag $Y_\bullet$ given by $Y_i = V(X_0, \dots, X_{i-1})$, the associated map corresponds to the lexicographic valuation on monomials given by

$$\nu_{Y_\bullet}(X_0^{a_0} \cdots X_{n-1}^{a_{n-1}} X_n^{a_n}) = (a_0, \dots, a_{n-1})$$

There exists a global convex body associated to a variety, which track the information of all Newton-Okounkov bodies. The following is a first result in this direction.

Proposition I.5.9 ([LM09, Proposition 4.1])**.** *Let X be an irreducible projective variety, and let D be a big divisor. For a fixed admissible flag $Y_\bullet$ on X, the following is verified.*

1. *If $D \equiv_{\mathrm{num}} D'$ are numerically equivalent, then $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(D')$.*
2. *For any integer $p > 0$, $\Delta_{Y_\bullet}(pD) = p \cdot \Delta_{Y_\bullet}(D)$ where $p \cdot \Delta_{Y_\bullet}(D)$ denotes the homotetic image of $\Delta_{Y_\bullet}(D)$ under scaling.*

The previous Proposition permits to define the Newton-Okounkov body $\Delta_{Y_\bullet}(\xi)$ of a big rational class $\xi \in \mathrm{Big}(X) \cap N_1(X)_{\mathbb{Q}}$. The next theorem asserts the existence of a *global Newton-Okounkov body.*

Theorem I.5.10 ([LM09, Theorem 4.5])**.** *There exists a closed convex cone $\Delta_{Y_\bullet}(X) \subset \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ such that for any $\xi \in \mathrm{Big} \cap N_1(X)_{\mathbb{Q}}$ we have $\mathrm{pr}_2^{-1}(\xi) \cap \Delta_{Y_\bullet} = \Delta_{Y_\bullet}(\xi)$, where $\mathrm{pr}_2 : \mathbb{R}^n \times N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ denotes the projection onto the second coordinate.*

Now we introduce the notion of restricted volume, which is highly related to Newton-Okounkov bodies.

Definition I.5.11. Let X be a smooth projective variety, $L \in \mathrm{Pic}(X)$ a line bundle, $F \subseteq X$ a subvariety of dimension d, and define

$$H^0(X|F; L) := \mathrm{Im}\left(H^0(X; L) \to H^0(X; L|_F)\right).$$

The restricted volume of L along F is

$$\mathrm{vol}_{X|F}(L) = \limsup_{m \to \infty} \frac{h^0(X|F; mL)}{m^d/d!}$$

Remark I.5.12. If D is big and nef, then $\mathrm{vol}_{X|F}(D) = (D^d \cdot F)$.

Theorem I.5.13 ([LM09, Theorem 4.26, Corollary 4.27])**.** *Let X be a normal projective variety of dimension n, let $F \subset X$ be an irreducible and reduced Cartier divisor and fix an admissible flag*

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{pt\}$$

with $Y_1 = F$. Let $\xi \in \mathrm{Big}(X)_{\mathbb{Q}}$ be a big rational class with Newton-Okounkov body $\Delta_{Y_\bullet}(\xi) \subseteq \mathbb{R}^n$. For the projection $\mathrm{pr}_1 : \Delta_{Y_\bullet}(\xi) \to \mathbb{R}$ we set

$$\Delta_{Y_\bullet}(\xi)_{\nu_1 = t} = \mathrm{pr}_1^{-1}(t) \subseteq \{t\} \times \mathbb{R}^{n-1}$$
$$\Delta_{Y_\bullet}(\xi)_{\nu_1 \geq t} = \mathrm{pr}_1^{-1}([t, +\infty)) \subseteq \mathbb{R}^n$$

and $\mu_F(\xi) = \sup\{s > 0 \mid \xi - s \cdot f \in \mathrm{Big}(X)\}$ where f is the numerical class of F. For any $t \in \mathbb{R}$ with $0 \leq t < \mu_F(\xi)$ we have

1. *$\Delta_{Y_\bullet}(\xi)_{\nu_1 \geq t} = \Delta_{Y_\bullet}(\xi - tf) + t \cdot \mathbf{e}_1$ where $\mathbf{e}_1 = (1, 0, \ldots, 0)$.*

2. *$\Delta_{Y_\bullet}(\xi)_{\nu_1 = t} = \Delta_{X|F}(\xi - tf)$.*

The following theorem gives a very useful description of the Newton-Okounkov body on surfaces that it will be used in Chapter III.

Theorem I.5.14. *Let S be a smooth projective surface and let $Y_\bullet : Y_0 = S \supseteq Y_1 = C \supseteq Y_2 = \{p\}$ be an admissible flag with $C \subset S$ an smooth curve through $p \in S$. A big $\mathbb{Q}$-divisor D on S has a Zariski decomposition $D = P(D) + N(D)$ with $P(D)$ being nef and $N(D)$ being either zero or effective with negative definite intersection matrix. The Newton-Okounkov body of D is given by*

$$\Delta_{Y_\bullet}(D) = \left\{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu_C(D),\ \alpha(t) \leq y \leq \beta(t)\right\}$$

where $D - tC = P_t + N_t$ is the Zariski decomposition, and

1. *$\nu \in \mathbb{Q}$ is the coefficient of C in $N(D)$,*

2. *$\mu_C(D) = \sup\{t > 0 : D - tC$ is big$\}$,*

3. *$\alpha(t) = \mathrm{ord}_p(N_t \cdot C)$,*

4. *$\beta(t) = \mathrm{ord}_p(N_t \cdot C) + P_t \cdot C$.*

Remark I.5.15. The Newton-Okounkov body of a divisor could be non-polyhedral. An example of this phenomena is given in [LM09, §6.3].

Chapter II

Fano varieties

In this chapter we present main facts about Fano varieties and their classification. The principal idea is to introduce the Δ-genus theory of Takao Fujita, which permited to classify polarized varieties in terms of numerical invariants. In particular, we'll obtain the classification of Fano varieties of large index.

II.1 General properties of Fano varieties

We start recalling the definition of a Fano variety.

Definition II.1.1. A Fano variety is a smooth complex projective variety X such that its anticanonical divisor K_X is ample.

Remark II.1.2. As we assume in the previous definition, a Fano variety is usually defined as a smooth variety. A singular normal projective variety X whose anticanonical divisor is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is usually called a *singular Fano variety*. It is also very common to specify the type of singularities that X has in its name (for example, a klt Fano variety).

Example II.1.3.

1. The unique Fano curve is the projective line $\mathbb{P}^1$.
2. Two-dimensional Fano varieties are the so-called *del Pezzo surfaces*, which are characterized below.

3. If $X = X_{d_1;\dots;d_r} \subset \mathbb{P}^n$ is a smooth complete intersection of degrees $d_1;\dots;d_r$ (i.e., $X = V(f_1;\dots;f_r)$ where $\deg(f_i) = d_i$), X is a Fano variety if and only if $\sum_{i=1}^r d_i \leq n$. This characterization follows from the adjunction formula because

$$\omega_X^{\vee} = \mathcal{O}_X(-K_X) = \mathcal{O}_X\left(n+1-\sum_{i=1}^r d_i\right).$$

Proposition II.1.4. *Let X be a Fano variety. Then*

1. $H^i(X;\mathcal{O}_X) = 0$ $\forall i > 0$.

2. $\mathrm{Pic}(X)$ *is a finitely generated torsion-free* $\mathbb{Z}$*-module and* $\mathrm{Pic}(X) = H^2(X;\mathbb{Z})$.

3. $\kappa(X) = -\infty$.

Proof. Assertion 1*.* is due to Kodaira vanishing theorem (view [Laz04, Theorem 4.2.1]), noting that $H^i(X;\mathcal{O}_X) = H^i(X;K_X - K_X) = 0$ $\forall i > 0$, and asssertion 3*.* is simply the observation that any power of K_X has no sections because of the ampleness of $-K_X$. Now we prove 2*.* From the exponential sheaf exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0,$$

we have the exact sequence

$$0 = H^1(X;\mathcal{O}_X) \to H^1(X;\mathcal{O}_X^*) \to H^2(X;\mathbb{Z}) \to H^2(X;\mathcal{O}_X) = 0;$$

where the vanishing comes from 1*.*, and we obtain

$$\mathrm{Pic}(X) = H^1(X;\mathcal{O}_X^*) = H^2(X;\mathbb{Z});$$

where the first isomorphism is well-known. This implies $\mathrm{Pic}(X)$ is finitely generated. We prove now $\mathrm{Pic}(X)$ is torsion-free, and note first that torsion-free elements are numerically trivial. Let $D \equiv_{\mathrm{num}} 0$ be a numerically trivial divisor and by Nakai-Moishezon criterion $-K_X + D$ is ample. Kodaira vanishing theorem shows that

$$H^i(X;D) = H^i(X;K_X + (-K_X + D)) = 0 \quad \forall i > 0.$$

As $D \equiv_{\mathrm{num}} 0$, it follows that $\chi(X;\mathcal{O}_X(D)) = \chi(X;\mathcal{O}_X) = 1$ and $H^0(X;D) = \mathbb{C}s$ for some section s. Then $H^0(X;nD) = \mathbb{C}s^n$ for any $n \in \mathbb{N}$ and if s is non-constant, D is a non-torsion element. Thus, if D is a torsion element its unique section is constant, and then $D \sim 0$. □

Now we present the characterization of de del Pezzo surfaces.

Proposition II.1.5. *Let X be a smooth del Pezzo surface. Then X is isomorphic to the blow-up of projective plane $\mathbb{P}^2$ at r points in general position*[1] *where $0 \leq r \leq 8$, or it is $X = \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let X be a smooth del Pezzo surface. If X is not a minimal surface (in which case there exists a curve $E \subset X$ such that $E \cong \mathbb{P}^1$ and $E^2 = -1$) by Castelnuovo's contractibility criterion (see [Bea83, Theorem II.17]) there exists a unique morphism $\varphi : X \to Y$ such that Y is smooth, $\mathrm{Exc}(\varphi) = E$ and $\varphi(E) = \{\mathrm{pt}\}$, i.e., we have $K_X = \varphi^* K_Y + E$. Since $-K_X$ is ample, by projection formula we calculate

$$\begin{aligned}(-K_Y)^2 = K_Y^2 = (\varphi^* K_Y)^2 = (K_X - E)^2 &= K_X^2 - 2K_X \cdot E + E^2 \\ &= K_X^2 - 2 \cdot (-1) - 1 \\ &= K_X^2 + 1 > 0,\end{aligned}$$

and if $C \subset X$ is an irreducible curve different from E,

$$(-K_Y) \cdot \varphi_* C = \varphi^*(-K_Y) \cdot C = (-K_X + E) \cdot C = -K_X \cdot C + E \cdot C > 0,$$

so by Nakai-Moishezon criterion $-K_Y$ is an ample divisor. This argument shows that we can suppose X is a minimal del Pezzo surface.

Let X be a minimal del Pezzo surface, and we claim X is rational. First, we note that $P_2(X) \overset{\text{def}}{=} h^0(X, 2K_X) = 0$, because if $D \geq 0$ is an effective divisor such that $D \sim 2K_X$ by Nakai-Moishezon criterion $-K_X \cdot D > 0$, but we also have $-K_X \cdot D = -2K_X^2 < 0$, a contradiction. Second, the irregularity is $q(X) \overset{\text{def}}{=} h^1(X, \mathcal{O}_X) = 0$ (Proposition II.1.4), and then X is rational because of Castelnuovo's rationality criterion ([Bea83, Theorem V.1]).

Classification of minimal rational surfaces ([Bea83, Theorem V.10]) implies that X is, either the projective plane $\mathbb{P}^2$, or a Hirzebruch surface $\mathbb{F}_n \overset{\text{def}}{=} \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)), n \neq 1$. Suppose $X = \mathbb{F}_n$ for some $n \in \mathbb{Z}$. This surface contains a rational curve $S_n \cong \mathbb{P}^1$ such that $S_n^2 = -n$ ([Bea83, Theorem IV.1]), and then genus formula gives

$$2g(S_n) - 2 = S_n^2 + K_X \cdot S_n \text{ and thus } -K_X \cdot S_n = 2 - n;$$

[1] This condition means there are no three collinear points, there are no six points lying in a conic, and there are no 8 points lying on a cubic with one of them being a double point.

and the ampleness of $-K_X$ forces that $n = 0$, so $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Remember that $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the blow-up of $\mathbb{P}^2$ at two points, so any non-minimal del Pezzo surface can be obtained as the blow-up of $\mathbb{P}^2$ at r points. Note that

$$9 = (-K_{\mathbb{P}^2})^2 = (-K_X)^2 + r$$

and then $0 \leq r \leq 8$. □

Definition II.1.6. Let X be a Fano variety. By the previous Proposition there exists the greatest rational number $r = \iota(X) > 0$ such that $-K_X = rH$ where H is an ample Cartier divisor. The number r is called the *index* of X and the divisor H is called a *fundamental divisor*.

Lemma II.1.7. *Let X be a Fano variety of dimension n. Then the index $\iota(X)$ satisfies the bound $\iota(X) \leq n+1$.*

Proof. By bend-and-break lemma X contains a rational curve C such that

$$\iota(X) \leq -K_X \cdot C \leq n+1.$$

□

II.2 Classification of Fano varieties

II.2.1 Definition of Δ-genus

In this section we will introduce the concept of Δ-genus defined by T. Fujita in [Fuj75]. We will present some basic properties and we will illustrate his method of classification using *ladders*. The main idea of T. Fujita is to consider projective varieties with a chosen polarization.

Definition II.2.1 (sectional and Δ-genus)**.** Let (X, H) be a polarized algebraic variety, i.e., $H \in \mathrm{Pic}(X)$ is an ample line bundle. Consider the Hilbert polynomial

$$\chi(X, tH) = \sum_{j=0}^{n} \chi_j(X, tH)\, t^{[j]}/j! \quad \text{where} \quad t^{[j]} := \frac{(t+j-1)!}{(t-1)!}.$$

We define the *degree* of H as $d := \chi_n(X, tH)$ and the *sectional genus* as $g(X, H) := 1 - \chi_{n-1}(X, tH)$. The Δ-*genus* of (X, H) is defined as $\Delta(X, H) := \dim(X) + d - h^0(X, H)$.

Lemma II.2.2. *Let $(X;H)$ be a polarized algebraic variety and let $D \in |L|$ be an irreducible and reduced member. Then $\chi_{j+1}(X;H) = \chi_j(D;H|_D)$ for every $j \geq 0$.*

Proof. The exact sequence

$$0 \to \mathcal{O}_X((t-1)H) \to \mathcal{O}_X(tH) \to \mathcal{O}_D(tH|_D) \to 0$$

gives the equality $\chi(D;tH) = \chi(X;tH) - \chi(X;(t-1)H)$. Comparing the coefficients shows $\chi_{j+1}(X;H) = \chi_j(D;H|_D)$. □

The following lemma justifies the name sectional genus.

Lemma II.2.3. *Let $(X;H)$ be a polarized algebraic variety and let $D \in |L|$ be an irreducible and reduced member.*

1. *If* $\dim(X) = 1$, $g(X;H) = h^1(X;\mathcal{O}_X)$.
2. $g(D;H|_D) = g(X;H)$.
3. $2g(X;H) - 2 = (K_X + (n-1)H) \cdot H^{n-1}$.

Proof.

1. By Riemann-Roch theorem for curves, $\chi(X;tH) = \chi(X;\mathcal{O}_X) + t\deg(H)$, and then $g(X;H) = 1 - \chi_0(X;tH) = 1 - \chi(X;\mathcal{O}_X) = h^1(X;\mathcal{O}_X)$.
2. This is a direct consequence of the previous Lemma.
3. This formula can be obtained from a more precise version of Riemann-Roch theorem (see [BHJ17, Theorem A.1]).

□

Proposition II.2.4. *Let $(X;H)$ be a polarized algebraic variety and let $D \in |L|$ be an irreducible and reduced member. Then $0 \leq \Delta(X;H) - \Delta(D;H|_D) \leq h^1(X;\mathcal{O}_X) \leq h^1(D) + h^1(X;-H)$ and the following facts are equivalents:*

1. *The restriction map $H^0(X;H) \to H^0(D;H|_D)$ is surjective.*
2. $\Delta(D;H|_D) = \Delta(X;H)$.

Proof. Consider the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_D(H) \to 0$ and the long exact sequence

$$0 \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X(H)) \xrightarrow{r} H^0(D,\mathcal{O}_D(H)) \to H^1(X,\mathcal{O}_X)$$

Then we have

$$h^1(X,\mathcal{O}_X) \geq h^0(D,\mathcal{O}_D(H)) - \dim\ker(\delta) = \Delta(X,H) - \Delta(D,H|_D)$$

and

$$0 \leq \dim\operatorname{coker}(r) = \Delta(X,H) - \Delta(D,H|_D) \leq h^1(X,\mathcal{O}_X)$$

From the exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

we have $h^1(X,\mathcal{O}_X) \leq h^1(X,-H) + h^1(D)$. □

The previous results justifies the following definition, which is the main concept in the theory of T. Fujita.

Definition II.2.5. A divisor $D \in |H|$ of a polarized variety (X,H) is called a *rung* if it is irreducible and reduced, and it is called *regular* if $\Delta(D,H|_D) = \Delta(X,H)$. A *ladder* in (X,H) is a sequence of varieties $X = D_n \supset D_{n-1} \supset \dots \supset D_1$ such that $\dim(D_i) = i$ and D_i is a rung of $(D_{i+1},H|_{D_{i+1}})$.

Theorem II.2.6 ([Fuj77, Theorem 4.1], [Fuj90, Theorem 3.5])**.** *Let (X,H) be a polarized variety such that $\Delta(X,H) \leq g(X,H)$ and $\dim\operatorname{Bs}|H| < 0$. If $d(X,H) \geq 2\Delta(X,H) - 1$ then*

1. *(X,H) has a regular ladder.*
2. *$g(X,H) = \Delta(X,H)$ and H is simply generated (in particular is very ample).*

The following theorem is a fundamental tool in the characterization by Δ-genus.

Theorem II.2.7 ([Fuj90, Theorem 4.2])**.** *Let (X,H) be a polarized algebraic variety. Then $\Delta(X,H) > \dim\operatorname{Bs}|H|$ where $\operatorname{Bs}|H|$ is the base locus of the linear system $|H|$ (here we put $\dim\operatorname{Bs}|H| = -1$ if $\operatorname{Bs}|H| = \emptyset$). In particular $\Delta(X,H) \geq 0$.*

II.2.2 Classification of varieties with Δ-genus zero

In this section we present the ideas presented by T. Fujita in its work [Fuj75], in which he gave the complete classification of polarized varieties (X, H) with $\Delta(X, H) = 0$. The goal is to prove the following classification theorem.

Theorem II.2.8 ([Fuj90, Theorem 5.10])**.** *Let (X, H) be a smooth polarized algebraic variety with $n = \dim(X) \geq 2$ and $\Delta(X, H) = 0$. Then*

1. *if $d(X, H) = 1$ then $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*
2. *if $d(X, H) = 2$, X is a hyperquadric in $\mathbb{P}^{n+1}$, and $H = \mathcal{O}_X(1)$.*
3. *if $d(X, H) \geq 3$, $X = \mathbb{P}(E)$, where E is a direct sum of positive degree line bundles on $\mathbb{P}^1$.*
4. *$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*

Proof. For the moment we'll prove the characterization for $d = d(X, H) \in \{1, 2\}$. Suppose $d = 1$ (resp. $d = 2$). Since $\Delta(X, H) = 0$, by Theorem II.2.7 we have $\mathrm{Bs}\,|H| = \emptyset$, and there is an associated morphism $\varphi = \varphi_H : X \to \mathbb{P}^n$ (resp. $\varphi : X \to \mathbb{P}^{n+1}$) such that $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_X(H)$ (resp. $\varphi^* \mathcal{O}_{\mathbb{P}^{n+1}}(1) = \mathcal{O}_X(H)$). Since H is a basepoint-free ample divisor, φ is a finite morphism and then $Y = \varphi(X)$ has $\deg(Y) = 1$ (resp. $\deg(Y) \geq 2$). By the projection formula (Theorem I.1.13) we see

$$d(X, H) = H^n = \deg(\varphi) \cdot \deg(Y)$$

we have $\deg(\varphi) = 1$. This means φ is birational and by Zariski's main theorem $X = \mathbb{P}^n$ (resp. $X = Y$ is a hyperquadric). $\square$

To prove the other characterizations we will need to study varieties with Δ-genus zero more deeply. In the rest of this section we assume (X, H) is a smooth polarized variety of dimension $n = \dim(X)$ with $\Delta(X, H) = 0$.

Lemma II.2.9. *If $\Delta(X, H) = 0$ then $g(X, H) = 0$ and $h^i(X, tH) = 0$ for $i > 0, t \geq 0$. In particular, if $n = \dim(X) = 1$, $X = \mathbb{P}^1$.*

Proof. We prove it by induction on n. If $n = 1$, we have $0 = \Delta(X, H) = 1 + d - h^0(X, H)$ and by Riemann-Roch theorem $h^0(X, H) - h^0(X, K_X - H) = d + 1 - g(X, H)$, and then

$$g(X, H) \overset{\text{def}}{=} h^0(X, K_X) = h^0(X, K_X - H).$$

Thus, necessarily we have $g(X,H) = 0$, since otherwise $H \sim 0$.

Suppose $n \geq 2$ and take $D \in |H|$ a general member of the linear system. As $\mathrm{Bs}\,|H| = \emptyset$, by Bertini's theorem D is smooth. By Proposition II.2.4, $0 \leq \Delta(D, H|_D) \leq \Delta(X,H) \leq 0$, so by induction $g(D, H|_D) = g(X,H)$. Using now induction in the cohomology vanishing claim, we have $h^i(D, tH|_D) = 0$ for $i > 0, t \geq 0$, and the exact sequence $0 \to \mathcal{O}_X((t-1)H) \to \mathcal{O}_X(tH) \to \mathcal{O}_D(tH|_D) \to 0$ proves

$$h^i(X,(t-1)H) = h^i(X,tH) \quad \text{for} \quad i > 0, t \geq 0,$$

and the ampleness of H implies cohomology vanishing for $t \gg 0$. □

Lemma II.2.10. *The dimension of the space of sections of $K_X + nH$ is $h^0(K_X+nH) = d-1$ where $d = d(X,H)$ is the degree.*

Proof. We prove by induction on n. If $n = 1$, by Riemann-Roch theorem

$$h^0(K_X+H) - \underbrace{h^0(-H)}_{=0} = \deg(K+H) + 1 - g(X,H) = d - 1$$

Now we take $D \in |H|$ a smooth general member. We obtain the exact sequence

$$\underbrace{H^0(X,K+(n-1)H)}_{=0} \to H^0(K_X+nH) \to H^0(D,K_X+nH) \to \underbrace{H^1(K_X+(n-1)H)}_{=0}$$

where the vanishing of the right hand side is by Kodaira vanishing theorem, and the sectional genus formula

$$(K_X+(n-1)L)\cdot L^{n-1} = 2g(X,H) - 2 = -2 < 0$$

gives us the left hand side vanishing. By adjunction formula $K_D = (K_X+H)|_D$ and then

$$H^0(X,K_X+nH) = H^0(D,(K_X+nH)|_D) = H^0(D,K_D+(n-1)H|_D)$$

so induction works. □

Lemma II.2.11. *The dimension of the base locus of $|K_X+nH|$ verifies* $\dim \mathrm{Bs}\,|K_X+nH| < n-1$ *when $d = d(X,H) \geq 3$.*

Proof. The case $n = 1$ is clear (because $X = \mathbb{P}^1$). For $n \geq 2$ take $D \in |H|$ a general smooth member. By induction, noting that $D \cap \mathrm{Bs}\,|K + nH| = \mathrm{Bs}\,|K_D + (n-1)H_D|$ we estimate

$$\dim \mathrm{Bs}\,|K_M + nH| \leq 1 + \dim(\mathrm{Bs}\,|K_M + nH| \cap D) = 1 + \dim \mathrm{Bs}\,|K_D + (n-1)H_D|.$$

□

Proof of Theorem 2.3.1 (continuation). Now we suppose $d \leq 3$. The rest of the proof will be done in fourth steps.

Step 1. We will prove the following statement:

When $n = 2$, we have $(K_X + 2H)^2 = 0$ unless $(X, H) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

Note that by Lemma II.2.10 it is verified that $(K_X + 2H)^2 \geq 0$, because there exists at least 2 effective divisors linearly equivalents to $K_X + 2H$. Sectional genus formula gives $(K_X + H) \cdot H = 2g(X, H) - 2 = -2$ so

$$(K_X + 2H)^2 = K_X^2 + 4(K_X \cdot H + H^2) = K_X^2 - 8 \geq 0.$$

On the other hand, by Noether's formula ([Bea83, I.14])

$$12 = 12\chi(\mathcal{O}_X) = K_X^2 + \chi_{\mathrm{top}}(X),$$

and $\chi_{\mathrm{top}}(X)$ (the topological Euler-Poincaré characteristic of X) is given by Betti numbers and

$$\chi_{\mathrm{top}}(X) = \sum_{i=0}^{5} (-1)^i b_i(X) = 2(b_0(X) - b_1(X)) + b_2(X) = 2 + b_2(X) \geq 3,$$

where we used that $b_1(X) = 2q(X) \overset{\text{def}}{=} 2h^1(X, \mathcal{O}_X) = 0$. The fact that $b_2(X) > 0$ is because the polarization H as an algebraic cycle defines a non-zero fundamental class $[H] \in H_{2n-2}(X, \mathbb{Z})$ and by Poincaré duality $H^2(X, \mathbb{Z}) \neq 0$. The two previous inequalities shows either $K_X^2 = 8$ or $K_X^2 = 9$. If $K_X^2 = 9$ then $b_2(X) = 1$ and the classification of surfaces gives $X = \mathbb{P}^2$. Since $(K_X + 2H)^2 = 1$, we obtain $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^2}(2)$. Henceforth we suppose $(X, H) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

Step 2. When $n \geq 3$, we claim that $(K_X + nH)^2 \cdot H^{n-2} = 0$. Note that it is enough to prove for $n = 3$, because if $D \in |H|$ is a smooth member, by induction the adjuntion formula shows

$$(K_X + nH)^2 \cdot H^{n-2} = (K_X + nH)|_D^2 \cdot H|_D^{n-3} = (K_D + (n-1)H|_D)^2 \cdot H|_D^{n-3}$$

Then we take $n = 3$ and $D \in |H|$ a smooth member, and note that by the previous step it is sufficient to prove $(D, H|_D) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Suppose by contradiction that we have $(D, H|_D) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. By Lefschetz's hyperplane section theorem (see [EH16, §C.4]) the map $H_2(D, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ is surjective, and $H_2(D, \mathbb{Z}) = \mathbb{Z}$. This implies $H_2(X, \mathbb{Z}) = \mathbb{Z}$ and by Proposition II.1.4

$$\mathrm{Pic}(X) = H^2(X, \mathbb{Z}) = H^2(D, \mathbb{Z}) = \mathrm{Pic}(D)$$

so there exists a line bundle $L \in \mathrm{Pic}(X)$ such that $L|_D = \mathcal{O}_{\mathbb{P}^2}(1)$. This implies $H = 2L$ and

$$4 = H|_D^2 = H^3 = 8L^3,$$

which is a contradiction. Then the claim $(K_X + nH)^2 \cdot H^{n-2} = 0$ follows.

Step 3. We will prove the next statement:

$\mathrm{Bs}|K + nH| = \emptyset$ and the image of the morphism defined by $|K + nH|$ is a curve.

Consider the rational map $\varphi = \varphi_{|K_X+nH|}$ associated to the complete linear system $|K_X + nH|$. Note that if $D_1, D_2 \in |K_X + nH|$ are two general members the Lemma II.2.11 shows $\dim(D_1 \cap D_2) < n - 1$, and if $D_1 \cap D_2 \neq \emptyset$ we have $(K_X + nH)^2 \cdot H^{n-2} = D_1 \cdot D_2 \cdot H^{n-2} > 0$ since H is ample. The previous calculation shows that $D_1 \cap D_2 = \emptyset$ and it follows that $\mathrm{Bs}|K_X + nH| = \emptyset$, so φ is in fact a morphism. Moreover, if $Y = \varphi(X)$ this also implies $\dim(Y) = 1$ (because two general hyperplane sections of Y doesn't intersect). In more detail, the computation

$$0 = D_1 \cdot D_2 = \varphi_*(\varphi^*\mathcal{O}_{\mathbb{P}^n}(1) \cdot D) = \mathcal{O}_{\mathbb{P}^n}(1) \cdot \varphi_*\varphi^* D = \mathcal{O}_{\mathbb{P}^n}^2 \cdot Y = \mathcal{O}_Y(1)^2$$

gives the desired conclusion.

Step 4. Take the morphism $\varphi : X \to Y = \varphi(X) \subseteq \mathbb{P}^{d-2}$ as before and denote $y = \deg(Y)$. If F is a general fiber of φ, note that by sectional genus formula

$$(K_X + (n-1)H) \cdot H^{n-1} = 2g(X, H) - 2 = -2 \implies (K_X + nH) \cdot H^{n-1} = d - 2$$

and thus we arrive to

$$d - 2 = (K_X + nH) \cdot H^{n-1} = yF \cdot H^{n-1}$$

where we have used the fact that any two fibers of φ are numerically equivalent. This allows us to calculate

$$0 \leq \Delta(Y, \mathcal{O}_Y(1)) \leq 1 + w - (d - 1) = y - (d - 2)$$

and we get the conditions

$$y = d - 2; \quad F \cdot H^{n-1} = 1 \quad \text{and} \quad \Delta(Y, \mathcal{O}_Y(1)) = 0.$$

Then, $Y = \mathbb{P}^1$ and any fiber F' of φ is irreducible and reduced because $H^{n-1} \cdot F = H^{n-1} \cdot F' = 1$. Since the restriction $H|_F$ is basepoint-free and $H|_F^{n-1} = 1$, the proof of the first case shows $(F, H|_F) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ for any fiber, and the conclusion follows, for instance, from [Wat14, Proposition 2.5]. $\square$

The previous classification theorem gives us in particular the following classification for Fano varieties of large index.

Corollary II.2.12 (Kobayashi-Ochiai)**.** *Let X be a Fano variety of dimension $n = \dim(X)$ and index $r = \iota(X) \geq n$. Then X is one of the following:*

1. $X = \mathbb{P}^n$ *if* $r = n+1$.
2. $X = Q \subset \mathbb{P}^{n+1}$ *is a hyperquadric if* $r = n$.

Proof. We know Hilbert polynomial $\chi(X, \mathcal{O}_X(tH))$ is a degree n polynomial in the variable t, $\chi(X, \mathcal{O}_X) = 1$ and by Serre duality

$$\chi(X, \mathcal{O}_X(-rH)) = \chi(X, \omega_X) = (-1)^n \chi(\mathcal{O}_X) = (-1)^n$$

By Kodaira vanishing theorem

$$h^i(X, \mathcal{O}_X(tH)) = h^i(X, K_X - (K_X - tH)) = h^i(X, K_X + (t+r)H) = 0 \quad \forall i > 0, \ t > -r.$$

Since we already know $n+1$ values of $\chi(X, \mathcal{O}_X(tH))$, this fixes the polynomial and we conclude:

$$\chi(X, \mathcal{O}_X(tH)) = \begin{cases} \binom{t+n}{n} & \text{if } r = n+1 \\ \binom{t+n+1}{n+1} - \binom{t+n-1}{n+1} & \text{if } r = n. \end{cases}$$

From the above formula we calculate

$$h^0(X, \mathcal{O}_X(H)) = \chi(X, \mathcal{O}_X(H)) = n + d$$

so $\Delta(X, H) = 0$, and when $r = n+1$ (resp. $r = n$) we have $d(X, H) = 1$ (resp. $d(X, H) = 2$). The conclusion follows from the classification of varieties with Δ-genus 0. $\square$

II.2.3 Classification of del Pezzo manifolds

Since we already understand Fano varieties X with index $\mathrm{ind}(X) \geq \dim(X)$, the next step will be jump to varieties with index $\mathrm{ind}(X) = \dim(X) - 1$. In this section we present the classification of such varieties, follow mainly the expositions in [Fuj80, Fuj81, Fuj90].

Definition II.2.13. A n-dimensional del Pezzo variety is a Fano variety X with index $\mathrm{ind}(X) = n - 1$.

The first crucial observation is that del Pezzo varieties can be characterized using Δ-genus invariant.

Theorem II.2.14 ([Fuj80, Theorem 1.9])**.** *Let (X, H) be a polarized variety of dimension $n = \dim(X)$. Then $-K_X = (n-1)H$ if and only if $\Delta(X, H) = g(X, H) = 1$.*

With this numerical characterization, T. Fujita completely classified del Pezzo varieties. This is summarized in the following theorem.

Theorem II.2.15 ([Fuj80, Fuj81])**.** *Let (X, H) be a del Pezzo variety of dimension $\dim(X) = n \geq 3$, i.e., $-K_X = (n-1)H$ with H ample, and let $d = H^n$ be its degree. Then $1 \leq d \leq 8$ and:*

1. *If $d = 1$, X is a hypersurface of degree 6 in $\mathbb{P}(1^n, 2, 3)$.*
2. *If $d = 2$, $X \xrightarrow{\pi} \mathbb{P}^n$ is a double cover branched along $B_4 \subset \mathbb{P}^n$, a smooth hypersurface of degree 4, and $H = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$.*
3. *If $d = 3$, X is a cubic hypersurface in $\mathbb{P}^{n+1}$, with $H = \mathcal{O}_X(1)$.*
4. *If $d = 4$, $X = X_{2,2} \subset \mathbb{P}^{n+2}$ is a smooth complete intersection of two quadrics.*
5. *If $d = 5$, $X = G \cap H$ is a smooth hyperplane section of*
$$\mathbb{G}(1, 4) = G \subset \mathbb{P}(\textstyle\bigwedge^2 \mathbb{C}^5) = \mathbb{P}^9.$$
6. *If $d = 6$, then X is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^2$, or $\mathbb{P}(T_{\mathbb{P}^2})$, where $T_{\mathbb{P}^2}$ denotes the tangent bundle of $\mathbb{P}^2$.*
7. *If $d = 7$, $X = \mathrm{Bl}_p(\mathbb{P}^3)$ is the blow-up of $\mathbb{P}^3$ at a point.*
8. *If $d = 8$, then $(X, H) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.*

The most complicated cases of this classification theorem are $d = 1, 5$, and we won't be discussed here. As an illustration we will finish this section discussing the easiest cases $d = 3, 4$, and in the following sections we will be interested on explicit constructions of del Pezzo fourfolds of degree 5.

Proof of the cases $d = 3, 4$. When the degree is $d = 3$, by definition we have $h^0(X, H) = n + 2$ and we know H is very ample by Theorem II.2.6. Then, directly we have $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree 3, i.e., is a cubic hypersurface with the polarization $H = \mathcal{O}_X(1)$.

In the case $d = 4$, we also have H is very ample and now we have $h^0(X, H) = n + 3$ and an embedding $X \xhookrightarrow{|H|} \mathbb{P}^{n+2}$. We can compute

$$h^0(X, 2H) = \chi(X, H) = \frac{1}{2}(n^2 + 7n + 8) = h^0(\mathbb{P}^{n+2}, 2H) - 2,$$

so there exists two hyperquadrics in $\mathbb{P}^{n+2}$ containing X, which gives the conclusion. $\square$

II.3 Birational geometry of Fano fourfolds

In this section, our attention turns to the 4-dimensional case. The primary objective is to gain a deeper understanding of the birational geometry of del Pezzo fourfolds with degree $d = 4, 5$. We present some results about these varieties which can be found in [PZ16, PZ17], and we also expand several calculations. We will freely use the notation and results from Intersection Theory as in [Ful84] and [EH16].

This study lays a critical foundation for the subsequent analysis of Fano-Mukai fourfolds of genus 9, which will be addressed in Chapter IV.

II.3.1 Geometry of fourfolds

We begin studying smooth projective varieties of dimension 4 in general. First, we stablish explicitly the *Hirzebruch-Riemann-Roch theorem* ([Har77, Theorem A.4.1]) in dimension 4.

Theorem II.3.1. *Let X be a smooth projective fourfold and D a divisor on X. Then*

$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{24}\left[D^4 + 2D^3 \cdot c_1(X) + D^2 \cdot \left(c_1(X)^2 + c_2(X)\right) + \right.$$
$$\left. + D \cdot c_1(X) \cdot c_2(X)\right] + \chi(\mathcal{O}_X).$$

Remark II.3.2. Note that if X is rationally connected[2] (e.g. if X is a Fano variety) then $\chi(X, \mathcal{O}_X) = 1$.

A fundamental tool for our purposes will be the knowledge of the Chow ring of a blow-up in dimension 4, and the explicit formulas for blowing-up Chern classes. The following theorem can be obtained from [Ful84, Theorem 15.4].

Lemma II.3.3. *Let X be a smooth projective fourfold, and $\pi : \widetilde{X} \to X$ the blow-up along a smooth subvariety $Z \subset X$ with exceptional divisor E.*

1. *If $Z = p \in X$ is a point and $H \in \mathrm{Pic}(X)$, the following relationships are verified:*

$$K_{\widetilde{X}} = \pi^* K_X + 3E \text{ and } c_2(\widetilde{X}) = \pi^* c_2(X) + 2E^2.$$

Moreover,

$$(\pi^*H)^4 = H^4, \quad (\pi^*H)^3 \cdot E = (\pi^*H)^2 \cdot E^2 = (\pi^*H) \cdot E^3 = 0, \text{ and } E^4 = -1.$$

2. *If $Z = C \subset X$ is a curve of genus $g(C)$, $A \in \mathrm{CH}^2(\widetilde{X})$ is the class of a fiber of the $\mathbb{P}^2$-bundle $\pi|_E : E \to C$, and $H \in \mathrm{Pic}(X)$, we have*

$$K_{\widetilde{X}} = \pi^* K_X + 2E \text{ and } c_2(\widetilde{X}) = \pi^* c_2(X) + (K_X \cdot C + 6 - 6g(C))A.$$

Moreover,

$$(\pi^*H)^4 = H^4, \quad (\pi^*H)^3 \cdot E = (\pi^*H)^2 \cdot E^2 = 0, \quad (\pi^*H) \cdot E^3 = H \cdot C, \text{ and }$$
$$E^4 = -K_X \cdot C + 2g(C) - 2.$$

3. *If $Z = S \subset X$ is a surface and $H \in \mathrm{Pic}(X)$, we have*

$$K_{\widetilde{X}} = \pi^* K_X + E \text{ and } c_2(\widetilde{X}) = \pi^* c_2(X) + \pi^* S + \pi^* K_X \cdot E.$$

Moreover,

$$(\pi^*H)^4 = H^4, \quad (\pi^*H)^3 \cdot E = 0, \quad (\pi^*H)^2 \cdot E^2 = -S \cdot H^2,$$
$$(\pi^*H) \cdot E^3 = -H|_S \cdot K_S + K_X \cdot H \cdot S, \text{ and }$$
$$E^4 = c_2(X) \cdot S + K_X|_S \cdot K_S - c_2(S) - K_X^2 \cdot S,$$

where $c_2(S) = \chi_{\mathrm{top}}(S) = 12\chi(S, \mathcal{O}_S) - K_S^2$.

[2] This means that for general $x, y \in X$ we can find a rational curve conneting these points.

II.3.2 Lines on del Pezzo fourfolds

Here we study the lines contained in a del Pezzo fourfold of degree $d \in \{3, 4, 5\}$ following [PZ16]. We prove the following characterization.

Lemma II.3.4 ([PZ16, Lemma 2.5])**.** *Let $W = W_d \hookrightarrow \mathbb{P}^{d+2}$ be a del Pezzo fourfold of degree $d \geq 3$. Then W contains a line and its normal bundle is one of the following:*

$$N_{l/W} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1);$$
$$N_{l/W} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Proof. Let $W = W_d \hookrightarrow \mathbb{P}^{d+2}$ be a del Pezzo fourfold of degree $d \geq 3$, and take $H_1, H_2 \subset W$ two general hyperplane sections. Then, by Theorem II.2.6 $F := H_1 \cap H_2 \hookrightarrow \mathbb{P}^d$ is a smooth del Pezzo surface of degree d, and by the classification of del Pezzo surfaces we know that such surface F contains lines $\mathbb{P}^1 = \ell \subset F$ and that they verify $-K_F \cdot \ell = 1$, i.e., ℓ is a (-1)-curve on F. In particular, we have that $N_{\ell/F} = \mathcal{O}_\ell(-1)$.

In order to compute the normal bundle $N_{\ell/W}$, note that since $F = H_1 \cap H_2$, we have $N_{F/W} = (\mathcal{O}_W(1) \oplus \mathcal{O}_W(1))|_F$, and hence $N_{F/W}|_\ell = \mathcal{O}_\ell(1)^{\oplus 2}$. By Birkhoff-Grothendieck theorem, we can write

$$N_{\ell/W} = \mathcal{O}_\ell(a) \oplus \mathcal{O}_\ell(b) \oplus \mathcal{O}_\ell(c) \text{ for some } a, b, c \in \mathbb{Z}.$$

Hence, the short exact sequence

$$0 \to N_{\ell/F} = \mathcal{O}_\ell(-1) \to N_{\ell/W} \to N_{F/W}|_\ell = \mathcal{O}_\ell(1)^{\oplus 2} \to 0$$

implies that $a + b + c = \deg(N_{\ell/W}) = -1 + 2 = 1$. On one hand, the non-zero morphism $\mathcal{O}_\ell(-1) \hookrightarrow N_{\ell/W} \to \mathcal{O}_\ell(a)$ induces $\mathcal{O}_\ell \to \mathcal{O}_\ell(a+1)$, i.e., a non-zero regular section of $\mathcal{O}_\ell(a+1)$. The existence of such a section implies that $a + 1 \geq 0$, and similarly we deduce that $b \geq -1$ and $c \geq -1$. On the other hand, the composition $\mathcal{O}_\ell(a) \hookrightarrow N_{\ell/W} \to \mathcal{O}_\ell(1)^{\oplus 2} \to \mathcal{O}_\ell(1)$ induces a non-zero regular section of $\mathcal{O}_\ell(1-a)$ and hence $1 - a \geq 0$. Similarly, we deduce that $b \leq 1$ and $c \leq 1$, and thus if we assume that $a \leq b \leq c$ we obtain that $(a, b, c) \in \{(0, 0, 1), (-1, 1, 1)\}$, i.e.,

$$N_{\ell/W} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \text{ or } N_{\ell/W} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}.$$

□

Remark II.3.5. If we consider a line $\ell \subset W$ on a del Pezzo fourfold of degree $d \in \{3;4;5\}$, we have $N_{\ell/W} = \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, by deformation theory this implies that the Hilbert scheme of lines on W, denoted $\mathcal{L}(W)$, is smooth of dimension 4. Moreover, Prokhorov and Zaindeberg observed in [PZ16, Corollary 2.6] that through any point $p \in W$ there passes a family of lines in W of dimension ≥ 1, and that this family has dimension 1 if $d \leq 4$.

II.3.3 Sarkisov link between W_4 and $\mathbb{P}^4$

To explore the geometry of W_4 in more detail, we will prove there is a way to link this variety with $\mathbb{P}^4$. First, we recall the construction of a linear projection from a projective variety.

Definition II.3.6. Consider a $(n+1)$-dimensional $\mathbb{C}$-vector space $V_{n+1} = \mathbb{C}^{n+1}$ and a projective variety $X \subset \mathbb{P}^n$ which contains properly a linear subspace $Y := \mathbb{P}(V_{m+1}) = \mathbb{P}^m \subsetneq X$. The *linear projection with center* Y corresponds to the rational map $\pi : X \dashrightarrow \mathbb{P}(V_{n+1}/V_{m+1}) = \mathbb{P}^{n-m-1}$.

In the case of a quartic del Pezzo fourfold, Prokhorov and Zaidenberg proved that the linear projection from a line fits in the following Sarkisov link.

Proposition II.3.7 ([PZ16, Proposition 3.1])**.** *Let $W = W_4 \subset \mathbb{P}^6$ be a quartic del Pezzo fourfold, i.e., an intersection of two quadrics in $\mathbb{P}^6$. Pick $H \in \mathrm{Pic}(W)$ a generator and a general line $\ell \subset W$. There exists a conmutative diagram*

$$\begin{tikzcd} & E \arrow[dl] & & \widetilde{W} \arrow[dl, "\sigma"'] \arrow[dr, "\varphi"] & & M \arrow[dr] & \\ \ell & & W \arrow[rr, dashed, "\pi"] & & \mathbb{P}^4 & & F \end{tikzcd}$$

where π is the linear projection with center ℓ, σ is the blow-up of ℓ with exceptional divisor E, and φ is the birational morphism defined by the linear system $|H - E|$. Furthermore, we have the following hold.

1. *The φ-exceptional locus is an irreducible divisor $M \subset \widetilde{W}$.*

2. *If $L \in \mathrm{Pic}(\mathbb{P}^4)$ is the ample generator, the folowing relations are verified*

$$\varphi^*L \sim H - E; \quad M \sim 2H - 3E;$$
$$H \sim 3\varphi^*L - M; \quad E \sim 2\varphi^*L - M.$$

3. *The image* $F := \varphi(M)$ *is a smooth surface of degree* 5 *and* φ *is the blow-up with center* F*.*

4. $W \setminus \sigma(M) \simeq \mathbb{P}^4 \setminus \varphi(E)$ *where* $\varphi(E)$ *is a quadric in* $\mathbb{P}^4$*.*

Proof. We will divide the proof in several steps.

Step 1. the rational map φ *is a morphism and* $\widetilde{W}$ *is a fano fourfold.* Since the linear projection $\pi : W \dashrightarrow \mathbb{P}^4$ is defined by $|H - I_\ell|$ (where I_ℓ denotes the sheaf of ideals of ℓ), and its schematic base locus is precisely given by $\ell \subset W$, a local computation easily shows that the linear system $|\sigma^*H - E|$ is basepoint-free, so φ is in fact a morphism. In particular, the divisor $D := \sigma^*H - E$ is nef. Now, since W is a del Pezzo fourfold, we have that $-K_W = 3H$ and

$$-K_{\widetilde{W}} = 3\sigma^*H - 2E \overset{\text{def}}{=} \sigma^*H + 2D.$$

We know that $\rho_{\widetilde{W}} = 2$, so the nef cone of $\widetilde{W}$ is 2-dimensional. The nef divisors σ^*H and $2D$ are linearly independent in $N^1(\widetilde{W})_{\mathbb{R}} = \mathbb{R}^2$ and thus their sum $-K_{\widetilde{W}}$ is contained in the interior of the nef cone and hence is ample (see Corollary I.1.24).

Now we can play the *2-ray game.* More precisely, $\mathrm{NE}(\widetilde{W})$ has two extremal rays R_1, R_2, and we can assume that $\sigma = \mathrm{cont}_{R_1}$. By the Cone Theorem, the second extremal ray R_2 corresponds to a second extremal contraction $\varphi := \mathrm{cont}_{R_2} : \widetilde{W} \to T$ onto a normal projective variety T.

Step 2. The extremal ray R_2 *is generated by the class* $[\widetilde{C}]$*, where* $\widetilde{C}$ *is the strict transform of a line* $\mathbb{P}^1 = C \subset W$ *meeting the fixed line* $\ell \subset W$ *at one point.* First of all, note that the line $C \subset W$ exists by Remark II.3.5. By the projection formula we have that

$$D \cdot \widetilde{C} = \sigma^*H \cdot \widetilde{C} - E \cdot \widetilde{C} = H \cdot C - E \cdot \widetilde{C} = 1 - 1 = 0,$$

and hence D is a nef but not ample divisor. Since $\widetilde{C}$ is not contracted by σ, we deduce that $[\widetilde{C}]$ spans an extremal ray of $\mathrm{NE}(\widetilde{W})$ and moreover D is a *supporting divisor* for the associated extremal contraction $\varphi : \widetilde{W} \to T$, i.e., $\varphi = \varphi_{|mD|}$ where $m \gg 0$ (see Corollary I.2.17). Since D is basepoint-free, we deduce that $m = 1$.

Step 3. $T = \mathbb{P}^4$ *and the morphism* $\varphi : \widetilde{W} \to T = \mathbb{P}^4$ *is birational.* By Theorem I.3.2, there is an exact sequence

$$0 \to \mathrm{Pic}(T) \xrightarrow{\varphi^*} \mathrm{Pic}(\widetilde{W}) \xrightarrow{(\cdot\widetilde{C})} \mathbb{Z}$$

and hence $\rho(T) = 1$. Moreover, since $D \cdot \widetilde{C} = 0$, there exists $L \in \mathrm{Pic}(T)$ such that $D = \sigma^*H - E = \varphi^*L$. Since D is basepoint-free and $\rho(T) = 1$, we have that L is necessarily an ample line bundle. The relations below follows from Lemma II.3.3:

$$(\sigma^*H)^4 = H^4 \stackrel{\text{def}}{=} 4; \quad (\sigma^*H)^3 \cdot E = 0; \quad (\sigma^*H)^2 \cdot E^2 = 0;$$
$$(\sigma^*H) \cdot E^3 = H^3 \cdot \ell \stackrel{\text{def}}{=} 1; \; E^4 = -K_W \cdot \ell - 2 \stackrel{\text{def}}{=} 1.$$

In particular, $D^4 = (\sigma^*(H) - E)^4 = 1$ and hence $\deg(\varphi) \cdot L^4 = (\varphi^*L)^4 = 1$, from which we deduce that $T = \mathbb{P}^4$ (because $L^4 = 1$) and that $\varphi : \widetilde{W} \to T$ is birational (as a consequence of $\deg(\varphi) = 1$).

Step 4. The φ-exceptional locus is an irreducible divisor $M \in |2\sigma^*H - 3E|$ and $F := \varphi(M) \subset \mathbb{P}^4$ is a quintic surface. First, we already now φ is a divisorial contraction, because it is birational. Then, a direct computation shows that

$$(\varphi^*L)^3 \cdot (a\sigma^*H - bE) = D^3 \cdot (a\sigma^*H - bE) = 3a - 2b;$$

and intersecting with the nef divisor σ^*H we observe that if the divisor $a\sigma^*H - bE$ is effective and non-zero then $a \geq 1$. Thus, the unique linear system contracted by φ is $M \stackrel{\text{def}}{=} 2\sigma^*H - 3E$ and it follows this linear system has a unique irreducible divisor. Since $D^2 \cdot M^2 = (2\sigma^*H - 3E) = -5 \neq 0$, it follows that D is contracted to a surface in $\mathbb{P}^4$. Moreover, if we denote $F := \varphi(R)$ for a divisor $R \in |M|$, we calculate (using Lemma II.3.3) that

$$\deg(F) \stackrel{\text{def}}{=} L^2 \cdot \varphi_*(R) = -(\varphi^*(L))^2 \cdot M^2 = -D^2 \cdot M^2 = 5;$$

i.e., $F \subset \mathbb{P}^4$ is a quintic surface.

The hypothesis that the line $\ell \subset W$ is general implies that both R and F are a smooth and φ is the blow-up of F (see [PZ16, Proposition 3.1] for details about that). ☐

As a worked example, we will use tools from intersection theory to calculate in detail the dimension $h^0(\widetilde{W}; D)$.

Lemma II.3.8. *With the notation of II.3.7, we have that* $\dim \mathrm{H}^0(\widetilde{W}; \mathcal{O}_{\widetilde{W}}(D)) = 6$.

Proof. First, observe that we can write $D = K_{\widetilde{W}} + A$ where $A = -K_{\widetilde{W}} + D$ is an ample divisor (since it is the sum of an ample and a nef divisor), and hence the Kodaira vanishing theorem implies that $\mathrm{H}^i(\widetilde{W}; \mathcal{O}_{\widetilde{W}}(D)) = 0$ for all $i \geq 1$.

We are left to compute

$$h^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(D)) = 1 + \frac{1}{24}\left[D^4 + 2D^3 \cdot c_1(\widetilde{W}) + D^2 \cdot \left(c_1(\widetilde{W})^2 + c_2(\widetilde{W})\right) + D \cdot c_1(\widetilde{W}) \cdot c_2(\widetilde{W})\right],$$

where

$$c_1(\widetilde{W}) = -K_{\widetilde{W}} = 3 \cdot H - 2E \text{ and } c_2(\widetilde{W}) = c_2(W) + (K_W \cdot \ell + 6)A = c_2(W) + 3A,$$

where $A \in \mathrm{CH}^2(\widetilde{W})$ is the class of a fiber of the $\mathbb{P}^2$-bundle $|_E : E \to \ell$. Here, since W is a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^6$, we can deduce from the Euler exact sequence for $T_{\mathbb{P}^6}$ and the short exact sequence defining the normal bundle $N_{W/\mathbb{P}^6}$ that the total Chern class of W is given by

$$c(W) = 1 + 3H + 5H^2 + 3H^3 + 3H^4,$$

which implies that $c_2(W) = 5H^2$ and hence $c_2(\widetilde{W}) = 5 \cdot H^2 + 3A$.

On one hand, it follows from II.3.3 that if $D = H - E$ and $c_1(\widetilde{W}) \overset{\text{def}}{=} -K_{\widetilde{W}} = 3 \cdot H - 2E$ then

$$D^4 = 1, \ 2D^3 \cdot c_1(\widetilde{W}) = 10, \text{ and } D^2 \cdot c_1(\widetilde{W})^2 = 20.$$

On the other hand, using that $c_2(\widetilde{W}) = 5 \cdot H^2 + 3A$ we observe that

$$\begin{aligned} D^2 \cdot c_2(\widetilde{W}) + D \cdot c_1(\widetilde{W}) \cdot c_2(\widetilde{W}) &= c_2(\widetilde{W}) \cdot (4 \cdot H^2 - 7 \cdot H \cdot E + 3E^2) \\ &= 80 + 3A \cdot (4 \cdot H^2 - 7 \cdot H \cdot E + 3E^2) \\ &= 89, \end{aligned}$$

where the last equality follows from Lemma II.3.9 below. We conclude therefore that

$$h^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(D)) = 1 + \frac{1}{24}(1 + 10 + 20 + 89) = 6.$$

□

Lemma II.3.9. *With the notation as above, we have that* $A \cdot H^2 = 0$, $A \cdot H \cdot E = 0$ *and* $A \cdot E^2 = 1$.

Proof. The first two assertions follows from the projection formula. In order to compute $A \cdot E^2$ we consider the diagram

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "g := |_E"'] & \widetilde{W} \arrow[d] \\ \ell \arrow[r, hook, "i"] & W \end{tikzcd}$$

and we write $= c_1(\mathcal{O}_E(1)) \in \mathrm{CH}^1(E)$. Here, $j\ [E] =$ and $j\ (\ \ j\ \ ) = (j\ \ )$ by the projection formula. Therefore, if $p \in \mathrm{CH}^1(\ell)$ denotes the class of a point in ℓ and $f_p := g\ (p) \in \mathrm{CH}^1(E)$, then we have

$$E^2 = \ \ j\ (\ ) \text{ and } E^2\ A = E^2 j\ g\ (p) = \ \ j\ (\ \ (j\ j\ )(f_p)) = \ \ j\ (\ \ ^2\ f_p) = \ \ (\ \ 1) = 1,$$

where $(j\ j\ )(f_p) = \ \ f_p$ follows from [Ful84, Proposition 2.6(c)] (cf. [Ful84, Theorem 3.3(b)]). □

Remark II.3.10. The previous computation seems to indicate that there is an inaccuracy on [PZ16, page 268], where it is claimed that $h^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(D)) = 5$.

II.3.4 Schubert calculus and the Chow ring of a Grassmannian

Before describing in detail a Sarkisov link for W_5, we provide a brief overview of the Chow ring of a Grassmannian, focusing on the Grassmannian of lines in $\mathbb{P}^4$.

Consider a k-vector space V with $\dim(V) = n+1$ and a complete flag $V_\bullet$ in V, i.e., a nested sequence of subspaces:

$$\{0\} \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} = V,$$

where $\dim(V_i) = i$.

Given two integers (a, b) such that $0 \leq b \leq a \leq n-1$, the Schubert cycle $\Sigma_{a,b}(V_\bullet) \subset G = \mathrm{Gr}(2, V)$ in the Grassmannian of lines $\mathrm{Gr}(2, V)$ is defined as the closed subvariety:

$$\Sigma_{a,b}(V_\bullet) = \{\Lambda \in G : \Lambda \cap V_{n-a} \neq 0 \text{ and } \Lambda \subset V_{n+1-b}\}.$$

This subvariety defines a class $\sigma_{a,b} := [\Sigma_{a,b}(V_\bullet)] \in CH^{a+b}(G)$ of codimension $a+b$ in the Chow ring $CH^*(G)$ which is independent of the chosen flag $V_\bullet$, and it is called a *Schubert class.*

Theorem II.3.11 ([EH16, Corollary 4.7])**.** *Schubert classes forms a free basis for the Chow ring $CH^*(G)$.*

As an example, we will give the complete description of the Chow ring $CH^*(G) = CH^*(\mathrm{Gr}(2, 5))$. Intersection formulas can be deduced from [Ful84, Proposition 14.6.1].

Example II.3.12. We think about elements in $\mathrm{Gr}(2,5)$ as lines in a projective space $\mathbb{P}^4$, so a complete flag corresponds to a fixed sequence of nested linear subspaces

$$\{p\} := \mathbb{P}^0 \subset \ell_0 := \mathbb{P}^1 \subset \Pi := \mathbb{P}^2 \subset \Lambda := \mathbb{P}^3 \subset \mathbb{P}^4$$

and the Schubert cycles are described as follows:

$$\begin{aligned}
&\Sigma_{1,0} = \{\ell \in G : \ell \cap \Pi \neq 0\}, && \Sigma_{2,0} = \{\ell \in G : \ell \cap \ell_0 \neq 0\}\\
&\Sigma_{3,0} = \{\ell \in G : p \in \ell\}, && \Sigma_{1,1} = \{\ell \in G : \Lambda \cap \mathbb{P}^2 \neq 0, \ell \subset \Lambda\}\\
&\Sigma_{2,1} = \{\ell \in G : \Lambda \cap \mathbb{P}^1 \neq 0, \ell \subset \Lambda\}, && \Sigma_{3,1} = \{\ell \in G : p \in \ell, \ell \subset \Lambda\}\\
&\Sigma_{2,2} = \{\ell \in G : \ell \subset \Pi\}, && \Sigma_{3,2} = \{\ell \in G : p \in \ell, \ell \subset \Pi\}\\
&\Sigma_{3,3} = \{\ell \in G : \ell = \ell_0\}.
\end{aligned}$$

Below we give explicitly the rules to calculate intersections in G (which can be obtained from [EH16, Proposition 4.9]):

$$\begin{gathered}
\sigma_{1,0}^2 = \sigma_{2,0} + \sigma_{1,1}\\
\sigma_{2,0}\sigma_{1,0} = \sigma_{3,0} + \sigma_{2,1}\\
\sigma_{1,0}\sigma_{1,1} = \sigma_{2,1}\\
\sigma_{1,1}^2 = \sigma_{2,2}\\
\sigma_{2,1}\sigma_{1,0} = \sigma_{3,1} + \sigma_{2,2}\\
\sigma_{3,0}\sigma_{1,0} = \sigma_{3,1}\\
\sigma_{2,0}^2 = \sigma_{3,1} + \sigma_{2,2}\\
\sigma_{2,0}\sigma_{1,1} = \sigma_{3,1}\\
\sigma_{2,1}\sigma_{1,1} = \sigma_{3,1}\sigma_{1,0} = \sigma_{2,2}\sigma_{1,0} = \sigma_{3,2}\\
\sigma_{3,2}\sigma_{1,0} = \sigma_{3,1}\sigma_{2,0} = \sigma_{3,0}^2 = \sigma_{2,1}^2 = \sigma_{2,2}\sigma_{1,1} = \sigma_{3,3}\\
\sigma_{3,1}\sigma_{1,1} = \sigma_{2,0}\sigma_{2,2} = 0
\end{gathered}$$

These calculations also permit to compute Chern classes of G. To do this we recall some constructions of vector bundles on Grassmannians.

Construction II.3.13. Let $G = \mathrm{Gr}(k, V)$ be the Grassmannian of k-planes on a vector space V of $\dim(V) = n$. The trivial vector bundle $\mathcal{V} := G \times V$ contains a subbundle $\mathcal{S}$, called the *universal subbundle of* G, defined at every point $[\Lambda] \in G$ as $\mathcal{S}_{[\Lambda]} = \Lambda \subset V$. This vector bundle permits to obtain another non-trivial vector bundle taking the quotient, i.e., $Q := V/S$ is a vector bundle called the *universal quotient bundle of* G. Some linear algebra computations permits to calculate the total Chern class of these vector bundles, and for $k = 2$ the formulas are the following:

$$c(Q) = 1 + \sigma_{1,0} + \sigma_{2,0} + \cdots + \sigma_{n-2}, \; c(S) = 1 - \sigma_{1,0} + \sigma_{1,1}$$
$$\text{and } c(S^\vee) = 1 + \sigma_{1,0} + \sigma_{1,1}.$$

A fundamental fact is that the tangent bundle T_G of the Grassmanian corresponds to $T_G = \mathrm{Hom}(S, Q) = S^\vee \otimes Q$ (see, e.g., [EH16, Theorem 3.5]). In order to express the Chern classes of the tangent bundle T_G in terms of the Chern classes of $S^\vee$ and Q, we can make use of the identity

$$\mathrm{ch}(T_G) = \mathrm{ch}(S^\vee) \cdot \mathrm{ch}(Q),$$

where $\mathrm{ch}(E)$ denotes the *Chern character* of a vector bundle E (see, e.g. [Ful84, §15.1] or [EH16, §14.2]).

Using the preceding identities in the case of $G = \mathrm{Gr}(2, 5)$, by direct computations we can derive the following lemma.

Lemma II.3.14. *The Chern classes of* $G = \mathrm{Gr}(2, 5)$ *are*

$$c_1(G) = 5\sigma_{1,0}, \qquad c_2(G) = 11\sigma_{2,0} + 12\sigma_{1,1}$$
$$c_3(G) = 15\sigma_{3,0} + 30\sigma_{2,1}, \quad c_4(G) = 35\sigma_{3,1} + 25\sigma_{2,2}$$

II.3.5 Explicit constructions of quintic del Pezzo fourfolds

In this section we give an account on the results and constructions by T. Fujita in [Fuj81] in the special case of fourfolds. We will denote by (W, H) a smooth polarized fourfold such that

$$d(W, H) \overset{\text{def}}{=} H^4 = 5 \text{ and } \Delta(W, H) \overset{\text{def}}{=} \dim(X) + d - h^0(W, H) = 9 - h^0(W, H) = 1.$$

Note that since $d(W, H) \geq 3$ it follows from Theorem II.2.6 that

$$g(W, H) \overset{\text{def}}{=} 1 + \frac{1}{2}(K_W + 3H) \cdot H^3 = 1.$$

Moreover, by Theorem II.2.14 we have that $-K_W = 3H$ and hence W is a quintic del Pezzo fourfold with $\mathrm{Pic}(W) = \mathbb{Z}[H]$ (cf. [Fuj81, Lemma 9.1]).

Example II.3.15. The following examples by Fujita [Fuj81, §7] provide different constructions of such varieties W based on objects from classical projective geometry.

1. **The Grassmannian** $\mathrm{Gr}(2,5)$**.** If we denote by $G \subset \mathbb{P}^9$ be the image of $\mathrm{Gr}(2,5)$ via the Plücker embedding into $\mathbb{P}(\wedge^2\mathbb{C}^5)$ and by H_G the corresponding very ample divisor on G, then $\dim(G) = 6$, $h^0(G, H_G) = 10$ and one can compute using Schubert calculus that $d(G, H_G) = 5$. Indeed, hyperplane sections of $\mathrm{Gr}(2,5)$ under its Plücker embedding are given by the Schubert cycle $H_G = \sigma_{1,0}$, and the formulas from previous sections shows $H_G^6 = \sigma_{1,0}^6 = 5$. Thus, (G, H_G) is a del Pezzo variety and any smooth codimension 2 linear section $W_L := G \cap L$, $L = \mathbb{P}^7$ is a quintic del Pezzo fourfold.

2. **The Plücker quadric** $Q \subset \mathbb{P}^5$**.** Let $Q := \{x \in \mathbb{P}^5;\ x_0x_5 - x_1x_4 + x_2x_3 = 0\}$ be the Plücker quadric in $\mathbb{P}^5$ and let $Q_0 = \{x_5 = 0\} \cap Q$ be the singular hyperplane section with $\mathrm{Sing}(Q_0) = \{p = [1;0;\ldots;0]\}$. We will construct a Sarkisov link from Q to a quintic del Pezzo fourfold W via the following diagram

$$\begin{tikzcd} & Y \arrow[dl, "\varphi_{\ell_p}"'] \arrow[dr, "\varphi_\sigma"] & \\ Z \arrow[d, "\varphi_\ell"'] & & X \arrow[d, "\varphi_p"] \\ W & & Q \arrow[ll, dotted, "\text{birational}"'] \end{tikzcd}$$

Here, we consider the blow-up $\varphi_p : X := \mathrm{Bl}_p(Q) \to Q$, with exceptional divisor $E_p \subset X$, and we denote by $D \subset X$ the strict transform of Q_0 on X. By the definition of blow-up coordinates we observe that

$$D = \{(\mathbf{z},\mathbf{t}) \in \mathbb{P}^4 \times \mathbb{P}^3;\ [z_1;z_2;z_3;z_4] = [t_1;t_2;t_3;t_4];\ z_1z_4 = z_2z_3;\ t_1t_4 = t_2t_3\},$$

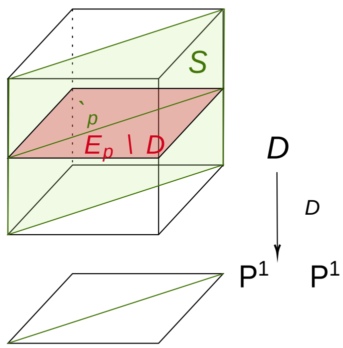
and hence $D = \{(\mathbf{z},\mathbf{u},\mathbf{v}) \in \mathbb{P}^4 \times \mathbb{P}^1 \times \mathbb{P}^1;\ [z_1;z_2] = [z_3;z_4] = [u_0;u_1];\ [z_1;z_3] = [z_2;z_4] = [v_0;v_1]\}$ lies in the image of the Segre embedding

$$\mathbb{P}^4 \times \mathbb{P}^1 \times \mathbb{P}^1 \xhookrightarrow{\mathrm{Id} \times \varphi_{|\mathcal{O}(1,1)|}} \mathbb{P}^4 \times \mathbb{P}^3.$$

Following T. Fujita's notation, we denote by $H_{\mathbf{x}}$ (resp. $H_{\mathbf{z}}$, etc) the line bundle $\mathcal{O}_{\mathbb{P}^5}(1)$ (resp. $\mathcal{O}_{\mathbb{P}^4}(1)$, etc) on the projective space with homogeneous coordinates $\mathbf{x} \in \mathbb{P}^5$ (resp. $\mathbf{z} \in \mathbb{P}^4$, etc). Then, $H_{\mathbf{x}}|_D = H_{\mathbf{z}}$ and $E_p \cap D = \{z_1 = z_2 = z_3 = z_4 = 0\} = \mathbb{P}^1 \times \mathbb{P}^1 \subset D$ is a section of the $\mathbb{P}^1$-bundle $\pi_D : D \to \mathbb{P}^1 \times \mathbb{P}^1$

induced by the projection pr_{23}. Moreover, $(D, H_{\mathbf{z}}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ where $E = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$.

If S is a smooth surface in the linear system $|\pi_D^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$, we consider the blow-up $\varphi_S : Y := \mathrm{Bl}_S(X) \to X$ with exceptional divisor $E_S \subset Y$, and we denote by $F_p \subset Y$ and $D_Y \subset Y$ the strict transforms of E_p and D on Y respectively. Now, since $\mathcal{O}_D(S)|_{E_p\cap D} = H_{\mathbf{u}} = (\mathrm{pr}_2|_D)^*\mathcal{O}_{\mathbb{P}^1}(1)$, we have that $S\cap E_p = \ell_p = \mathbb{P}^1$ is a line on $E_p = \mathbb{P}^3$ and we can assume that $\ell_p = \{t_1 = t_3 = 0\}$ up to change of coordinates.



Hence, $F_p = \mathrm{Bl}_{\ell_p}(E_p) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is the blow-up of $\mathbb{P}^3$ along a line. Explicitly, we have that $F_p = \{([t_1:\dots:t_4],\mathbf{s}) \in \mathbb{P}^3\times\mathbb{P}^1;\ [t_1:t_3] = [s_0:s_1]\}$ and ℓ_p is a section of the $\mathbb{P}^2$-bundle $F_p \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1$. We observe that

$$\mathcal{O}_Y(F_p)|_{F_p} = \varphi_S^*(E_p)|_{F_p} = -H_{\mathbf{t}},$$

and hence F_p can be blown-down to ℓ_p. More precisely, it follows from Moishezon contraction theorem [Mis69] that $(Y, F_p) = (\mathrm{Bl}_{\ell_p}(Z), E_{\ell_p})$ for some[3] smooth projective variety Z with $\ell_p \subset Z$. If D_Z is the image of D_Y by the blow-down $\varphi_{\ell_p} : Y \to Z$ then

$$D_Z = \{(\mathbf{z},\mathbf{s}) \in \mathbb{P}^4\times\mathbb{P}^1;\ [z_1:z_3] = [z_2:z_4] = [s_0:s_1]\}.$$

As before, we can check (see [Fuj81, §7.7] for details) that D_Z can be blown-down to $\ell = \mathbb{P}^1$ with respect to the $\mathbb{P}^2$-bundle structure $D_Z \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1$. More precisely, there is a smooth projective variety W such that $\ell \subset W$ and $(Z, D_Z) = (\mathrm{Bl}_\ell(W), E_\ell)$,

[3]Alternatively, if we denote by Σ the image of S in Q, then we have that $\ell_p = S\cap E_p$ is a (-1)-curve on S and Σ is its blow-down. We observe that $(\Sigma, H_{\mathbf{x}})$ is a polarized manifold with $d(\Sigma, H_{\mathbf{x}}) = 0$ and $\Delta(\Sigma, H_{\mathbf{x}}) = 0$ and hence it follows from [Fuj75] that $\Sigma = \mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at a point. Now, one observes that $Z = \mathrm{Bl}_\Sigma(Q)$ and D_Z is the strict transform of Q_0.

and if we consider $\varphi_\ell : Z \to W$ then it can be checked that $D_Z + H_{\mathbf{x}} \sim \varphi_\ell H$, where H is an ample line bundle in W such that $-K_W = 3H$ and $H^4 = 5$.

3. **The twisted cubic $C \subset \mathbb{P}^3 \subset \mathbb{P}^4$.** Consider $\mathbb{P}^4$ with homogeneous coordinates $\mathbf{x} = [x_0 : \ldots : x_4]$ and let $D = \{x_4 = 0\} = \mathbb{P}^3$ be a hyperplane. Let $C := \nu_3(\mathbb{P}^1)$ be the twisted cubic embedded in $\mathbb{P}^3 = D$ via the Veronese embedding of degree 3 and assume, for instance, that

$$\mathbb{P}^1_{\mathbf{t}} = C = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 = D,\ x_0x_2 = x_1^2,\ x_0x_3 = x_1x_2,\ x_1x_3 = x_2^2\},$$

where $H_{\mathbf{x}}|_C \sim 3H_{\mathbf{t}}$. Let $\varphi : \widetilde{W} := \mathrm{Bl}_C(\mathbb{P}^4) \to \mathbb{P}^4$ be the blow-up along C, with exceptional divisor $E_C \subset \widetilde{W}$, and let $\widetilde{D}$ be the strict transform of D on $\widetilde{W}$. Note that the sections $\langle x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2\rangle \subset \mathrm{H}^0(\widetilde{W}, 2\varphi^* H_{\mathbf{x}} - E_C)$ define a linear system Λ on $\widetilde{D} = \mathrm{Bl}_C(D)$ which is base point free. Let $\pi : \widetilde{D} \to \mathbb{P}^2$ be the associated morphism, and note that this defines a $\mathbb{P}^1$-bundle structure on $\widetilde{D}$ since the twisted cubic $C \subset D = \mathbb{P}^3$ is not a complete intersection: if we denote by F any fiber of π, then there are precisely two quadrics $Q_1, Q_2 \subset D$ containing C such that F is the intersection of their strict tranforms on $\widetilde{D}$. Since C is linearly non-degenerate (i.e., is not contained in any plane in $\mathbb{P}^3 = D$), we have that both Q_1 and Q_2 are irreducible and hence $Q_1 \cap Q_2$ is a curve of degree 4 in $\mathbb{P}^3$ containing the cubic C, i.e., $Q_1 \cap Q_2 = C \cup \ell_F$ where $\ell_F \cong \mathbb{P}^1$ is a *residual line*. Here, the restriction of $\widetilde{D} \to D$ to F induces an isomorphism $F \to \ell_F$ and thus $\pi : \widetilde{D} \to \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle.

Now, remark that $\mathcal{O}_{\widetilde{W}}(2\varphi^* H_{\mathbf{x}} - E_C)|_F = \mathcal{O}_F$ since $F \cong \mathbb{P}^1$ and $(2\varphi^* H_{\mathbf{x}} - E_C) \cdot F = 0$ and hence $\mathcal{O}_{\widetilde{W}}(\widetilde{D})|_F \overset{\text{def}}{=} \mathcal{O}_{\widetilde{W}}(\varphi^* H_{\mathbf{x}} - E_C)|_F = \mathcal{O}_{\widetilde{W}}(-\varphi^* H_{\mathbf{x}})|_F = \mathcal{O}_F(-1)$ and hence $\widetilde{D}$ can be blow-down with respecto to π by the Moishezon contraction theorem. More precisely, there is a smooth projective variety W such that $(\widetilde{W}, \widetilde{D}) = (\mathrm{Bl}_\Xi(W), E_\Xi)$, where $\Xi \cong \mathbb{P}^2$. Since $(2\varphi^* H_{\mathbf{x}} - E_C)|_F = \mathcal{O}_F$, we have that $2\varphi^* H_{\mathbf{x}} - E_C \sim \psi^* H$ for some line bundle $H \in \mathrm{Pic}(W)$, where $\psi : \widetilde{W} \to W$ is the blow-down of $\widetilde{D}$ to Ξ.

Since $\rho(\widetilde{W}) = 2$ and $2\varphi^* H_{\mathbf{x}} - E_C = (\varphi^* H_{\mathbf{x}} - E_C) + \varphi^* H_{\mathbf{x}}$ is the sum of two numerically independent nef divisors, we deduce that $\psi^* H$ is ample and hence $H \in \mathrm{Pic}(W)$ is ample, as $\rho(W) = 1$. Note that

$$\psi^*(K_W + 3H) = K_{\widetilde{W}} - \widetilde{D} + 3\psi^* H = (-5\varphi^* H_{\mathbf{x}} + 2E_C) - (\varphi^* H_{\mathbf{x}} - E_C) + 3(2\varphi^* H_{\mathbf{x}} - E_C) = 0$$

and hence $-K_W = 3H$, i.e., W is a smooth del Pezzo fourfold. Finally, we compute (see §1) that

$$\begin{aligned} d(W,H) = H^4 &= (2\varphi^* H_x - E_C)^4 \\ &= 16\varphi^* H_x^4 - 32\varphi^* H_x^3 \cdot E_C + 24\varphi^* H_x^2 \cdot E_C^2 - 8\varphi^* H_x \cdot E_C^3 + E_C^4 \\ &= 5, \end{aligned}$$

since

$$\varphi^* H_x^4 = H_x^4 = 1,\ \varphi^* H_x^3 \cdot E_C = \varphi^* H_x^2 \cdot E_C^2 = 0,\ \varphi^* H_x \cdot E^3 = H_x^3 \cdot C = 3$$
$$\text{and } E^4 = -K_{\mathbb{P}^4} \cdot C + 2g(C) - 2 = 13.$$

The main theorem presented in this section is the following theorem by T. Fujita.

Theorem II.3.16 ([Fuj81, Theorem 7.9])**.** *All quintic del Pezzo fourfolds are isomorphic to each other.*

The proof is based on constructing explicit Sarkisov links via linear projections from suitable planes on W (cf. [Tod30] where the case of linear sections of $\mathrm{Gr}(2,5) = \mathbb{G}(1,4)$ is discussed).

Lemma II.3.17 ([Fuj81, Lemma 10.1])**.** $W \subset |H| = \mathbb{P}^7$ *contains a plane* $\mathbb{P}^2 = S \subset W$.

We distinguish between two types of planes on W depending on the nature of the associated linear projection.

Definition II.3.18. Let $S \subset W \subset \mathbb{P}^7$ be a plane. We say that:

- S is of **vertex type** if $\pi_S : W \dashrightarrow \mathbb{P}^4$ is not surjective.
- S is of **non-vertex type** if $\pi_S : W \dashrightarrow \mathbb{P}^4$ is surjective.

Proposition II.3.19 ([Fuj81, 10.7])**.** *If* $S \subset W$ *is a plane of non-vertex type then* $\mathrm{Bl}_S(W) = \mathrm{Bl}_C(\mathbb{P}^4)$ *where* $C \subset \mathbb{P}^3 \subset \mathbb{P}^4$ *is the twisted cubic, i.e.,* W *is obtained as in Example II.3.15(3).*

The previous Proposition will be proved in an alternative way in the next section (see II.3.25). The next ingredient in the proof of Theorem II.3.16 is the following result allowing to distinguish planes of vertex type and non-vertex type using *lines*.

Lemma II.3.20 ([Fuj81, Lemma 10.8])**.** *Let $S \subset W$ be a plane and let $\mathbb{P}^1 = \ell \subset S$ be a line. Let us consider $\sigma : \widetilde{W} := \mathrm{Bl}_\ell(W) \to W$ and let $\varphi : \widetilde{W} \to X \subset \mathbb{P}^5$ be the regular morphism associated to the basepoint-free linear system $|\sigma^* H - E|$. If we denote by $\widetilde{S} \subset \widetilde{W}$ the strict transform of S, then $p_S := \varphi(\widetilde{S}) \in$ is a point on X and*

$$S \text{ is of vertex type if and only if } p_S \text{ is a vertex of } X \ (i.e., \operatorname{Sing}(X) = \{p_S\}).$$

Moreover, if $S_1, S_2 \subset W$ are planes such that $S_1 \cap S_2 = \ell$ is a line, then either S_1 or S_2 is of non-vertex type.

The last ingredient consist in analyzing planes of vertex type.

Proposition II.3.21 ([Fuj81, 10.21-10.25])**.** *Let $S \subset W$ be a plane of vertex type and let $\sigma : \widetilde{W} := \mathrm{Bl}_S(W) \to W$. Then, we have that*

1. *The image of $\varphi_{|\sigma^* H - E|} : \widetilde{W} \to \mathbb{P}^4$ is a* smooth *quadric $Q \subset \mathbb{P}^4$.*
2. *There exists a fiber*[4] *$F \subset \widetilde{W}$ of $\varphi_{|\sigma^* H - E|} : \widetilde{W} \to Q$ with* $\dim(F) = 2$.
3. *The image $\sigma(F)$ is a plane $S_F \subset W$ such that $S \cap S_F$ is a line.*

Putting all the geometric information together, we can deduce the main result of T. Fujita.

Proof of Theorem II.3.16. Let $S \subset W$ be a plane. If S is of non-vertex type then it follows from Proposition II.3.19 that W can be obtained from the twisted cubic $C \subset \mathbb{P}^3 \subset \mathbb{P}^4$. If S is of vertex type, it follows from Proposition II.3.19 and Lemma II.3.20 that S_F is of non-vertex type and hence we are in the previous situation. Finally, we observe (by explicit linear algebra computations) that all the pairs $(C, \mathbb{P}^3)$ in $\mathbb{P}^4$ are projectively equivalent (i.e., the natural action of $\mathrm{PGL}_5(\mathbb{C})$ on such pairs is transitive) and hence all quintic del Pezzo fourfolds W are isomorphic. □

[4] In particular, the morphism $\varphi_{|\sigma^* H - E|} : \widetilde{W} \to Q$ is not flat. It would be natural to study if $\widetilde{W}$ is the projectivization of a *Banica sheaf* over the smooth 3-dimensional quadric Q (see [BW96] for details).

II.3.6 Sarkisov link between W_5 and $\mathbb{P}^4$ after Fujita and Todd

In this section we follow the original articles by Todd [Tod30] and Fujita [Fuj81], and the work of Prokhorov and Zaidenberg [PZ16].

For the rest of the section we take $W = W_5$ as a del Pezzo fourfold of degree 5. By the analysis of the previous section we already know that W corresponds to the image of $\mathrm{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$ under the Plücker embedding, so $W = W_5 \subset \mathbb{P}^7$. We also know that $-K_W = 3H$ and $\mathrm{Pic}(W) = \mathbb{Z}[H]$.

The coputation of Chern classes of $G = \mathrm{Gr}(2, 5)$ in the previous section allows us to compute first Chern classes of W.

Lemma II.3.22. *The Chern classes of $W = W_5$ are*

$$c_1(W) = 3\sigma_{1,0}|_W, \quad c_2(W) = 4\sigma_{2,0}|_W + 5\sigma_{1,1}|_W$$

Proof. Since W corresponds to a codimension 2 linear section of G, this implies that it corresponds to the vanishing of two sections of its polarization $\mathcal{O}_G(H) = \mathcal{O}_G(1)$. In particular, this means that its normal bundle is given by $N_{W/G} = \mathcal{O}_G(H)|_W^{\oplus 2}$, and then its total Chern class is,

$$c(N_{W/G}) = (1 + c_1(\mathcal{O}_G(H)|_W))^2 = (1 + \sigma_{1,0}|_W)^2.$$

By definition of the normal bundle we have an exact sequence of vector bundles

$$0 \to T_W \to T_G|_W \to N_{W/G} \to 0,$$

and by the functoriality of the total Chern classes, we attain

$$c(G) = c(W) \cdot c\left(N_{W/G}\right) = c(W) \cdot (1 + \sigma_{1,0}|_W)^2,$$

which permits to obtain the result. □

The following result is a very well-known characterization about planes in W due to [Tod30].

Proposition II.3.23 ([Tod30])**.** *Let $W = W_5 \subset \mathbb{P}^7$ be a del Pezzo fourfold of degree 5. Then the following hold.*

1. *W contains a unique $\sigma_{2,2}$-plane Ξ and a one-parameter family of $\sigma_{3,1}$-planes.*

2. *Any $\sigma_{3,1}$-plane Π meets Ξ along a tangent line to a fixed conic $C \subset \Xi$.*

3. *Any two $\sigma_{3,1}$-planes Π_1, Π_2 meet at a point $p \in \Xi \setminus C$.*

4. *Let R be the union of all $\sigma_{3,1}$-planes. Then R is a singular hyperplane section of W with $\mathrm{Sing}(R) = \Xi$.*

Lemma II.3.24. *Let $\Lambda \subset W$ be a plane. Then $c_1(N_{\Lambda/W}) = 0$ and $c_2(N_{\Lambda/W}) = 2$ (respectively $c_2(N_{\Lambda/W}) = 2$) if Λ is of type $\sigma_{2,2}$ (respectively of type $\sigma_{3,1}$).*

Using the previous description of W, we will study the blow-up $\widetilde{W} = \mathrm{Bl}_{\Xi}(W)$ of W at its unique $\sigma_{2,2}$-plane, so we have the following situation

$$\begin{tikzcd} \mathbb{P}(N_{\Xi/W}) = E \arrow[r, hook, "j"] \arrow[d, "\pi_E"] & \widetilde{W} \arrow[d, "\sigma"] \\ \mathbb{P}^2 = \Xi \arrow[r, hook, "i"] & W \end{tikzcd}$$

The crucial fact about this blow-up, again by Prokhorov and Zaidenberg, is that it fits in a Sarkisov link with $\mathbb{P}^4$, in a similar fashion that in the case of W_4. Specifically, we have the following characterization, which is in fact the same as the description given by T. Fujita.

Proposition II.3.25 ([PZ17, Theorem 2.1])**.** *Let $\Xi \subset W$ be the unique $\sigma_{2,2}$-plane. There exists a commutative diagram*

$$\begin{tikzcd} & E \arrow[dl] & & \widetilde{W} \arrow[dl, "\sigma"'] \arrow[dr, "\varphi"] & & \widetilde{R} \arrow[dr] & \\ \Xi & & W \arrow[rr, dashed, "\psi"] & & \mathbb{P}^4 & & C \end{tikzcd}$$

where

1. *$\sigma : \widetilde{W} \to W$ is the blow-up of Ξ, $\psi : W \dashrightarrow \mathbb{P}^4$ is the projection from Ξ and $\varphi : \widetilde{W} \to \mathbb{P}^4$ is the blow-up of a rational normal cubic curve $C \subset \mathbb{P}^4$.*

2. *$\varphi : \widetilde{W} \to \mathbb{P}^4$ is the morphism associated to the linear system $|H - E|$, where $E = \sigma^{-1}(\Xi)$ is the exceptional divisor of σ.*

3. *$\varphi(E) = \mathbb{P}^3 = \langle C \rangle$ is the linear span of the curve C.*

4. *the exceptional divisor $\widetilde{R} = \varphi^{-1}(C)$ of φ is equal to the strict transform of R and $\widetilde{R} \sim H - 2E$.*

Proof. In this proof we will omit the computations of intersection numbers, because they are presented in the next Lemma for reasons of order.

By the same argument as in the case of W_4, we have that the blow-up $\widetilde{W}$ is a Fano manifold. Indeed, the divisor $D := H - E$ is nef and $-K_{\widetilde{W}} = 2H - D$ is ample. It follows that we have a morphism $\varphi = \varphi_{|D|} : \widetilde{W} \to \mathbb{P}^4$ induced by the linear system $|D|$, and the calculations $D^4 = D^3 \cdot E = 1$ implies that $\varphi(\widetilde{W}) = \mathbb{P}^4$ and $\varphi(E) = \mathbb{P}^3$. Besides, note that the strict transform $\widetilde{R}$ of the hyperplane section $R \subset W$ described in II.3.23 verifies that $\widetilde{R} \sim H - kE$ for some $k \geq 2$. Indeed, $\pi^*R = \widetilde{R} + kE$ with $k \geq 2$ because $\mathrm{Sing}(R) = \Xi$, and since $R \in |H|$, we obtain $\widetilde{R} \sim H - kE$ for some $k \geq 2$. Moreover, since $\widetilde{R}$ is irreducible and reduced, previous calculations implies $(H - E)^3 \cdot \widetilde{R} = 2 - k \geq 0$, so in fact $k = 2$.

On the other hand, note that if $H \subset \mathbb{P}^4$ is a generic hyperplane, we have $\varphi^* H = H - E = E + R$, because $R \in |H - 2E|$, and as E is the fixed part (in particular $h^0(\widetilde{W}; E) = 1$), R is the movable part of $\varphi^* H$. We can note $C \overset{\text{def}}{=} \varphi(R)$ is contained in a unique hyperplane in $\mathbb{P}^4$. Indeed, if we take the restriction $H^0(\widetilde{W}; \varphi^* H) \xrightarrow{\mathrm{rest}_R} H^0(R; H)$, and if $h \in H^0(\mathbb{P}^4; \mathcal{O}_{\mathbb{P}^4}(1))$ is a hyperplane, note

$$h \in \ker(\mathrm{rest}_R) \iff \forall x \in R, (\varphi^* h)(x) = h(\varphi(x)) = 0 \iff C \subset \{h = 0\}$$

Thus, if C is contained in more than 1 hyperplane, $\dim(\ker(\mathrm{rest}_R)) \geq 2$ and by the exact sequences

$$0 \to \mathcal{O}_{\widetilde{W}}(-\widetilde{R}) \to \mathcal{O}_{\widetilde{W}} \to \mathcal{O}_{\widetilde{R}} \to 0 \quad \otimes \quad \mathcal{O}_{\widetilde{W}}(E + \widetilde{R})$$
$$0 \to \mathcal{O}_{\widetilde{W}}(E) \to \mathcal{O}_{\widetilde{W}}(H) \to \mathcal{O}_{\widetilde{R}}(H) \to 0$$

we obtain the exact sequence

$$0 \to H^0(\widetilde{W}; E) \to H^0(\widetilde{W}; H) \xrightarrow{\mathrm{rest}_R} H^0(R; H) \to \cdots$$

so $\ker(\mathrm{rest}_R) = h^0(\widetilde{W}; E) = 1$, which is a contradiction, and the unique hyperplane containing $C \overset{\text{def}}{=} \varphi(\widetilde{R})$ is $\varphi(E) = \mathbb{P}^3$.
We can describe C as a twisted cubic curve. First, since

$$H^2 \cdot C = (H - E)^2 \cdot (H - 2E)^2 = (H)^4 + 13(H)^2 \cdot E^2 + 4E^4 = 0$$

we deduce $\dim C = 1$, so C is an irreducible curve. Let F be a general fiber of $\widetilde{R} \to C$ and denote $d = \deg(C)$. Then

$$d(H\ )^2\ \ F = (H\ )^2\ \ (H\ \ \ \ E)\ \ \widetilde{R} = (H\ )^2\ \ (H\ \ \ \ E)\ \ (H\ \ \ \ 2E) = 3$$

and since C is not a line (it is contained in a unique hyperplane) it follows that $\deg(C) = 3$. Moreover, as C is a degree 3 curve in $\mathbb{P}^3$ which is not contained in a $\mathbb{P}^2$, it is rational[5], so in fact C is a twisted cubic curve in $'(E)$.

This computation also says that $(H\ )^2\ \ F = 1$ for a general fiber F, and in fact since C is rational we have $(H\ )^2\ \ F = 1$ for any fiber F of $'j_{\widetilde{R}}$. By the relative version of Nakai-Moishezon criterion (see [KM98, Theorem 1.42]) this implies $\ \ H$ is $'$-ample and moreover $'j_{\widetilde{R}}$ is a $\mathbb{P}^2$-bundle.

Finally, note that if $L\ 2\ \mathrm{Pic}(\mathbb{P}^4)$ is the ample generator, $\widetilde{R} = \ \ H\ \ \ 2E = '\ L\ \ \ E$ and then $O_{\widetilde{W}}(\widetilde{R})j_F = O_{\widetilde{W}}(\ \ E)j_F = O_F(\ \ 1)$. Moishezon contraction theorem ([Mis69, Fuj81]) implies that $'$ is the blow-up along C. □

As we did in the case of quartic del Pezzo fourfolds, as an example we will calculate in detail the dimension $h^0(W; D)$.

Lemma II.3.26. *With the notation of Theorem II.3.25, we have* $h^0(W; D) = 5$.

Proof. By Kodaira vanishing theorem we have $\ (\widetilde{W}; O_{\widetilde{W}}(D)) = h^0(\widetilde{W}; O_{\widetilde{W}}(D))$, and since $\widetilde{W}$ is Fano, we know that $\ (\widetilde{W}; O_{\widetilde{W}}) = 1$. Then by Hirzebruch-Riemann-Roch theorem we have the formula

$$h^0(\widetilde{W}; O_{\widetilde{W}}(D)) = 1 + \frac{1}{24}\Bigg[\underbrace{D^4}_{\text{(i)}} + 2\underbrace{D^3\ \ c_1(\widetilde{W})}_{\text{(ii)}} + \underbrace{D^2\ \ c_1(\widetilde{W})^2}_{\text{(iii)}} + \underbrace{D^2\ \ c_2(\widetilde{W})}_{\text{(iv)}} + \underbrace{D\ \ c_1(\widetilde{W})\ \ c_2(\widetilde{W})}_{\text{(v)}}\Bigg]$$

In the following we calculate every term in the formula.

(i) Using formulas for the Chow ring of blow-up, we have

$$H\ \ = H^4 \overset{\text{def}}{=} 5,\ (H\ )^3\ \ E = 0,\ (H\ )^2\ \ E^2 = \ \ \Xi\ \ H^2 = \ \ 1.$$

[5] In general, if $C\ \ \ \mathbb{P}^n$ is a $\deg(C) = n$ curve which is not contained in a hyperplane, we can consider the projection from a general point $x\ 2\ C$, $\ : \mathbb{P}^n$ 99K $\mathbb{P}^{n\ \ 1}$ and note that $\deg(\ (X)) = n\ \ \ 1$. Inductively, we reduce to $\mathbb{P}^2$.

By adjunction and Lemma II.3.24, we have $! \quad = \; !_W j \qquad c_1\left(N_{\;=W}\right)$, so $O_W(H)j \quad = O \;(1)$ and we obtain

$$H \quad E^3 = \quad Hj \quad K \;\; + K_W \quad H \;\; \Xi = \quad O\;(1) \;\; O\;(\;\;3) \quad 3H^2 \;\; \Xi = 3 \quad 3 = 0.$$

We can also calculate E^4 noting the following[6]

$$E^4 = (\;\;1)^4 \;\; {}^1 s_4 \;\; {}_2(N \;\; {}_{=W}) \;\; Hj^0 = \quad (c_1(N \;\; {}^2_{=W} \quad c_2(N \;\; {}_{=W})) = 2.$$

Using all this calculations we compute

$$D^4 = (H \;\;)^4 \quad 4\widetilde{H}^3 \;\; E + 6(H \;\;)^2 \;\; E^2 \quad 4H \quad E^3 + E^4 = 5 \quad 6 + 2 = 1.$$

(ii) Because $c_1(\widetilde{W}) \stackrel{\text{def}}{=} \quad K_{\widetilde{W}} = 3H \qquad E$, we directly obtain

$$D^3 \;\; c_1(\widetilde{W}) = 3(H \;\;)^4 + 12(H \;\;)^2 \;\; E^2 + E^4 = 15 \quad 12 + 2 = 5.$$

(iii) A direct calculation shows

$$D^2 \;\; c_2(\widetilde{W}) = 9(H \;\;)^4 + 22(H \;\;)^2 \;\; E^2 + E^4 = 45 \quad 22 + 2 = 25.$$

(iv) In this case the term corresponds to

$$\begin{aligned} D^2 \;\; c_2(\widetilde{W}) = H \qquad c_2(\widetilde{W}) + H \qquad \Xi \quad 2H \quad E \qquad c_2(W) \quad 2H \quad E \qquad \Xi \, + \\ 6(H \;\;)^2 \;\; E^2 + E^2 \qquad c_2(W) + E^2 \qquad \Xi \end{aligned}$$

By projection formula $(H \;\;)^2 \qquad \Xi = \quad (H^2 \;\; \Xi) = 1$ and using Schubert calculus

$$\begin{aligned} (H \;\;)^2 \qquad c_2(W) &= H^2 \;\; c_2(W) \\ &= \;\; {}_{1;0}j^2_W \;\; (4 \;\; {}_{2;0}j_W + 5 \;\; {}_{1;1}j_W) \\ &= (\;\; {}_{2;0}j_W + \;\; {}_{1;1}j_W) \;\; (4 \;\; {}_{2;0}j_W + 5 \;\; {}_{1;1}j_W) \\ &= (13 \;\; {}_{3;1} + 9 \;\; {}_{2;2})j_W \\ &= (13 \;\; {}_{3;1} + 9 \;\; {}_{2;2}) \qquad {}_{1;0} \\ &= 13 + 9 = 22 \end{aligned}$$

Directly, we have

$$E^3 \qquad K_W = \quad 3H \quad E^3 = 0,$$
$$E^2 \;\; H \qquad K_W = \quad 3E^2 \;\; (H \;\;)^2 = 3 \;\; \text{and} \;\; H \qquad K_W \;\; E = \quad 3E \;\; H^3 = 0.$$

□

[6]In dimension 4, the second Segre class is given by $s_2(E) = c_1(E)^2 \quad c_2(E)$.

Chapter III

K-stability

The aim of this chapter is to provide an introduction to K-stability and to outline the broader context of this fascinating topic. As we will soon see, K-stability originates in Differential Geometry but has grown over the years into a vibrant and multifaceted area of research. Its study requires a diverse array of tools and perspectives, ranging from group actions to valuations, making it both challenging and rewarding.

We begin by exploring the motivations behind the concept and then survey its development, starting from the foundational definitions to some of the modern techniques that have emerged. Along the way, we will touch upon a remarkable and unexpected connection: the deep interplay between K-stability and the Minimal Model Program. Finally, the chapter concludes with a collection of examples in dimension 2, where we demonstrate K-stability computations using the methods introduced throughout this survey.

This chapter is the result of a series of lectures I gave jointly with Pedro Montero during the *Algebraic Geometry Seminar* at *Pontificia Universidad Católica de Chile*, mainly based on lectures notes by Harold Blum and Kristin DeVleming. I am deeply grateful to Giancarlo Urzúa and the seminar participants for giving me the opportunity to take part in this enriching experience.

III.1 The Calabi problem for Fano varieties

To study projective (smooth) varieties X from the point of view of Differential Geometry, we consider Hermitian metrics h (instead of Riemannian metrics), which in turn are associated with

$$\omega \overset{\text{loc}}{=} \frac{i}{2}\sum_{ij} h_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j \text{ a } (1,1)\text{-form with } h = (h_{ij}) \text{ Hermitian matrix}$$
$$\text{where } h_{ij} : X \to \mathbb{C} \text{ are } C^\infty\text{-functions.}$$

We say that (X, ω) is a **Kähler variety** if $\mathrm{d}\omega = 0$.

Example III.1.1. In $\mathbb{P}^n$, the **Fubini-Study metric** associated to[1]

$$\omega_{\mathrm{FS}} := \frac{i}{2}\partial\overline{\partial}\log\|z\|^2$$

is a Kähler metric. Thus, for any $X \subset \mathbb{P}^n$ smooth projective subvariety, we have that $\omega := \omega_{\mathrm{FS}}|_X$ is a Kähler metric on X, i.e., *every smooth projective variety is a Kähler variety.*

Remark III.1.2. If $X = \mathbb{P}^1$, $\mathbb{C}/\Lambda$ or $\mathbb{D}/\Gamma$ is a Riemann surface, there are classical metrics (Fubini-Study, Euclidean, Poincaré, respectively) on X with constant curvature ($+1$, 0, -1, respectively).

In 1954, Eugenio Calabi proposed studying the existence of a Kähler metric ω on every smooth projective variety X such that

$$\operatorname{Ric}(\omega) = \lambda\omega \text{ for some } \lambda \in \{-1, 0, 1\} \text{ (Kähler-Einstein Equation)}$$

where $\operatorname{Ric}(\omega) \overset{\text{loc}}{=} -i\partial\overline{\partial}\log\det(h_{ij})$ is the Ricci curvature of ω.

Example III.1.3. In the affine chart $U_0 = \{Z = [Z_0 ; Z_1 ; Z_2] \in \mathbb{P}^2 ; Z_0 \neq 0\} = \mathbb{A}^2$ of $\mathbb{P}^2$ with coordinates (z_1, z_2), we have $\omega_{\mathrm{FS}} \overset{\text{loc}}{=} \frac{i}{2}\partial\overline{\partial}\log(1+|z_1|^2+|z_2|^2) = \frac{i}{2}\sum_{ij} h_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j$ where

$$h = (h_{ij}) = \frac{\lambda}{(1+|z_1|^2+|z_2|^2)^2}\begin{pmatrix} 1+|z_2|^2 & -\overline{z}_1 z_2 \\ -z_1\overline{z}_2 & 1+|z_1|^2 \end{pmatrix},$$

so we have $\det(h_{ij}) = \lambda^2(1+|z_1|^2+|z_2|^2)^{-3}$ and then $\operatorname{Ric}(\omega_{\mathrm{FS}}) \overset{\text{def}}{=} 6\lambda\,\omega_{\mathrm{FS}}$. Since Ric is invariant under rescalings $\omega \mapsto \lambda_0^{-1}\omega$, we can normalize to obtain $\lambda = 1$. Similarly, $\operatorname{Ric}(\omega_{\mathrm{FS}}) = 2\pi(n+1)\omega_{\mathrm{FS}}$ in $\mathbb{P}^n$.

Theorem III.1.4 (Kodaira's Theorem)**.** *The Ricci curvature* $\operatorname{Ric}(\omega)$ *defines a real* $(1,1)$*-form such that* $[\operatorname{Ric}(\omega)] = 2\pi c_1(X) \overset{\text{def}}{=} 2\pi[-K_X] \in \mathrm{H}^{1,1}(X, \mathbb{R})$*. Thus, in the case* $\lambda = -1$ *(resp.* $\lambda = 1$*) we have that* $[K_X] = [\omega]$ *(resp.* $[-K_X] = [\omega]$*) is cohomologous to a positive* $(1,1)$*-form. Kodaira's embedding theorem ensures that* K_X *(resp.* $-K_X$*) is* **ample***. Thus, the Kähler-Einstein equation implies that:*

[1] Here, $\partial f = \sum \frac{\partial f}{\partial z^i}\,\mathrm{d}z^i$ and $\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}^j}\,\mathrm{d}\overline{z}^j$.

1. X *is* **canonically polarized** *(i.e., K_X ample) if* $= 1$.
2. X *is* **Calabi-Yau** *if* $= 0$.
3. X *is* **Fano** *(i.e., K_X ample) if* $= 1$.

The existence of Kähler-Einstein metrics on **all** canonically polarized and Calabi-Yau varieties are fundamental results in Geometric Analysis by Aubin and Yau, respectively.

Theorem III.1.5 (Calabi-Yau theorem, [Szé14]). *Let $(M; !)$ be a compact Kähler manifold, and let be a real $(1; 1)$-form representing $c_1(M)$. Then there exists a unique Kähler metric on M with $[\ ] = [!]$ such that* $\mathrm{Ric}(\ ) = 2$.

On the other hand, we will see that **not every** *Fano variety admits a Kähler-Einstein metric.*

Example III.1.6. Let X be a smooth Fano variety. Then,

- $(\dim(X) = 1)$ $X = \mathbb{P}^1$ is Kähler-Einstein.
- $(\dim(X) = 2)$ $X = \mathbb{P}^1 \ \mathbb{P}^1$, $\mathbb{P}^2$ or $\mathrm{Bl}_{p_1;:::;p_r}(\mathbb{P}^2)$ blow-up at r 8 points in general position. We will see that all of them are Kähler-Einstein **except for** $\mathrm{Bl}_p(\mathbb{P}^2) = \mathbb{F}_1$ and $\mathrm{Bl}_{p_1;p_2}(\mathbb{P}^2)$.
- $(\dim(X) = 3)$ Iskovskikh, Mori and Mukai classified the 3-folds of Fano into $17 + 88 = 105$ families. In 2023, in [ACC+23] was proved that for exactly 78 families, the general member admits a Kähler-Einstein metric.

Historically, the concept of K-stability was introduced by G. Tian in [Tia97], as a criterion to characterize the existence of Kähler-Einstein metrics on Fano manifolds. This definition depends on the sign of an analytic invariant, called the *generalized Futaki invariant*, which in turn has its origins in [Fut83]. In the last article it is defined a linear functional on the Lie algebra of vector fields of a Kähler manifold, and it is proved that the kernel of that functional provides an obstruction for the existence of Kähler-Einstein metrics on Fano manifolds.

Later, in [Don02] it is given a definition of K-stability using purely algebraic geometry terms, and that definition makes sense for polarized varieties. The following is the fundamental theorem that justify the importance of K-stability.

Theorem III.1.7 (Chen-Donaldson-Sun 2014, Tian 2015). *A smooth Fano variety X admits a Kähler-Einstein metric if and only if $(X;\ K_X)$ is K-polystable.*

III.2 Definition of K-stability

The purpose of this section is to present the very definition of K-(semi/poly)stability, which involves the definition of two crucial concepts: test configurations and the Donaldson-Futaki invariant. First, we present the complete definition (which is very obscure) and then we will develop the basic aspects of the mentioned concepts.

Nowadays, the definition of K-stability looks like the following form. This is the definition stated in [ACC+23].

Definition III.2.1. Let X be a Fano klt variety of dimension $\dim(X) = n \geq 2$ and let $L := -K_X$ be the anticanonical divisor. A test configuration of the polarized pair (X, L) consists of the following data:

1. a normal variety $\mathcal{X}$ equipped with a $\mathbb{G}_m$-action.
2. a flat $\mathbb{G}_m$-equivariant morphism $\pi : \mathcal{X} \to \mathbb{P}^1$ where we consider the natural action $\mathbb{G}_m \curvearrowright \mathbb{P}^1, (t, [x; y]) \mapsto [tx; y]$.
3. a $\mathbb{G}_m$-equivariant π-ample $\mathbb{Q}$ line bundle $\mathcal{L} \to \mathcal{X}$ and a $\mathbb{G}_m$-equivariant isomorphism
$$\left(\mathcal{X}\setminus\pi^{-1}(0), \mathcal{L}|_{\mathcal{X}\setminus\pi^{-1}(0)}\right) \cong \left(X \times (\mathbb{P}^1\setminus\{0\}), \mathrm{pr}_1^*(L)\right)$$
where pr_1 denotes the projection to the first factor and $0 = [0; 1]$.

For such a test configuration, we define the Donaldson-Futaki invariant
$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n}\left(\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1} + \frac{n}{n+1}\mathcal{L}^{n+1}\right).$$

Denote $\mathcal{X}_0 = \pi^{-1}(0)$ and $\mathcal{X}_\infty = \pi^{-1}(\infty)$ where $\infty := [1; 0]$. The test configuration is said to be

1. *trivial* if there is a $\mathbb{G}_m$-equivariant isomorphism
$$\left(\mathcal{X}\setminus\mathcal{X}_\infty, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_\infty}\right) \cong \left(X \times (\mathbb{P}^1\setminus\infty), \mathrm{pr}_1^*(L)\right),$$
2. *product-type* if there is an isomorphism
$$\left(\mathcal{X}\setminus\mathcal{X}_\infty, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_\infty}\right) \cong \left(X \times (\mathbb{P}^1\setminus\infty), \mathrm{pr}_1^*(L)\right).$$

A Fano variety X is said to be

1. K-semistable if for every test configuration $(\mathcal{X};\mathcal{L})$ we have $\mathrm{DF}(\mathcal{X};\mathcal{L}) \geq 0$.

2. K-stable if for every non-trivial test configuration $(\mathcal{X};\mathcal{L})$ we have $\mathrm{DF}(\mathcal{X};\mathcal{L}) > 0$.

3. K-polystable if it is K-semistable and $\mathrm{DF}(\mathcal{X};\mathcal{L}) \equiv 0$ if and only if $(\mathcal{X};\mathcal{L})$ is product-type.

Remark III.2.2. By definition we have the implications

$$X \text{ is K-stable} \implies X \text{ is K-polystable} \implies X \text{ is K-semistable.}$$

This definition arises from a considerable amount of theory, and now we are going to unravel this definition.

III.2.1 Test configurations

In this section we consider X as a (separated, finite type over $k = \mathbb{C}$) projective scheme and $L \in \mathrm{Pic}(X)$ as an ample line bundle.

Definition III.2.3. Suppose there is an action $\sigma : \mathbb{G}_m \times X \to X$ of the affine algebraic group $\mathbb{G}_m$ and let $L \in \mathrm{Pic}(X)$. A $\mathbb{G}_m$-linearization of L is an action of $\mathbb{G}_m$ on the total space $\mathbf{V}(L)$ of L that makes the projection $\mathbf{V}(L) \to X$ a $\mathbb{G}_m$-equivariant morphism and such that the action on the fibers is linear. More formally, it is an action $\tilde{\sigma} : \mathbb{G}_m \times \mathbf{V}(L) \to \mathbf{V}(L)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbf{V}(L) \arrow[r, "\tilde{\sigma}"] \arrow[d, "\mathrm{Id}_{\mathbb{G}_m}\times\pi"'] & \mathbf{V}(L) \arrow[d, "\pi"] \\
\mathbb{G}_m \times X \arrow[r, "\sigma"] & X
\end{tikzcd}$$

and such that the zero section $\mathbf{0}_L \subset \mathbf{V}(L)$ is $\mathbb{G}_m$-invariant.

Remark III.2.4. Since $\mathrm{Pic}(\mathbb{G}_m) = \{1\}$, every $L \in \mathrm{Pic}(X)$ admits a linearization and the possible classes of linearizations are parametrized by the character group $\mathsf{X}(\mathbb{G}_m) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{gr}}(\mathbb{G}_m;\mathbb{G}_m) = \mathbb{Z}$. For example, if $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(-1)$ then

$$\mathbf{V}(L) \stackrel{\text{def}}{=} \{([x_0;x_1];\ \lambda(x_0;x_1));\ [x_0;x_1] \in \mathbb{P}^1;\ \lambda \in \mathbb{C}\};$$

and the $\mathbb{G}_m$-action given by $t \cdot ([x_0;x_1];\ \lambda(x_0;x_1)) := ([x_0;tx_1];\ \lambda(x_0;tx_1))$ determines a $\mathbb{G}_m$-linearization of L.

Now, our starting definition of test configuration will be the following.

Definition III.2.5 (test configuration)**.** A **test configuration** of (X, L) is a pair $(\mathcal{X}, \mathcal{L})$ along with

1. a proper and flat morphism $\pi : \mathcal{X} \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$,
2. a $\mathbb{G}_m$-action on $\mathcal{X}$ such that π is equivariant for the standard action $(a, t) \mapsto at$ of $\mathbb{G}_m$ on $\mathbb{A}^1$,
3. a $\mathbb{Q}$-line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ that is π-ample and $\mathbb{G}_m$-linearized on $\mathcal{X}$,
4. an isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L)$ between the general fiber $\mathcal{X}_1 := \pi^{-1}(1)$ and the original polarized variety.

The idea behind this definition comes from Geometric Invariant Theory (GIT). That we are doing is to consider $\mathbb{G}_m$-equivariant degenerations of a polarized variety, and then performing a limit process. In summary, this process can be thought like use GIT limits in the Hilbert scheme.

Example III.2.6. Let (X, L) be as in the previous definition.

1. The **trivial test configuration** $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1}) := (X, L) \times \mathbb{A}^1$ is the one where the action of $\mathbb{G}_m$ on $X_{\mathbb{A}^1}$ is the product action, with the action on X being **trivial** and the action on $\mathbb{A}^1$ being the standard action.
2. A **product test configuration** is $(X_{\mathbb{A}^1}, L_{\mathbb{A}^1})$ as before, except that the action of $\mathbb{G}_m$ on X is not necessarily trivial. If $\mathrm{Aut}(X) = \{1\}$, then every product configuration is trivial.
3. Let $Z \subseteq X$ be a closed subscheme and let $\mu : \mathcal{X} := \mathrm{Bl}_{Z \times 0}(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$. Then $\pi := \mathrm{pr}_{\mathbb{A}^1} \circ \mu : \mathcal{X} \to \mathbb{A}^1$ is a proper and flat morphism[2]. Also, $\mathcal{X}_0 = E + F$, where $E = \mu^{-1}(Z \times 0)$ is the exceptional divisor and where $F = \mathrm{Bl}_Z(X)$ is the strict transform of $X \times 0$. If $Z \subseteq X$ is $\mathbb{G}_m$-invariant (e.g., by the trivial action on X), then there is an induced action of $\mathbb{G}_m$ on $\mathcal{X}$, and since $-E$ is μ-ample, we have that $\mathcal{L} := \mu^* L_{\mathbb{A}^1} \otimes \mathcal{O}_{\mathcal{X}}(-tE)$ is π-ample for $0 < t \ll 1$. Thus, $(\mathcal{X}, \mathcal{L})$ is a test configuration.

[2] Known as the **deformation to the normal cone** in *Intersection Theory* (see [Ful84, §5.1])

4. Let $r \in \mathbb{N}^{\geq 1}$ be such that $rL := L^{\otimes r}$ is very ample, i.e., $\iota : X \hookrightarrow \mathbb{P}(V^{\vee})$ is an embedding, where $V = \mathrm{H}^0(X, rL)$. A group morphism $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ induces a test configuration $(\mathcal{X}_\rho, \mathcal{L}_\rho)$ where $\mathcal{X}_\rho$ is the Zariski closure in $\mathbb{P}(V^{\vee}) \times \mathbb{A}^1$ of the image of $X \times \mathbb{G}_m \hookrightarrow \mathbb{P}(V^{\vee}) \times \mathbb{G}_m, (x, a) \mapsto (\rho(a)x, a)$ and where $\mathcal{L}_\rho := \frac{1}{r}\mathcal{O}_{\mathcal{X}_\rho}(1)$. All test configurations of (X, L) are obtained in this way (Ross–Thomas, 2007).

Note the definition doesn't require that a test configuration to be projective, so a natural question is if it we can define canonically a compactification process for a test configuration (this will be very important later in order to write the Donaldson-Futaki invariant in terms of intersection numbers).

Construction III.2.7 (Compactification)**.** Given a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L), we can consider the $\mathbb{G}_m$-equivariant families $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ and $(X, L) \times (\mathbb{P}^1 \setminus \{0\}) \to (\mathbb{P}^1 \setminus \{0\})$, where the $\mathbb{G}_m$ action on $(X, L) \times (\mathbb{P}^1 \setminus \{0\})$ corresponds to the product of the trivial action on (X, L) and the standard action on $\mathbb{P}^1 \setminus \{0\}$. We have a $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0}) \cong (X, L) \times (\mathbb{A}^1 \setminus \{0\})$$
$$(p, s) \mapsto (a^{-1} \cdot p, a^{-1} \cdot s) \times \{a\}$$

where $a = \pi(p)$, and therefore this isomorphism allows us to glue the two previous families, obtaining the *compactification* $\bar{\pi} : (\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$. This compactification has the following properties:

1. the morphism $\bar{\pi} : (\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$ is flat, proper, and $\mathbb{G}_m$-equivariant.
2. the $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}$ is $\bar{\pi}$-ample and $\mathbb{G}_m$-linearized.
3. the fiber over ∞ corresponds to $\left(\overline{\mathcal{X}}_\infty, \overline{\mathcal{L}}_\infty\right) \cong (X, L)$.

Example III.2.8. Consider the product test configuration $\mathcal{X} = \mathbb{P}^1 \times \mathbb{A}^1$ induced by the action

$$t \cdot [x : y] = [t^d x : y]$$

for some $d \in \mathbb{Z}$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1) \times \mathbb{A}^1$. The compactification of this test by definition results to be

$$\overline{\mathcal{X}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$$

III.2.2 Donaldson-Futaki invariant

Here we will give a more understandable Donaldson-Futaki invariant, which is defined in terms of weights of a group action. After that, we will derive an intersection formula for this invariant, which will give the formula presented in Definition III.2.1. First, we recall the following definition.

Definition III.2.9. Consider V a finite-dimensional k-vector space with an action $\mathbb{G}_m \curvearrowright V$. The weight of the action is defined as

$$\mathrm{wt}(V) = \sum_{\lambda \in \mathbb{Z}} \lambda \dim (V_\lambda),$$

where $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$ is the weight decomposition of V where

$$V_\lambda := \{v \in V \mid t \cdot v = t^\lambda v \text{ for all } t \in \mathbb{G}_m(k)\}.$$

Remark III.2.10. If $\mathbb{G}_m$ acts on V, a vector space with $\dim(V) = n$, then there is an induced $\mathbb{G}_m$-action on $\det(V) := \bigwedge^n V$, and it verifies $\mathrm{wt}(\det(V)) = \mathrm{wt}(V)$. Indeed, if $s_1, \dots, s_n \in V$ is a basis and $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ are such that $t \cdot s_i = t^{\lambda_i} s_i$, then

$$t \cdot s_1 \wedge \dots \wedge s_n = t^{\sum_i \lambda_i} s_1 \wedge \dots \wedge s_n$$

and thus $\mathrm{wt}(\det(V)) = \mathrm{wt}(V)$.

From now on, we consider X as a complex projective variety with $\dim(X) = n$, and $(\mathcal{X}, \mathcal{L})$ a test configuration of a polarized pair (X, L). For $m \in \mathbb{N}$ such that $m\mathcal{L}$ is a line bundle, we define

$$N_m := \dim \mathrm{H}^0(\mathcal{X}_0, m\mathcal{L}_0) \quad \text{and} \quad w_m := \mathrm{wt}\, \mathrm{H}^0(\mathcal{X}_0, m\mathcal{L}_0)$$

It is known that the values N_m (the Hilbert polynomial of $\mathcal{X}_0$) are given by a polynomial with rational coefficients of degree n. Furthermore, it is possible to prove that the values of w_m are given by a rational polynomial of degree $n+1$, so we have an expansion

$$\frac{w_m}{mN_m} = F_0 + F_1 m^{-1} + F_2 m^{-2} + \dots$$

for $m > 0$ sufficiently divisible.

Definition III.2.11 (Futaki Invariant)**.** The *Donaldson-Futaki invariant* of $(\mathcal{X};\mathcal{L})$ is defined as

$$\mathrm{DF}(\mathcal{X};\mathcal{L}) := -2F_1$$

Example III.2.12. Consider the same product test configuration of Example III.2.8, i.e., $\mathcal{X} = \mathbb{P}^1 \times \mathbb{A}^1$ induced by the action

$$t \cdot [x:y] = [t^d x : y]$$

for some $d \in \mathbb{Z}$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathbb{A}^1$. In this case, we see that

$$\mathrm{H}^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(m)) = \mathbb{C}[X;Y]_m \text{ with } t \cdot (x^k y^{m-k}) = t^{dk} x^k y^{m-k};$$

so $w_m = \left(\frac{m(m+1)}{2}\right) d$, $N_m = m+1$, and $\mathrm{DF}(\mathcal{X};\mathcal{L}) = 0$ (and thus $\mathbb{P}^1$ is **not** K-stable).

Lemma III.2.13. *Let $\mathbb{G}_m \curvearrowright \mathbb{P}^1$ be the action given by $t \cdot [x:y] = [tx:y]$. Given a $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbb{P}^1}(m)$, then*

$$\mathrm{wt}\,(\mathcal{O}_{\mathbb{P}^1}(m)_0) - \mathrm{wt}\,(\mathcal{O}_{\mathbb{P}^1}(m)_\infty) = m$$

where $0 := [0:1]$ *and* $\infty := [1:0]$.

Proposition III.2.14. *If $(\mathcal{X};\mathcal{L})$ is a test configuration of $(X;L)$ and $n = \dim(X)$, there exist $a_i; b_i \in \mathbb{Q}$ such that*

$$N_m := \dim \mathrm{H}^0(\mathcal{X}_0; m\mathcal{L}_0) = a_0 m^n + a_1 m^{n-1} + \dots + a_n$$
$$w_m := \mathrm{wt}\, \mathrm{H}^0(\mathcal{X}_0; m\mathcal{L}_0) = b_0 m^{n+1} + b_1 m^n + \dots + b_{n+1}$$

for all $m > 0$ sufficiently divisible. Moreover,

$$a_0 = \frac{L^n}{n!} \quad \textit{and} \quad b_0 = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}$$

and if $\mathcal{X}$ is normal, also

$$a_1 = -\frac{L^{n-1} \cdot K_X}{2(n-1)!} \quad \textit{and} \quad b_1 = -\frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{2n!}$$

Proof. Serre's vanishing theorem implies that

$$\mathrm{H}^i(\overline{\mathcal{X}}_t; m\overline{\mathcal{L}}_t) = 0 \qquad \forall i > 0; m \gg 0; \forall t \in \mathbb{P}^1.$$

The cohomology base change theorem (see [Har77, Theorem II.12.11]) implies that for an m such that the above holds, we also have that

1. $R^i\pi_*\mathcal{O}_{\overline{X}}(m\overline{L}) = 0$ for all $i > 0$.
2. $\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})$ is locally free.
3. $\pi_*\mathcal{O}_{\overline{X}}(m\overline{L}) \otimes k(t) \to \mathrm{H}^0(X_t, mL_t)$ is an isomorphism for all $t \in \mathbb{P}^1$.

Conditions (2) and (3) allow us to state that

$$N_m := \dim \mathrm{H}^0(X_0, mL_0) = \dim \mathrm{H}^0(X_1, mL_1) = \dim \mathrm{H}^0(X, mL)$$

thus the statement about N_m and the formulas for a_0, a_1 are obtained from the Riemann-Roch theorem.

On the other hand, we can consider the line bundle $\det\left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right)$, which is a $\mathbb{G}_m$-linearized bundle over $\mathbb{P}^1$. From condition (3), we deduce

$$\mathrm{wt}\left(\det\left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right)_0\right) = \mathrm{wt}\left(\det \mathrm{H}^0(X_0, mL_0)\right) = w_m$$

and since $\mathbb{G}_m$ acts trivially on the fiber $(\overline{X}, \overline{L})_\infty$, we have $\mathrm{wt}\left(\det\left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right)_\infty\right) = 0$, and the previous lemma implies

$$\det\left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right) \cong \mathcal{O}_{\mathbb{P}^1}(w_m)$$

The Hirzebruch-Riemann-Roch theorem for vector bundles on curves, combined with conditions (1), (2), (3) and Leray's direct image theorem, allows us to make the following calculation

$$\begin{aligned} w_m = \deg\left(\mathcal{O}_{\mathbb{P}^1}(w_m)\right) = \deg\left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right) &= \chi\left(\mathbb{P}^1, \left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right)\right) - \mathrm{rk}\left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right) \\ &= \chi\left(\mathbb{P}^1, \left(\pi_*\mathcal{O}_{\overline{X}}(m\overline{L})\right)\right) - N_m \\ &= \chi\left(\overline{X}, \mathcal{O}_{\overline{X}}(m\overline{L})\right) - N_m \end{aligned}$$

The above allows us to conclude that the values w_m are given by a polynomial with rational coefficients of degree $n+1$, so we only need to find its coefficients. The coefficient b_0 is obtained directly from the Riemann-Roch theorem, as $\deg(N_m) = n$. To finish, note that

$$2L^n = 2\overline{L}^n \cdot \mathcal{O}_{\overline{X}}(X_1) = \overline{L}^n \cdot \pi^*\mathcal{O}_{\mathbb{P}^1}(2) = -\overline{L}^n \cdot \pi^*K_{\mathbb{P}^1}$$

and from the Riemann-Roch theorem (assuming $\overline{X}$ is normal) we see that

$$w_m = \frac{\overline{L}^{n+1}}{(n+1)!}m^{n+1} - \frac{\overline{L}^n \cdot K_{\overline{X}} + 2L^n}{2n!}m^n + O(m^{n-1})$$

The last two calculations allow us to deduce b_1. $\square$

Theorem III.2.15 (Wang-Odaka)**.** *If $(\mathcal{X};\mathcal{L})$ is a test configuration of $(X;L)$ and $\mathcal{X}$ is normal, then*

$$\mathrm{DF}(\mathcal{X};\mathcal{L}) = \frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{V} + \bar{S}\frac{\overline{\mathcal{L}}^{n+1}}{(n+1)V}$$

where $V = L^n$ and $\overline{S} = nV^{-1}(-K_X \cdot L^{n-1})$.

Proof. It suffices to note that $\mathrm{DF}(\mathcal{X};\mathcal{L}) = \frac{2(b_0a_1 - b_1a_0)}{a_0^2}$. □

Since Wang-Odaka's formula is valid for normal tests, we expect (and we need) to have a way to *normalize* test configurations, and that the normalization be well-behaved in terms of the Donaldson-Futaki invariant.

Construction III.2.16. Let $(\mathcal{X};\mathcal{L})$ be a test configuration of a polarized variety $(X;L)$ with X normal. We define the normalization of $(\mathcal{X};\mathcal{L})$ as $(\widetilde{\mathcal{X}};\widetilde{\mathcal{L}})$ where $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is the normalization morphism of $\mathcal{X}$ and $\widetilde{\mathcal{L}} := \pi^*\mathcal{L}$. Indeed, this result in a test configuration since the composition $\widetilde{\mathcal{X}} \to \mathcal{X} \to \mathbb{A}^1$ is proper and flat, and by the universal property of normalization there exists a unique morphism $\tilde{\rho} : \mathbb{G}_m \times \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ such that the diagram

$$\begin{tikzcd} \mathbb{G}_m \times \widetilde{\mathcal{X}} \arrow[r, dashed, "\exists!"] \arrow[d, "\mathrm{Id}\times\pi"'] & \widetilde{\mathcal{X}} \arrow[d, "\pi"] \\ \mathbb{G}_m \times \mathcal{X} \arrow[r, "\rho"] & \mathcal{X} \end{tikzcd}$$

commutes and $\widetilde{\mathcal{L}}$ is ample over $\mathbb{A}^1$ since π is finite. This implies $\tilde{\rho}$ defines a $\mathbb{G}_m$-action on $\widetilde{\mathcal{X}}$ and since $\mathcal{X}|_{\mathbb{A}^1\setminus\{0\}}$ was already normal, we have an isomorphism $\widetilde{\mathcal{X}}|_{\mathbb{A}^1\setminus\{0\}} \cong X \times (\mathbb{A}^1 \setminus \{0\})$.

The previous construction is well-behaved for our purposes in the sense of the following lemma.

Lemma III.2.17. *Let $(X;L)$ be a polarized variety. If $(\mathcal{X};\mathcal{L})$ is a test configuration, then*

$$\mathrm{DF}(\widetilde{\mathcal{X}};\widetilde{\mathcal{L}}) \leq \mathrm{DF}(\mathcal{X};\mathcal{L}).$$

Proof. For reasons of notation, in this proof we denote the normalization by $(\mathcal{X}^\nu;\mathcal{L}^\nu)$, and we denote $w_m = \mathrm{wt}\, H^0(\mathcal{X}_0; m\mathcal{L}_0); N_m = \dim H^0(\mathcal{X}_0; m\mathcal{L}_0)$. Note that for $m >$

0 sufficiently divisible, proof of Proposition III.2.14 tells us $N_m = \mathrm{H}^0(X, mL) = N_m$ and also

$$w_m = (\overline{X}, \mathcal{O}_{\overline{X}}(m\overline{L})) \quad N_m.$$

We denote by $^- : \overline{X} \to \overline{X}$ the normalization morphism of the compactified test. Now consider the exact sequence

$$\begin{array}{l} 0 \to \mathcal{O}_{\overline{X}} \to {}^- \mathcal{O}_{\overline{X}} \to F \to 0 \qquad = \quad \mathcal{O}_{\overline{X}}(m\overline{L}) \\ 0 \to \mathcal{O}_{\overline{X}}(m\overline{L}) \to {}^-(\mathcal{O}_{\overline{X}} (m\overline{L})) \to F \quad \mathcal{O}_{\overline{X}}(m\overline{L}) \to 0 \end{array}$$

where $F := {}^- \mathcal{O}_{\overline{X}} / \mathcal{O}_{\overline{X}}$, and where we have used

$$({}^- \mathcal{O}_{\overline{X}}) \quad \mathcal{O}_{\overline{X}}(m\overline{L}) = {}^-(\mathcal{O}_{\overline{X}} \quad {}^- \mathcal{O}_{\overline{X}}(m\overline{L})) = {}^- \mathcal{O}_{\overline{X}} (m\overline{L}).$$

As $^-$ is an affine morphism, we have that $R^{i-} F = 0$ for all $i > 0$, and then $H^i(\overline{X}, F) = H^i(\overline{X}, {}^- F)$ for all $i \quad 0$. Using this and Serre's vanishing theorem we can compute

$$\begin{aligned} (\overline{X}, \mathcal{O}_{\overline{X}} (m\overline{L})) &= (\overline{X}, {}^- \mathcal{O}_{\overline{X}} (m\overline{L})) \\ &= (\overline{X}, \mathcal{O}_{\overline{X}}(m\overline{L})) + (\overline{X}, F \quad \mathcal{O}_{\overline{X}}(m\overline{L})) \\ &= (\overline{X}, \mathcal{O}_{\overline{X}}(m\overline{L})) + H^0(\overline{X}, F \quad \mathcal{O}_{\overline{X}}(m\overline{L})) \end{aligned}$$

As a result we obtain

$$w_m = w_m + \dim \mathrm{H}^0 \left(\overline{X}, F \quad \mathcal{O}_{\overline{X}}(m\overline{L})\right) \quad w_m$$

which permits to conclude $\mathrm{DF}(X, L) \quad \mathrm{DF}(X, L)$. □

Remark III.2.18. It is important to understand how these notions of stability are related.

1. If there is a non-trivial $\mathbb{G}_m \curvearrowright X$ action and a $\mathbb{G}_m$-linearization of L, then (X, L) is not K-stable. Indeed, the action and its dual action give rise to two non-trivial test configurations $(X, L), (X', L')$ that satisfy $\mathrm{DF}(X, L) + \mathrm{DF}(X', L') = 0$, so one of these numbers is non-positive.

2. If X is a Fano variety over $\mathbb{C}$, then:

   (a) $(X, \quad K_X)$ is K-polystable if and only if X admits a Kähler-Einstein metric.

(b) $(X, -K_X)$ is K-stable if and only if X admits a Kähler-Einstein metric and $\mathrm{Aut}(X)$ is finite.

(c) $(X, -K_X)$ is K-semistable if and only if there exists a test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ such that $\mathcal{X}_0$ is a (possibly singular) Fano variety that admits a Kähler-Einstein metric.

III.3 Valuations and Test Configurations

The aim of this section is to present a correspondence between test configurations of polarized complex varieties (X, L) and valuations of its rational function field $\mathbb{C}(X)$. The spirit behind this is the fact that the rational function field is a birational invariant, so having such a correspondence would exhibit the birational nature of K-stability. This section will be divided in the proof of the two following correspondences

$$\left\{\begin{array}{c}\text{Test configurations}\\ \text{of } (X, L)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Filtrations of}\\ R(X, L) := \bigoplus_{m \geq 0} \mathrm{H}^0(X, mL)\}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Valuations}\\ \text{of } \mathbb{C}(X)\end{array}\right\}$$

These correspondences where deeply studied in [BHJ17].

III.3.1 Test configurations and Rees algebras

Here we discuss the first half of the correspondences. We start with the following observation.

Remark III.3.1. Given a vector space V, there is a bijection between linear actions of $\mathbb{G}_m$ on V and $\mathbb{Z}$-gradings on V. Given an action of $\mathbb{G}_m$ on V, there is a **weight decomposition** $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$ where

$$V_\lambda := \{v \in V;\ a \cdot v = a^\lambda v \text{ for all } a \in \mathbb{G}_m\}.$$

Conversely, a $\mathbb{Z}$-grading $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$ allows us to define $a \cdot v := \sum a^\lambda v_\lambda$, for any $v = \sum v_\lambda$.

Definition III.3.2. Let V be a finite-dimensional vector space (e.g., $\mathrm{H}^0(X, mL)$). A **$\mathbb{Z}$-filtration** of V is a collection of subspaces $\{F^\lambda\}_{\lambda \in \mathbb{Z}} \subseteq V$ such that

1. $F^{\lambda+1}V \subseteq F^\lambda V$ for all $\lambda \in \mathbb{Z}$, i.e., it is a **decreasing** filtration,

2. $F^\lambda V = 0$ for all $\lambda \gg 0$, and

3. $F^\lambda = V$ for all $\lambda \ll 0$.

The **Rees algebra** associated with the filtration $F^\bullet$ is the finitely generated and torsion-free $k[t]$-module given by $\mathrm{Rees}(F^\bullet) := \bigoplus_{\lambda \in \mathbb{Z}} F^\lambda V t^{-\lambda}$, with $k[t]$-module structure given by $t \cdot (vt^{-\lambda}) := vt^{-\lambda+1}$.

Construction III.3.3 (Rees correspondence)**.** There is a bijective correspondence between

$$\left\{ \begin{array}{c} \mathbb{G}_m\text{-linearized vector} \\ \text{bundles } \mathcal{V} \to \mathbb{A}^1 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \mathbb{Z}\text{-filtrations of finite-dimensional} \\ \text{vector spaces } V \end{array} \right\}.$$

Indeed, given a vector space V and a $\mathbb{Z}$-filtration $F^\bullet V$, its Rees algebra $R := \mathrm{Rees}(F^\bullet)$ induces a locally free sheaf $\widetilde{R}$ on $\mathbb{A}^1 = \mathrm{Spec}(k[t])$, and then it induces a vector bundle $\mathcal{V} := \mathbf{V}(\widetilde{R}) \to \mathbb{A}^1$. Since R admits a $\mathbb{Z}$-grading compatible with the $\mathbb{Z}$-grading of $k[t]$, we have that $\mathcal{V} \to \mathbb{A}^1$ is a $\mathbb{G}_m$-linearized vector bundle.

Given a $\mathbb{G}_m$-linearized vector bundle $\mathcal{V} \to \mathbb{A}^1$, there is an induced $\mathbb{G}_m$ action on its global sections, which gives a weight decomposition

$$\mathrm{H}^0(\mathbb{A}^1, \mathcal{V}) = \bigoplus_{\lambda \in \mathbb{Z}} \mathrm{H}^0(\mathbb{A}^1, \mathcal{V})_\lambda .$$

It is important to note that the $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}]) \stackrel{\text{def}}{=} \{t \neq 0\} \subseteq \mathbb{A}^1$ action on the global sections is given by the **dual** representation $t \cdot \sigma(x) := \sigma(t^{-1} \cdot x)$ and thus $t \in k[t]$ acts with weight -1 on the $k[t]$-module $\mathrm{H}^0(\mathbb{A}^1, \mathcal{V})$, i.e., $\mathrm{H}^0(\mathbb{A}^1, \mathcal{V})_\lambda \xrightarrow{t} \mathrm{H}^0(\mathbb{A}^1, \mathcal{V})_{\lambda-1}$ is an **injective** $k[t]$-module morphism.

We can construct $F^\bullet V$ geometrically as follows: Let $V := \mathcal{V}_1$ be the general fiber of $\mathcal{V} \to \mathbb{A}^1$ and

$$F^\lambda V := \mathrm{Im}\left(\mathrm{H}^0(\mathbb{A}^1, \mathcal{V})_\lambda \xrightarrow{\mathrm{ev}_1} V, \; s \mapsto s(1)\right)$$

where $F^\lambda V \subseteq F^{\lambda-1} V$ since t is an injective morphism. Also, $F^\lambda V = 0$ (resp. $F^\lambda V = V$) for $\lambda \gg 0$ (resp. $\lambda \ll 0$) since $\mathrm{H}^0(\mathbb{A}^1, \mathcal{V})$ is a finitely generated $k[t]$-module (resp. $\mathrm{Im}(\mathrm{H}^0(\mathbb{A}^1, \mathcal{V}) \xrightarrow{\mathrm{ev}_1} V) = V$).

Remark III.3.4. The above construction has two consequences that will help us with calculations.

1. Since $\mathcal{V} \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant, we have $\mathcal{V}_{\mathbb{A}^1\setminus\{0\}} = V \times (\mathbb{A}^1\setminus\{0\})$. On the other hand, since $R \otimes_{k[t]} k[t]/\langle t\rangle = R/tR$, there is an isomorphism

$$V_0 = \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda V / F^{\lambda+1} V \overset{\text{def}}{=} \mathrm{gr}_F V.$$

2. The inclusion $\mathrm{H}^0(\mathbb{A}^1, \mathcal{V}) = \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda V t^{-\lambda} \hookrightarrow \mathrm{H}^0(\mathbb{A}^1\setminus\{0\}, \mathcal{V}) = \bigoplus_{\lambda\in\mathbb{Z}} V t^{-\lambda}$ implies that

$$s \in F^\lambda V \iff \bar{s} t^{-\lambda} \in \mathrm{H}^0(\mathbb{A}^1, \mathcal{V}), \text{ where } \bar{s} \in \mathrm{H}^0(\mathbb{A}^1\setminus\{0\}, \mathcal{V}) \text{ is a } \mathbb{G}_m\text{-invariant section such that } \mathrm{ev}_1(\bar{s}) = s.$$

To use filtrations in the context of test configurations $(\mathcal{X} \to \mathbb{A}^1, \mathcal{L})$ of a polarized scheme (X, L), we consider $r \in \mathbb{N}_{\geq 1}$ such that $rL \in \mathrm{Pic}(X)$ and denote $R := R(X, rL) := \bigoplus_{m\in\mathbb{N}} R_m$ with $R_m := \mathrm{H}^0(X, mrL)$ a finite-dimensional vector space. Let's see that we can construct a **graded $\mathbb{Z}$-filtration** $F^\bullet R$, that is, a $\mathbb{Z}$-filtration $F^\bullet R_m$ for all $m \in \mathbb{N}$ such that $F^\lambda R_m \cdot F^\mu R_n \subseteq F^{\lambda+\mu} R_{m+n}$. To do this, we note that:

1. By the projection formula, $\mathrm{H}^0(\mathcal{X}, mr\mathcal{L}) = \mathrm{H}^0(\mathbb{A}^1, \mathcal{V})$ where $\mathcal{V} := \pi_*(\mathcal{L}^{\otimes mr})$ is a $\mathbb{G}_m$-linearized vector bundle.

2. There is a canonical restriction morphism $\mathrm{ev}_1 : \mathrm{H}^0(\mathcal{X}, mr\mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, mr\mathcal{L})_{t=1} = \mathrm{H}^0(X, mrL)$.

Thus, we can define $F^\lambda_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X, mrL) := \mathrm{Im}\left(\mathrm{H}^0(\mathcal{X}, mr\mathcal{L})_\lambda \xrightarrow{\mathrm{ev}_1} \mathrm{H}^0(X, mrL)\right)$, and this filtration is **finitely generated**. More precisely, the Rees correspondence gives us an isomorphism of $k[t]$-modules

$$\mathrm{H}^0(\mathcal{X}, mr\mathcal{L}) = \bigoplus_{\lambda\in\mathbb{Z}} \mathrm{H}^0(\mathcal{X}, mr\mathcal{L})_\lambda \xrightarrow[\text{Rees}]{\simeq} \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda \mathrm{H}^0(X, mrL) t^{-\lambda}$$

compatible with the grading, and then

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}^0(\mathcal{X}, mr\mathcal{L}) \simeq \bigoplus_{n\in\mathbb{N}} \bigoplus_{\lambda\in\mathbb{Z}} F^\lambda_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X, mrL) \overset{\text{def}}{=} \mathrm{Rees}(F^\bullet_{\mathcal{X},\mathcal{L}} R(X, rL)),$$

where the latter is a finitely generated $k[t]$-algebra since $\mathcal{L}$ is relatively ample over $\mathbb{A}^1$.

Theorem III.3.5. *There is a correspondence*[3] *between test configurations $(\mathcal{X}, \mathcal{L})$ of the polarized variety (X, L) and graded $\mathbb{Z}$-filtrations $F^\bullet$ of $R(X, L)$ for some $r > 0$.*

[3] bijective, if we declare two filtrations equivalent if they coincide on $\mathrm{H}^0(X, mL)$ for all m sufficiently divisible.

Proof. By the previous discussion, it suffices to note that $\mathrm{Rees}(\mathcal{F}_{\mathcal{X},\mathcal{L}} R(X; rL))$ induces the test configuration

$$\mathcal{X} := \mathrm{Proj}_{n\in\mathbb{N}}\left(\bigoplus_{n\in\mathbb{N}}\bigoplus_{\lambda\in\mathbb{Z}} \mathcal{F}^{\lambda}_{\mathcal{X},\mathcal{L}}\, \mathrm{H}^0(X; mrL)\right) \xrightarrow{\pi} \mathbb{A}^1$$

where the morphism π is obtained since the degree $m = 0$ component of the Rees algebra is $k[t]$. By construction, π is projective and $\mathcal{L} := \frac{1}{k}\mathcal{O}_{\mathcal{X}}(k)$ is a $\mathbb{Q}$-ample line bundle for $k \gg 0$. Moreover, the fact that there is a $\mathbb{G}_m$-equivariant isomorphism $(\mathcal{X}; \mathcal{L})|_{\mathbb{A}^1\setminus\{0\}} \simeq (X; L) \times (\mathbb{A}^1 \setminus \{0\})$ follows from the above Remark. □

Corollary III.3.6. *Let $(\mathcal{X}; \mathcal{L})$ be a test configuration of the polarized variety $(X; L)$. Then, if $\mathcal{X}$ is reduced and irreducible, then X also is.*

Proof. Let $\mathcal{F}^{\bullet} R$, with $R = R(X; rL)$ for $r > 0$, be the $\mathbb{Z}$-filtration associated with the test configuration $(\mathcal{X}; \mathcal{L})$ such that $\mathcal{X} \simeq \mathrm{Proj}(\mathrm{Rees}(\mathcal{F}^{\bullet} R))$. The result follows directly from the fact that $\mathrm{Rees}(\mathcal{F}^{\bullet} R) \subseteq R[t; t^{-1}]$. □

Fact: An analogous analysis, using the characterization of normality by Serre's R_1 and S_2 conditions, implies that if X is **normal** and $\mathcal{X}_0$ is **reduced**, then $\mathcal{X}$ is normal.

III.3.2 Valuations

Throughout this section, k will be an algebraically closed field with $\mathrm{char}(k) = 0$.

Definition III.3.7. Let K/k be a finitely generated field extension, i.e., its transcendence degree $\mathrm{tr.deg}\, K/k < +\infty$ is finite. A (real) valuation is a function $v : K^{\times} \to \mathbb{R}$ such that

1. $v(fg) = v(f) + v(g)$ for all $f, g \in K^{\times}$, i.e., $v : K^{\times} \to (\mathbb{R}, +)$ is a group homomorphism.

2. $v(f + g) \geq \min\{v(f), v(g)\}$ for all $f, g \in K^{\times}$.

3. $v|_{k^{\times}} = 0$.

Additionally, we define $v(0) = +\infty$.

Context. Given a normal algebraic variety X over k, we consider $K = K(X)$ the field of rational functions of X, which is a finitely generated extension of k with $\mathrm{tr.deg}\, K/k = \dim(X)$. We denote by Val_X the set of all valuations of the extension K/k.

Definition III.3.8. For a valuation v of K/k, we define the following list of invariants.

1. The *valuation ring* $\mathcal{O}_v := \{f \in K \mid v(f) \geq 0\}$, a local ring with maximal ideal $\mathfrak{m}_v := \{f \in K \mid v(f) > 0\}$.
2. The *residue field* $k(v) := \mathcal{O}_v/\mathfrak{m}_v$.
3. The *transcendence degree* $\mathrm{tr.deg}(v) = \mathrm{tr.deg}_k\, k(v)$.
4. The *value group* $\Gamma_v := v(K^*) \subset \mathbb{R}$ and its *rational rank* $\mathrm{rat.rk}(v) := \dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$.

Example III.3.9.

1. Let $x \in X$ be a smooth point of a variety of dimension n. We define the order of vanishing of a regular function $f \in \mathcal{O}_{X,x} \setminus \{0\}$ at x as

   $$\mathrm{ord}_x(f) := \max\{d \in \mathbb{N} \mid f \in \mathfrak{m}_x^d\}$$

   We can extend this function to a valuation $K^* \to \mathbb{R}$ by defining

   $$\mathrm{ord}_x(f/g) := \mathrm{ord}_x(f) - \mathrm{ord}_x(g).$$

   In this case, we note that $\Gamma_v = \mathbb{Z}$ and therefore $\mathrm{rat.rk}(\mathrm{ord}_x) = 1$.

2. Consider $X = \mathbb{A}^2_{x,y}$. Given $f = \sum_{a,b \in \mathbb{N}} c_{a,b} x^a y^b$ where $c_{a,b} \in k$, we define the valuation v of $K(x,y)$ by

   $$v(f) = \min\{a + b\sqrt{2} \mid c_{a,b} \neq 0\}$$

   The values $v(x) = 1, v(y) = \sqrt{2}$ are the weights of the action.

3. **Divisorial valuations.** A divisor E over X (see Definition I.4.1) corresponds to a proper, birrational morphism $\pi : Y \to X$ with Y normal and $E \subset Y$ a prime

divisor. In this situation, the local ring $\mathcal{O}_{Y,E}$ of E is a discrete valuation ring (DVR), whose associated valuation is

$$\mathrm{ord}_E : K^* \to \mathbb{Z}, \quad f \mapsto \mathrm{ord}_E(\pi^* f)$$

i.e., it corresponds to computing the order of vanishing of the pullbacked function along the subvariety E.

Definition III.3.10 (Center of a valuation)**.** If $v \in \mathrm{Val}_X$ is a valuation, the *center of* v is the point $\xi \in X$ such that $v \geq 0$ on $\mathcal{O}_{X,\xi}$ and $v > 0$ on $\mathfrak{m}_\xi$. The center of the valuation v will be denoted $c_X(v)$.

Remark III.3.11. The fact that $c_X(v)$ exists is equivalent to $X \to \mathrm{Spec}(k)$ being proper, and if this center exists it is unique if X is a separated variety (cf. valuative criterions of properness and separatedness).

Example III.3.12.

1. The valuation ord_x associated with the order of vanishing at a smooth point $x \in X$ is a divisorial one, whose center corresponds to $c_X(\mathrm{ord}_x) = x$. Indeed, $\mathrm{ord}_x = \mathrm{ord}_F$ where F corresponds to the exceptional divisor of the blow-up of X at x. We can perform this calculation locally. Consider $u_1, \dots, u_n \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ local coordinates, and a function $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha u^\alpha \in \mathcal{O}_{X,x}$ where $u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$. By definition, $d := \mathrm{ord}_x(f) = \min\{|\alpha| ; c_\alpha \neq 0\}$. Consider the blow-up $\pi : \widetilde{X} := \mathrm{Bl}_x X \to X$ given by:

$$\widetilde{X} \overset{\mathrm{loc}}{=} \{(u, [y]) \in X \times \mathbb{P}^{n-1} \mid u_i y_j = u_j y_i \quad \forall i, j = 1, \dots, n\}$$

In the open set $y_i \neq 0$ we have coordinates $u_j = u_i y_j$, and the exceptional divisor is given by $F = \{u_i = 0\}$. We compute that

$$\pi^* f(x_1, \dots, x_n) = u_i^d \widetilde{f}$$

for some regular function $\widetilde{f}$ such that $u_i \nmid \widetilde{f}$, so $\mathrm{ord}_F(f) = d$.

2. If $E \subset X$ is a prime divisor of X with generic point $\xi \in X$, the valuation $v = \mathrm{ord}_E$ is such that $v \geq 0$ on $\mathcal{O}_{X,\xi}$ and $v > 0$ on its maximal ideal.

3. More generally, if $E \subset Y \to X$ is a divisor over X then $\overline{c_X(v)} = \pi(E)$.

The example of divisorial valuations raises the question of how to characterize a valuation as divisorial. A theorem by Zariski shows that this can be done numerically in terms of transcendence degree and rational rank. The proof of this fact is a consequence of [KM98, Lemma 2.45].

Theorem III.3.13. *Let v be a valuation of K. Then v is divisorial if and only if* $\operatorname{tr.deg}(v) = n - 1$ *and* $\operatorname{rat.rk}(v) = 1$.

Working locally, we can have a notion of valuating sections of line bundles.

Construction III.3.14. Let $v \in \mathrm{Val}_X$. Given a line bundle $L \in \operatorname{Pic}(X)$ we can make sense of $v(s)$ for sections $s \in \mathrm{H}^0(X, L)$:

In a neighborhood U of the center $\xi = c_X(v)$ we can trivialize L, i.e., fix an isomorphism $L|_U \cong U \times \mathbb{A}^1$ in which a local section $s \in \mathrm{H}^0(U, L|_U)$ is represented by a regular function $s : U \to \mathbb{A}^1$. In this way we can define $v(s)$ by evaluating this local representation, which is well-defined since two trivializations of L differ by a unit $a \in k^*$, and therefore if s', s'' are two local representations of a section s we have $s' = as''$ for some $a \in k^*$, and then $v(s') = v(as') = v(s'')$. Furthermore, $v(s) > 0$ if and only if $s(\xi) = 0$.

Similarly, we can evaluate v on a Cartier divisor D by considering the valuation of the local equation of D around $c_X(v)$.

Since the function field is a birational invariant, any test configuration $(\mathcal{X}, \mathcal{L})$ of a polarized pair (X, L) has function field $k(\mathcal{X}) = K(X)(t)$ since $\mathcal{X} \setminus \mathcal{X}_0 = X \times (\mathbb{A}^1 \setminus \{0\})$. Thus, it becomes natural to study valuations of $K(X)(t)$.

Theorem III.3.15 (Generalized Abhyankar's inequality)**.** *Let $k \subseteq K' \subseteq K$ be field extensions, and let v be a valuation of K/k. Then*

$$\operatorname{tr.deg}(v) + \operatorname{rat.rk}(v) \leq \operatorname{tr.deg}(v') + \operatorname{rat.rk}(v') + \operatorname{tr.deg} K/K'$$

where $v' = v|_{K'}$ is the restriction of the valuation v to K'.

Proposition III.3.16. *Let v be a valuation of $K(X)(t)$. If v is divisorial, its restriction $r(v) = v|_{K(X)}$ to $K(X)$ is either divisorial or trivial.*

Proof. Abhyankar's inequality implies that

$$\operatorname{tr.deg}(v) + \operatorname{rat.rk}(v) \leq \operatorname{tr.deg}(r(v)) + \operatorname{rat.rk}(r(v)) + 1 \leq n + 1$$

Since v is divisorial, in particular $\mathrm{tr.deg}(v) + \mathrm{rat.rk}(v) = n + 1$, so

$$\mathrm{tr.deg}(r(v)) + \mathrm{rat.rk}(r(v)) = n$$

It is clear that $\mathrm{rat.rk}(r(v)) \leq \mathrm{rat.rk}(v) = 1$, from which we conclude. $\square$

Remark III.3.17. There is a natural action $\mathbb{G}_m \curvearrowright K(X)(t)$ given by

$$a \cdot f = \sum_{\lambda \in \mathbb{Z}} a^{\lambda} f_{\lambda} t^{\lambda}$$

where $f = \sum_{\lambda \in \mathbb{Z}} f_{\lambda} t^{\lambda}$ with $f_{\lambda} \in K(X)$.

The following notion of equivariant valuations results to be very useful to understand valuations of $X \times \mathbb{A}^1$.

Definition III.3.18. A valuation of $K(X)(t)$ is $\mathbb{G}_m$-equivariant if $v(f) = v(a \cdot f)$ for every $f \in K(X)(t), a \in \mathbb{G}_m$. We denote by $\mathrm{Val}^{\mathbb{G}_m}_{X \times \mathbb{A}^1}$ the set of equivariant valuations of $X \times \mathbb{A}^1$.

Example III.3.19. Let w be a valuation of $K(X)$ and $s \in \mathbb{R}_{\geq 0}$. We can define a valuation w_s of $K(X)(t)$ by

$$w_s(f) := \min\{w(f_{\lambda}) + \lambda s\}$$

where $f = \sum_{\lambda \in \mathbb{Z}} f_{\lambda} t^{\lambda}$.

1. This valuation is $\mathbb{G}_m$-equivariant, given that

$$w(a^{\lambda} f_{\lambda}) = w(a^{\lambda}) + w(f_{\lambda}) = w(f_{\lambda}) \quad \forall a \in \mathbb{G}_m$$

2. If w has a center in X then

$$c_{X \times \mathbb{A}^1}(w_s) = \begin{cases} c_X(w) \times 0 & \text{if } s > 0 \\ c_X(w) \times \mathbb{A}^1 & \text{if } s = 0 \end{cases}$$

Note that there is a bijection between the valuations of X and the $\mathbb{G}_m$-equivariant valuations of $X \times \mathbb{A}^1$, given explicitly by

$$\begin{aligned} \mathrm{Val}_X \times \mathbb{R} &\to \mathrm{Val}^{\mathbb{G}_m}_{X \times \mathbb{A}^1} \\ (w, s) &\mapsto w_s \\ (v|_{K(X)}, v(t)) &\leftarrow\!\shortmid v \end{aligned}$$

Now we use our knowlegde of valuations to establish a link with test configurations.

Let $(\mathcal{X},\mathcal{L})$ be a normal test configuration of $(X;L)$. We have a canonical birational map $\mathcal{X} \dashrightarrow X \times \mathbb{A}^1$, and we consider $\mathcal{Y}$ as the normalization of the graph of this map. Then there is a diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} \arrow[rr, dashed] & & X \times \mathbb{A}^1 \end{tikzcd};$$

and given $E \subseteq \mathcal{X}_0$ an irreducible component of $\mathcal{X}_0$, this induces a divisorial valuation ord_E of the field $K(\mathcal{X})(t)$, whose restriction to $K(X)$ will be denoted $v_E = r(\mathrm{ord}_E)$. In this context, we can rewrite the graded $\mathbb{Z}$-filtration associated to a test configuration in terms of valuations.

Proposition III.3.20. *For $m > 0$ sufficiently large*

$$\mathcal{F}^{\lambda}_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X;mL) = \bigcap_{E \subseteq \mathcal{X}_0} \left\{ s \in \mathrm{H}^0(X;mL) \mid v_E(s) + m\,\mathrm{ord}_E(D) \geq \lambda\,\mathrm{ord}_E(t) \right\}$$

where D denotes the $\mathbb{Q}$-divisor on $\mathcal{Y}$ supported on $\mathcal{Y}_0$ such that $f^*\mathcal{L} = g^*(L \times \mathbb{A}^1) + D$.

Proof. Recall that the filtration of $\mathrm{H}^0(X;mL)$ defined earlier corresponds by definition to

$$\mathcal{F}^{\lambda}_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X;mL) = \left\{ s \in \mathrm{H}^0(X;mL) \mid t^{-\lambda} \bar{s} \in \mathrm{H}^0(\mathcal{X};m\mathcal{L}) \right\}$$

where $\bar{s} \in \mathrm{H}^0(\mathcal{X} \setminus \mathcal{X}_0; m\mathcal{L})$ denotes the $\mathbb{G}_m$-invariant section such that its restriction to $t = 1$ is $\bar{s}_1 = s$. At the same time, it defines a rational section of $L_{\mathbb{A}^1} = L \times \mathbb{A}^1$, which we denote $\bar{s}_{m\mathcal{L}}$ and $\bar{s}_{mL_{\mathbb{A}^1}}$. Now, since $\mathcal{X}$ is normal, $\bar{s}t^{-\lambda} \in \mathrm{H}^0(\mathcal{X};\mathcal{L})$ if and only if $\mathrm{ord}_E(\bar{s}t^{-\lambda}) \geq 0$ for all E irreducible components of $\mathcal{X}_0$. We calculate that

$$\begin{aligned} \mathrm{ord}_E\left(\bar{s}t^{-\lambda}_{m\mathcal{L}}\right) = \mathrm{ord}_E(\bar{s}_{m\mathcal{L}}) - \lambda\,\mathrm{ord}_E(t) &= \mathrm{ord}_E(\bar{s}_{f^* m\mathcal{L}}) - \lambda\,\mathrm{ord}_E(t) \\ &= \mathrm{ord}_E(\bar{s}_{g^* mL_{\mathbb{A}^1}(D)}) - \lambda\,\mathrm{ord}_E(t) \\ &= \mathrm{ord}_E(\bar{s}_{g^* mL_{\mathbb{A}^1}}) + m\,\mathrm{ord}_E(D) - \lambda\,\mathrm{ord}_E(t) \\ &\overset{\text{def}}{=} v_E(s) + m\,\mathrm{ord}_E(D) - \lambda\,\mathrm{ord}_E(t) \end{aligned}$$

$\square$

III.4 K-stability and the Minimal Model Program

In this section we will discuss generally speaking some intrinsic relationships that have been discovered between K-stability and the singularities appearing on the MMP. The ideas and results presented here are mainly the work of Y. Odaka, presented in [Oda13a, Oda12, Oda13b].

Theorem III.4.1 ([Oda13a, Theorem 4.1],[Oda12, Theorem 2.10],[Oda13b, Theorem 1.2])**.** *Let X be a projective normal variety and $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ ample. Then,*

1. *If $K_X \sim_{\mathbb{Q}} 0$, then X is klt (resp. lc) $\iff$ (X, L) is K-stable (resp. K-semistable).*
2. *If $L = K_X$, then X is lc $\iff$ (X, L) is K-stable $\iff$ (X, L) is K-semistable.*

The fact that a K-semistable variety has only log-canonical singularities is an astonishing fact, and is the content of [Oda13b, Theorem 1.2]. As an illustration, here we show the forward implications of the previous theorem.

We begin by modifying the intersection formula for the Donaldson-Futaki invariant. Let $(\mathcal{X}, \mathcal{L})$ be a normal test configuration of (X, L). We have

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \overline{\mathcal{X}} \arrow[rr, dashed] & & X \times \mathbb{P}^1 \end{tikzcd}$$

where $\mathcal{Y}$ corresponds to the normalization of the graph of the birational map $\overline{\mathcal{X}} \dashrightarrow X \times \mathbb{P}^1$.

Proposition III.4.2. *With the above notation,*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{\overline{\mathcal{L}}^n \cdot f^*\left(K_{\mathcal{Y}/X \times \mathbb{P}^1} + g^* \mathrm{pr}_1^* K_X\right)}{V} + \bar{S}\frac{\overline{\mathcal{L}}^{n+1}}{(n+1)V}$$

where $\overline{S} := nV^{-1}(-K_X \cdot L^n)$.

Proof. By Wang-Odaka's formula, it suffices to prove that:

$$f_*\left(K_{\mathcal{Y}/X \times \mathbb{P}^1} + g^* \mathrm{pr}_1^* K_X\right) = K_{\mathcal{X}/\mathbb{P}^1}.$$

To do this, observe that:

$$\begin{aligned} K_{\mathcal{Y}/X \times \mathbb{P}^1} + g^* \mathrm{pr}_1^* K_X &= K_{\mathcal{Y}} - g^*(K_{X \times \mathbb{P}^1} - \mathrm{pr}_1^* K_X) \\ &= K_{\mathcal{Y}} - g^* \mathrm{pr}_2^*(K_{\mathbb{P}^1}) \\ &= K_{\mathcal{Y}} - f^* \bar{\pi}^*(K_{\mathbb{P}^1}). \end{aligned}$$

Finally,

$$f\ \left(K_{Y/X\ \mathbb{P}^1} + g\ \mathrm{pr}_1 K_X\right) = f\ K_Y \quad f\ f \quad (K_{\mathbb{P}^1}) = K_{\overline{X}} \quad {}^{-}\ K_{\mathbb{P}^1} = K_{X/\mathbb{P}^1}.$$

□

Proposition III.4.3.

1. *If X is lc, then $K_{Y/X\ \mathbb{P}^1}$ is effective.*

2. *If X is klt, then $K_{Y/X\ \mathbb{P}^1}$ is effective and has support on Y_0.*

Proof. Let X be lc. Then $(X \quad \mathbb{P}^1; X \quad 0)$ is lc. Indeed, if $f : Y \to X$ is a log resolution of X, then $K_{Y/X} = \sum_i a_i E_i$ for certain prime divisors $E_i \quad Y$ such that $a_i \quad 1$. Now, $f_{\mathbb{P}^1} : Y \quad \mathbb{P}^1 \to X \quad \mathbb{P}^1$ is a log resolution of $(X \quad \mathbb{P}^1; X \quad 0)$ and

$$K_{Y\ \mathbb{P}^1/X\ \mathbb{P}^1} \quad f_{\mathbb{P}^1}(X \quad 0) = \sum_{i=1}^{r} a_i E_i \quad Y \quad 0$$

has coefficients $\quad 1$. Thus, the pair $(X \quad \mathbb{P}^1; X \quad 0)$ is lc. Moreover, since

$$K_{Y/X\ \mathbb{P}^1} \quad g\ (X \quad 0) = K_{Y/X\ \mathbb{P}^1} \quad Y_0$$

has coefficients $\quad 1$ and since $\mathrm{Supp}\left(K_{Y/X\ \mathbb{P}^1}\right) \quad \mathrm{Exc}(g) = Y_0$, it follows that $K_{Y/X\ \mathbb{P}^1}$ has coefficients $\quad 0$. If X is klt, note that no $E_i \quad \mathrm{Exc}(g)$ vanishes since $a(Y_0; X \quad \mathbb{P}^1) > \quad 1$. □

Proof of (⇒) in Theorem 4.1.1. Fix $(X ; L\ )$ a non-trivial test configuration. Suppose X is lc with $K_X \quad_{\mathbb{Q}} 0$. Then, the Donaldson-Futaki invariant reduces to

$$\mathrm{DF}(X ; L\ ) = \frac{\overline{L}^n \quad f\ \left(K_{Y/X\ \mathbb{P}^1}\right)}{V}.$$

Now, since $f\ (K_{Y/X\ \mathbb{P}^1})$ is effective with $\mathrm{Supp}\left(f\ K_{Y/X\ \mathbb{P}^1}\right) \quad X_{0,\mathrm{red}}$ and $L\ |_{Y_0}$ is ample, it follows from the Nakai-Moishezon criterion that $\overline{L}^n \quad f\ \left(K_{Y/X\ \mathbb{P}^1}\right) \quad 0$, and hence X is K-semistable.

Now, suppose X is klt. Note that the condition $\mathrm{codim}\left(\mathrm{Exc}\left(X \dashrightarrow X \quad \mathbb{A}^1\right)\right) \quad 2$ implies that $(X ; L\ )$ is trivial. In this case, $\mathrm{codim}(\mathrm{Exc}(X \dashrightarrow X \quad \mathbb{A}^1)) = 1$, so $f\ (K_Y/K_{X\ \mathbb{P}^1}) = X_0$ and $f\ (K_Y/K_{X\ \mathbb{P}^1}) \neq 0$. Thus, $\mathrm{DF}(X ; L\ ) > 0$ when $(X ; L\ )$ is non-trivial. This completes the case when $K_X \quad_{\mathbb{Q}} 0$.

Assume $L = K_X$ and that X is lc. Thus, the intersection formula results in:

$$\mathrm{DF}(X ; L\ ) = \frac{\overline{L}^n \quad f\ \left(K_{Y/X\ \mathbb{P}^1}\right)}{V} + \frac{\left(f\ \overline{L}^n \quad g\ L_{\mathbb{A}^1}\right) \quad \frac{n}{n+1}\overline{L}^{n+1}}{(n+1)V}$$

By the same argument used earlier, the first term is ≥ 0. The fact that the second term is > 0 is a consequence of [BHJ17, Proposition 7.8] and uses non-Archimedean methods. □

The following is a more general statement about the singularities of a K-semistable variety.

Theorem III.4.4 ([Oda13b, Theorem 1.2]). *Let X be a normal variety with K_X a $\mathbb{Q}$-Cartier divisor and L an ample $\mathbb{Q}$-line bundle on X. If (X, L) is K-semistable, then X is lc.*

The proof of this amazing theorem requires a considerable amount of techniques from the Minimal Model Program, and the strategy of the proof consists in to show that if X is not lc, then (X, L) admits a destabilizing test configuration, i.e., such that it has negative Donaldson-Futaki invariant. The key statement in this direction is the *existence of a log canonical model*.

Theorem III.4.5 ([OX12]). *If X is a normal variety such that K_X is $\mathbb{Q}$-Cartier, then there exists a proper birational morphism $f : Y \to X$ such that*

1. $(Y, \Delta_Y := \mathrm{Exc}(f) = E_1 + \dots + E_k)$ *is lc.*
2. $K_Y + \Delta_Y$ *is f-ample.*

The pair (Y, Δ_Y) is known as the log canonical model of X, and it is unique up to isomorphism.

Example III.4.6. Let $X = \{h = 0\} \subset \mathbb{A}^{n+1}$ with an isolated singularity at the origin, i.e., h is homogeneous. The blow-up of X at 0 gives a log resolution $Y \to X$ with $K_{Y/X} \sim (n - \deg(h))F$ with F an exceptional divisor. Thus, X is not lc when $\deg(h) > n + 1$, and then Y satisfies the conditions of Odaka-Xu's theorem.

Idea of the proof of Theorem III.4.4. We will assume that X is not lc and show that there is a test configuration $(\mathcal{X}, \mathcal{L})$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$. Consider Y the log canonical model of X and the divisor $E := K_{Y/X} + \Delta_Y$. Since E is nef, the *negativity lemma* (see [KM98, Lemma 3.39]) implies that $-E$ is effective. Consider the ideal sheaf $\mathcal{I} := f_* \mathcal{O}_Y(-mE)$ for $m > 0$ sufficiently divisible, and $Z \subset X$ the closed subscheme defined by the ideal I. Then[4] $Y = \mathrm{Bl}_Z(X)$, so $E = \mathrm{Exc}(f)$, and hence $K_{Y/X} = -E - \Delta_Y$ has all its coefficients < -1. Given the closed $Z \subset X$ above, consider now the ideal:

$$\mathcal{I} = I_{Z\times\mathbb{A}^1} + t^N \mathcal{O}_{X\times\mathbb{A}^1} \subseteq \mathcal{O}_{X\times\mathbb{A}^1}$$

where $N \in \mathbb{N}_{\geq 1}$, and we define $\mathcal{X}$ as the normalization of the blow-up of $X \times \mathbb{A}^1$ along $\mathcal{I}$:

$$\mathcal{X} := \widetilde{\mathrm{Bl}}_{\mathcal{I}}\, X \times \mathbb{A}^1 \xrightarrow{g} X \times \mathbb{A}^1.$$

We can write $\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-F)$ for some Cartier divisor $F \in \mathrm{Pic}(\mathcal{X})$, and we then have that $\mathcal{L}_\varepsilon := g^* L_{\mathbb{A}^1} - \varepsilon F$ is ample over $\mathbb{A}^1$ for $0 < \varepsilon \ll 1$ and thus $(\mathcal{X}; \mathcal{L}_\varepsilon)$ is a test configuration of $(X; L)$, which indeed satisfies that $K_{\mathcal{X}/X\times\mathbb{A}^1}$ has (for N sufficiently large) only negative coefficients.

We then claim that $\mathrm{DF}(\mathcal{X}; \mathcal{L}_\varepsilon) < 0$ when $0 < \varepsilon \ll 1$. Indeed, the intersection formula implies

$$\mathrm{DF}\left(\mathcal{X}; \mathcal{L}_\varepsilon\right) = \frac{\overline{\mathcal{L}}_\varepsilon^{\,n} \cdot K_{\overline{\mathcal{X}}/X\times\mathbb{P}^1}}{V} + \frac{\overline{\mathcal{L}}_\varepsilon^{\,n} \cdot g^* p_1^* K_X}{V} + \frac{\bar{S}}{n+1}\frac{\overline{\mathcal{L}}_\varepsilon^{\,n+1}}{V},$$

resulting in $\mathrm{DF}\left(\mathcal{X}; \mathcal{L}_\varepsilon\right)$ being a polynomial in ε, and it suffices to analyze the lower order term. This last analysis is done by Odaka, who proves that

$$\mathrm{DF}\left(\mathcal{X}; \mathcal{L}_\varepsilon\right) = c\,\varepsilon^d + \text{higher order terms},$$

for some rational number $c < 0$. This proves that for $\varepsilon \ll 1$, the test configuration $(\mathcal{X}; \mathcal{L}_\varepsilon)$ is destabilizing, and hence $(X; L)$ is **not** K-semistable. □

We conclude this section by stating a *klt version of Theorem III.4.1.*

Theorem III.4.7 ([Oda13b, Theorems 1.4-1.5])**.**

1. *If X is Fano and $(X; -K_X)$ is K-semistable, then X is klt.*

2. *If X is Calabi-Yau and $(X; L)$ is K-semistable, then X is klt.*

Remark III.4.8. The last theorem is crucial to the study of K-stability, because represent a bound in the family of varieties we have to delve: *it is enough to study klt Fano varieties.*

[4] For more details, see [BHJ17, Lemma 1.13].

III.5 Valuative criterion of K-stability

The goal of this section is to prove the valuative criterion of K-stability, which provides an amazing tool to show whether a variety is (or not) K-stable. Using the ideas introduced in §III.3, we will be able to rewrite the K-stability property as a numerical condition on divisors over the variety. Initially, K. Fujita and C. Li found a valuative criterion in [Li17, Fuj19b], which characterizes K-semistability, and afterwards in [BX19] characterizations of K-(poly)stability were given.

Let (X, L) be a polarized normal projective variety of $\dim(X) = n$, and let $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ be a test configuration.

Recall III.5.1. We saw that an irreducible component $E \subset \mathcal{X}_0$ defines a $\mathbb{G}_m$-equivariant divisorial valuation $\mathrm{ord}_E \in \mathrm{Val}^{\mathbb{G}_m}_{X \times \mathbb{A}^1}$, and we defined $v_E := \mathrm{ord}_E|_{K(X)}$. Additionally, by normalizing the graph of the natural rational map between $\mathcal{X}$ and the trivial test configuration, we obtain a diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} \arrow[rr, dashed] & & X \times \mathbb{A}^1 \end{tikzcd} \qquad \text{where } f^*\mathcal{L} \simeq g^* L_{\mathbb{A}^1} + D \text{ with } \mathrm{Supp}(D) \subseteq \mathcal{Y}_0.$$

In this context, we have already seen that

$$\mathcal{F}^{\lambda}_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X, mL) = \bigcap_{E \subseteq \mathcal{X}_0} \{ s \in \mathrm{H}^0(X, mL) : \mathrm{ord}_F(\bar{s} t^{-\lambda}) \overset{\text{def}}{=} v_F(s) + \overbrace{m\,\mathrm{ord}_F(D)}^{=:\theta_1 \text{ fixed}} - \overbrace{\lambda\,\mathrm{ord}_F(t)}^{=:\theta_2 \text{ fixed}} \geq 0 \}.$$

Definition III.5.2. Let $v = \mathrm{ord}_E : K(X)^{*} \to \mathbb{Z}$ be a divisorial valuation induced by a divisor over $E \subset Y \to X$. Then, v filters the algebra $R := R(X, L) \overset{\text{def}}{=} \bigoplus_{m \geq 0} \mathrm{H}^0(X, mL)$ by defining

$$\mathcal{F}^{\lambda}_v \mathrm{H}^0(X, mL) := \{ s \in \mathrm{H}^0(X, mL),\ v(s) \geq \lambda \}.$$

Warning. The numerical characterization of Zariski divisorial valuations **does not** only consider surjective valuations. More precisely, if $v : K(X)^{*} \to \mathbb{Z}$ is divisorial and $\mathrm{Im}(v) = c\mathbb{Z}$ with $c \in \mathbb{N}_{\geq 1}$ then $v = c\,\mathrm{ord}_E$. Thus, in the previous definition, $v(s) \geq \lambda$ if and only if $\mathrm{ord}_E(s) \geq \lceil \frac{\lambda}{c} \rceil$.

Definition III.5.3 ([Fuj19b])**.** We say that $v = c\,\mathrm{ord}_E$ (or that the divisor E) is **dreamy** if $\mathcal{F}_v R(X, L)$ is finitely generated, i.e., the Rees algebra

$$\mathrm{Rees}(\mathcal{F}_v R) = \bigoplus_{m \in \mathbb{N},\, \lambda \in \mathbb{Z}} \mathrm{H}^0(Y, m\pi^* L - \lceil \frac{\lambda}{c} \rceil E)$$

is finitely generated.

Example III.5.4 ([BCHM10]). If Y is **log-Fano** (i.e., there exists $\Delta_Y \geq 0$ effective with coefficients ≤ 1 such that (Y,Δ_Y) is klt[5] and $-(K_Y+\Delta_Y)$ is ample), then every divisor $E \subset Y$ is dreamy (Y is a **Mori Dream Space**).

Theorem III.5.5. *Consider (X,L) with X Fano klt and $L=-K_X$. Then, there is a bijection between:*

1. *Normal test configurations $(\mathcal{X},\mathcal{L})$ of (X,L) with $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$ and $\mathcal{X}_0$* **reduced and irreducible***.*

2. *Dreamy divisorial valuations $v : K(X) \to \mathbb{Z}$.*

Proof. (1) $\mapsto$ (2) is given by $\mathcal{X}_0 \mapsto v_{\mathcal{X}_0} \overset{\text{def}}{=} \mathrm{ord}_{\mathcal{X}_0}|_{K(X)}$. Here, $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$ and the filtration induced in $R = R(X,L)$ is given by

$$F^{\lambda}_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X,mL) = \{s \in \mathrm{H}^0(X,mL);\ v_{\mathcal{X}_0}(s) + m\,\mathrm{ord}_{\mathcal{X}_0}(D) \geq \lambda\,\mathrm{ord}_{\mathcal{X}_0}(t)\},$$

with $\mathrm{ord}_{\mathcal{X}_0}(D) := -A$, $\mathrm{ord}_{\mathcal{X}_0}(t) \overset{\text{def}}{=} 1$, i.e., $F^{\lambda}_{\mathcal{X},\mathcal{L}} \mathrm{H}^0(X,mL) = F^{\lambda+mA}_{v_{\mathcal{X}_0}} \mathrm{H}^0(X,mL)$ and $R_{(\mathcal{X},\mathcal{L})} := \mathrm{Rees}(F_{\mathcal{X},\mathcal{L}} R) = \mathrm{Rees}(F_{v_{\mathcal{X}_0}} R)$ as $k[t]$-algebras. Since $R_{(\mathcal{X},\mathcal{L})}$ is finitely generated, $v_{\mathcal{X}_0}$ is a dreamy valuation.

(2) $\mapsto$ (1) is given by $v \mapsto \mathcal{X} := \mathrm{Proj}_{\mathbb{A}^1}(\mathrm{Rees}(F_v R))$. Here, $\mathcal{X}_0$ is given by the Proj of the algebra

$$\begin{aligned}\mathrm{Rees}(F_v R) \otimes_{k[t]} k[t]/\langle t\rangle &= \frac{\mathrm{Rees}(F_v R)}{t \cdot \mathrm{Rees}(F_v R)} \\ &\overset{\text{def}}{=} \bigoplus_{m\in\mathbb{N},\,\lambda\in\mathbb{Z}} \frac{F_v^{\lambda} \mathrm{H}^0(X,mL)}{F_v^{\lambda+1} \mathrm{H}^0(X,mL)} \\ &\overset{\text{def}}{=} \bigoplus_{m\in\mathbb{N},\,\lambda\in\mathbb{Z}} \mathrm{gr}^{\lambda}_{F_v} \mathrm{H}^0(X,mL).\end{aligned}$$

Given that if $\bar{s},\bar{t} \neq 0$ have degrees λ and μ, respectively, then $\overline{st}$ is nonzero of degree $\lambda+\mu$, and we deduce that $\mathcal{X}_0$ is irreducible and reduced. In particular, since X is normal, $\mathcal{X}$ is irreducible and normal. Finally, the previous construction implies that $(\mathcal{X},\mathcal{L})$ satisfies $v_{\mathcal{X}_0} = v$. $\square$

Warning. In the previous context:

[5] i.e., the pair (Y,Δ_Y) is dlt.

1. Since $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$ and $K_{\mathbb{A}^1} = 0$ then $D \stackrel{\text{def}}{=} g^*(L_{\mathbb{A}^1}) + f^*\mathcal{L} \stackrel{\text{def}}{=} g^*(-K_{X\times\mathbb{A}^1}) - f^*(-K_{\mathcal{X}}) - K_{\mathcal{X}} \stackrel{\text{def}}{=} K_{\mathcal{X}/X} - K_{\mathcal{X}/(X\times\mathbb{A}^1)}$. Then, if $\widetilde{X_0} = f^{-1}_* X_0$ it follows that by definition of log-discrepancy

$$A \stackrel{\text{def}}{=} \mathrm{ord}_{X_0}(D) = \overbrace{\mathrm{coeff}_{\widetilde{X_0}}(K_{\mathcal{X}/X})}^{\stackrel{\text{def}}{=}0} - \mathrm{coeff}_{\widetilde{X_0}}(K_{\mathcal{X}/(X\times\mathbb{A}^1)})$$

$$\stackrel{\text{def}}{=} -(A_{X\times\mathbb{A}^1}(\widetilde{X_0}) - 1) = 1 - (cA_X(E) + \overbrace{\mathrm{ord}_{X_0}(t)}^{\stackrel{\text{def}}{=}1});$$

and thus $A = -cA_X(E) \stackrel{\text{def}}{=} -A_X(v_{X_0})$, where $v_{X_0} = c\,\mathrm{ord}_E$ is a divisorial valuation on X induced by X_0.

2. Li and Xu proved in [LX14], that it suffices to check K-stability of Fano varieties by considering **special test configurations** $(\mathcal{X}, \mathcal{L})$, i.e., those with $\mathcal{L} = -K_{\mathcal{X}/\mathbb{A}^1}$ and X_0 a klt Fano variety.

Definition III.5.6 (β-invariant)**.** Let X be a klt Fano variety and $r \in \mathbb{N}_{\geq 1}$ such that $-rK_X$ is Cartier. For $v := c\,\mathrm{ord}_E : K(X)^* \to \mathbb{Z}$ a divisorial valuation, we define the invariant $\beta(v) := A_X(v) - S_X(v)$, where

$$S_X(v) = \limsup_{m\to+\infty} \frac{\sum_{\lambda\in\mathbb{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{\mathcal{F}_v} \mathrm{H}^0(X, -mrK_X)}{m \dim \mathrm{H}^0(X, -mrK_X)}$$

and where $A_X(v) = cA_X(E)$, with $A_X(E)$ the log-discrepancy of the divisor $E \subset Y \to X$.

Proposition III.5.7. *Let $v = c\,\mathrm{ord}_E$ (i.e., $v = v_{X_0} \stackrel{\text{def}}{=} \mathrm{ord}_{X_0}|_{K(X)}$) a dreamy valuation, and $(\mathcal{X}, \mathcal{L})$ the associated test configuration, with X_0 reduced and irreducible. Then,*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = A_X(v) - S_X(v) = c(A_X(E) - S_X(E)).$$

Proof. Let $(\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$ be the associated projective test configuration. Considering $\mathcal{L} = -K_{\mathcal{X}}$ and $\overline{\mathcal{L}} = -K_{\overline{\mathcal{X}}/\mathbb{P}^1}$ the formula for the Donaldson-Futaki invariant using $w_m/mN_m = F_0 + F_1 m^{-1} + \cdots$ reduces to

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \stackrel{\text{def}}{=} -2F_1 = -\frac{1}{(n+1)(-K_X)^n}(-K_{\overline{\mathcal{X}}/\mathbb{P}^1})^{n+1} \stackrel{\text{def}}{=} -\frac{b_0}{a_0} \stackrel{\text{def}}{=} -F_0.$$

And the term F_0 is simply calculated by observing that if $v = v_{X_0} = c\,\mathrm{ord}_E$ then

$$
\begin{aligned}
w_m \stackrel{\text{def}}{=} \mathrm{wt}\,\mathrm{H}^0(X_0; \quad mK_{X=\mathbb{A}^1}j_{X_0}) &\stackrel{\text{def}}{=} \sum_{2Z} \quad \dim \mathrm{gr}_{F_{X\,;L}} \, \mathrm{H}^0(X; \quad mK_X) \\
&= \sum_{2Z} \quad \dim \mathrm{gr}_{F_v}^{+mA} \, \mathrm{H}^0(X; \quad mK_X) \\
&= \sum_{2Z}(\qquad mA) \dim \mathrm{gr}_{F_v} \, \mathrm{H}^0(X; \quad mK_X) \\
&= \quad mA_X(v) \underbrace{\dim \mathrm{H}^0(X; \quad mK_X)}_{\stackrel{\text{def}}{=} N_m} + \sum_{2Z} \quad \dim \mathrm{gr}_{F_v} \, \mathrm{H}^0(X; \quad mK_X)
\end{aligned}
$$

and then $\quad F_0 = \quad \limsup_{m\,!\,+1} \frac{w_m}{mN_m} = A_X(v) \quad S_X(v)$. □

Theorem III.5.8. *If X is a klt Fano variety of $\dim(X) = n$ and $E \quad Y \; ! \; X$ is a prime divisor over X then*

$$S_X(E) = \frac{1}{(\quad K_X)^n} \int_0 \mathrm{vol}(\qquad K_X \quad tE)\, \mathrm{d}t$$

where $\quad = \sup f t\ 2\ \mathrm{R}\ ^0; \qquad K_X \quad tE$ big divisor$g$ *is the pseudo-effective threshold.*

Proof. Let $v = \mathrm{ord}_E$ and assume that $\quad K_X$ is Cartier (to avoid writing $\quad rK_X$ throughout the proof). If $v \; := \dim F_v \; \mathrm{H}^0(X; \quad mK_X)$, we obtain a telescoping sum that calculates $\sum_{\;2Z} \quad \mathrm{gr}_{F_v} \, \mathrm{H}^0(X; \quad mK_X)$

$$\sum_{2Z} \quad (v \quad v_{\;+1}) = \sum_{=0}^{+1} v \stackrel{\text{def}}{=} \sum_{=1}^{+1} h^0(Y; \quad m \quad K_X \qquad E) \stackrel{\text{def}}{=} \int_0^{+1} h^0(Y; \quad m \quad K_X \quad dteE)\, \mathrm{d}t;$$

and thus

$$\sum_{2Z} \quad \mathrm{gr}_{F_v} \, \mathrm{H}^0(X; \quad mK_X) = m \int_0^{+1} h^0(Y; \quad m \quad K_X \quad dmteE)\, \mathrm{d}t:$$

Then, $S_X(E)$ is given by

$$
\begin{aligned}
\limsup_{m\,!\,+1} \int_0^{+1} \frac{h^0(Y; \quad m \quad K_X \quad dmteE)=(m^n=n!)}{h^0(X; \quad mK_X)=(m^n=n!)}\, \mathrm{d}t &= \int_0^{+1} \frac{\mathrm{vol}(\qquad K_X \quad tE)}{\mathrm{vol}(\quad K_X)}\, \mathrm{d}t \\
&\stackrel{\text{def}}{=} \int_0 \frac{\mathrm{vol}(\qquad K_X \quad tE)}{(\quad K_X)^n}\, \mathrm{d}t
\end{aligned}
$$

by the Dominated Convergence Theorem. □

The above can be summarized in the following fundamental result[6], by Chi Li (2017) and Kento Fujita (2019).

[6] Our calculations, along with the Li-Xu Theorem (2014), allow proving it for dreamy divisors. Moreover, Blum, Liu, Xu, and Zhou proved in 2019 that if $\;_X(E) < 0$ for an arbitrary divisor E, then this inequality holds for a dreamy divisor.

Theorem III.5.9 (Valuative criterion for K-stability, [Fuj19b, BX19])**.** *Let X be a klt Fano variety. Then, X is K-stable (resp. K-semistable) $\Leftrightarrow$ $\beta_X(E) > 0$ (resp ≥ 0) for every (dreamy) divisor E over X.*

Example III.5.10. Let $X := \mathrm{Bl}_p(\mathbb{P}^2) \xrightarrow{\pi} \mathbb{P}^2$ with exceptional divisor $E \subset X$ and let L be the pullback of a line. Then, $K_X = \pi^* K_{\mathbb{P}^2} + E = -3L + E$ and we then calculate $S_X(E)$ as

$$\begin{aligned} S_X(E) = \frac{1}{(-K_X)^2}\int_0^{+\infty} \mathrm{vol}(-K_X - tE)\,\mathrm{d}t &= \frac{1}{8}\int_0 \mathrm{vol}(3L - E - tE)\,\mathrm{d}t \\ &= \int_0^2 (9 - (1+t)^2)\,\mathrm{d}t \\ &= \frac{7}{6}. \end{aligned}$$

Since E and X are smooth, $A_X(E) = 1$ and thus $\beta_X(E) = 1 - \frac{7}{6} = -\frac{1}{6} < 0$. Hence, X **is not** K-semistable (and therefore not K-polystable either) and thus $\mathrm{Bl}_p(\mathbb{P}^2)$ **does not admit** Kähler-Einstein metrics.

Example III.5.11 ([Fuj18])**.** Let X be a K-semistable klt Fano variety and let $p \in X$ be a smooth point. Let $\pi : Y := \mathrm{Bl}_p(X) \to X$ with exceptional divisor $E \subset Y$, where $A_X(E) = (n-1) + 1 = n$ and where it holds[7] that

$$\mathrm{vol}_Y(\pi^*(-K_X) - tE) \geq (-K_X)^n - t^n$$

and thus

$$\beta_X(E) = A_X(E) - S_X(E) = n - S_X(E) \geq 0$$

is equivalent to

$$\begin{aligned} n \geq \frac{1}{(-K_X)^n}\int_0^{\infty} \mathrm{vol}_Y(\pi^*(-K_X) - tE)\,\mathrm{d}t &\geq \frac{1}{(-K_X)^n}\int_0^{\sqrt[n]{(-K_X)^n}} ((-K_X)^n - t^n)\,\mathrm{d}t \\ &= \frac{n}{n+1}\sqrt[n]{(-K_X)^n} \end{aligned}$$

and thus we have that X satisfies the inequality $(-K_X)^n \leq (n+1)^n$.

Remark III.5.12. The inequality $(-K_X)^n \leq (n+1)^n$ is true for every smooth Fano variety of $\dim(X) = n \leq 3$, but $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(n))$ does not satisfy it for $n \geq 4$ (and thus is not K-semistable). Furthermore, using results by Yuchen Liu and Ziquan Zhuang (see [Fuj18]), it can be proved that in the K-semistable case the equality $(-K_X)^n = (n+1)^n$ is equivalent to $X = \mathbb{P}^n$.

[7] It suffices to compare dimensions in the exact sequence $0 \to H^0(X; \mathcal{O}_X(-mK_X) \otimes \mathfrak{m}_p^{mt}) \to H^0(X; -mK_X) \to \mathcal{O}_X/\mathfrak{m}_p^{mt} \to 0$.

III.6 Invariants and and the Abban-Zhuang estimate

In this section, two different techniques for proving K-stability will be illustrated, for which the invariants and will be introduced. Tian's -invariant, introduced in [Tia87], gave a characterization of the existence of Kahler-Einstein metrics on smooth Fano varieties. The original definition, of analytic nature, is different than the presented here[8]. Besides, -invariant introduced in [FO18], give another characterization of this fact. In the following we discuss these two invariants and their relationship.

Construction III.6.1 ([FO18, BJ20]). The valuative criterion for K-stability states that a klt Fano variety is K-stable (resp. K-semistable) $\iff$ $(X) > 1$ (resp. 1), where:

$$(X) := \inf_{E \quad Y \to X} \frac{A_X(E)}{S_X(E)},$$

i.e., the infimum is taken over all divisors over X. The **-invariant** was originally defined by Fujita-Okada as a certain limit of **log-canonical thresholds** of m-*basis type divisors* and then Blum-Jonsson proved that it coincided with the previous expression, and additionally it satisfies:

$$\frac{n+1}{n} \quad (X) \qquad (X) \qquad (n+1) \quad (X),$$

where $n = \dim(X)$ and where

$$(X) = \inf\{\operatorname{lct}(X, D),\ 0 \quad D \quad _{\mathbb{Q}} \quad K_X\}$$

is the **-invariant** of Tian, where

$$\operatorname{lct}(X, D) = \sup\{c \in \mathbb{R}^{\ 0},\ (X, cD) \text{ is lc}\}.$$

Theorem III.6.2 ([Tia87]). *Let X be a klt Fano variety of dimension $n = \dim(X)$. If*

$$(X) > (\ \ ) \frac{n}{n+1}$$

then X is K-(semi)stable.

Example III.6.3. We will use Tian's criterion to prove that a degree 1 del Pezzo surface X is K-stable. Recall that $X = \mathrm{Bl}_{p_1,\dots,p_8}(\mathbb{P}^2)$ is the blow-up of $\mathbb{P}^2$ at 8 points in

[8] Demailly and Kollar proved that both definitions agree.

general position[9]. Denoting $\pi : X \to \mathbb{P}^2$ the blow-up, the canonical divisor corresponds to:

$$-K_X = \pi^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{8} E_i$$

where E_i are the exceptional divisors. Thus, the linear system $|-K_X|$ corresponds to the linear system of (strict transforms of) cubics passing through $p_1, \dots, p_8$.

Consider $D \sim_{\mathbb{Q}} -K_X$ (i.e., there exists $m \in \mathbb{N}$ such that $mD \sim (-K_X)$ are linearly equivalent) and note that D is reduced (i.e., it can only have multiplicities of 1). Indeed, if $\mathrm{Supp}(D) \in |-K_X|$ this is directly true since $D \geq \mathrm{Supp}(D)$ (considering $\mathrm{Supp}(D)$ as a cycle). If $\mathrm{Supp}(D) \notin |-K_X|$, since $-K_X$ is free of base points, for any $x \in D$ there exists $C \in |-K_X|$ with $x \in D$. Then

$$D \cdot C = (-K_X)^2 = 1$$

and therefore D must be reduced. This fact implies that it suffices to compute the lct when D is a curve. The condition of passing through $p_1, \dots, p_8$ implies that D is irreducible, and then the possibilities are reduced to:

$$D \text{ smooth: } \mathrm{lct}(X; D) \stackrel{\text{def}}{=} \sup\{c \in \mathbb{R}_{\geq 0}; 1 - \mathrm{ord}_E(cD) \geq 0\} = 1$$

$$D \text{ nodal: } \mathrm{lct}(X; D) = 1$$

$$D \text{ cusp: } \mathrm{lct}(X; D) = \frac{5}{6}$$

The involved log-canonical thresholds were calculated in the Example I.4.16. In any case, $\delta(X) > 2/3$ and therefore X is K-stable.

Remark III.6.4. I. Cheltsov (see [Che08]) shows that $\delta(X) \geq 2/3$ for every del Pezzo surface of degree ≤ 4. Moreover, K. Fujita (see [Fuj19a]) shows that $\delta(X) = \frac{n}{n+1}$ implies K-stability for *smooth* Fano varieties.

Using the language of filtrations, valuations, and Newton-Okounkov bodies, Abban and Zhuang (see [AZ22]) prove one of the most currently used methods to estimate the δ-invariant via *adjunction*. The first observation is that the valuative criterion allows extending the definition of K-stability to log Fano pairs.

[9] This means that there are no 3 collinear points, no 6 points lying on a conic, and there is no nodal or cuspidal cubic passing through the 8 points such that one of them is exactly the singular point.

Definition III.6.5. Given a log Fano pair (X,Δ) of dimension $n = \dim(X)$ and a divisor $E \subset Y \to X$ over X, we define

$$\delta(X,\Delta;E) = \frac{A_{(X,\Delta)}(E)}{S_{(X,\Delta)}(E)}$$

where

$$S_{(X,\Delta)}(E) = \frac{1}{(-K_X-\Delta)^n}\int_0^{\infty} \mathrm{vol}(\pi^*(-K_X-\Delta)-tE)\,\mathrm{d}t$$

We say that (X,Δ) is K-stable (resp. K-semistable) if

$$\delta(X,\Delta) = \inf\{\delta(X,\Delta;(E);E \subset Y \to X \text{ divisor over } X\} > 1\ (\geq 1).$$

Remark III.6.6. If X is an algebraic variety and $L \in \mathrm{Pic}(X)$ is ample, $V_\bullet := \{V_m = \mathrm{H}^0(X,mL)\}_{m\geq 0}$ is the associated linear serie, and if $E \subset Y \to X$, the filtration $(\mathcal{F}_E V_m)_t := \{s \in V_m, \mathrm{ord}_E(s) \geq mt\}$ is defined and

$$\mathrm{vol}(\mathcal{F}_E V_m)_t = \lim_{m\to+\infty}\frac{\dim((\mathcal{F}_E V_m)_t)}{m^n/n!}.$$

Then

$$S(V_\bullet;E) := \frac{1}{\mathrm{vol}(V_\bullet)}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}_E V_m)_t\,\mathrm{d}t = S_{(X,\Delta)}(E),$$

considering $L = -K_X - \Delta$.

Construction III.6.7 (Abban-Zhuang, [AZ22]). Let (X,Δ) be a klt pair with $\Delta \geq 0$, and let $E \subset Y \to X$ be a divisor over X of **plt type**, i.e., $-E$ is π-ample and (Y,Δ_Y+E) is a plt pair[10], where Δ_Y is defined by the condition

$$K_Y + \Delta_Y = \pi^*(K_X+\Delta) + (A_{(X,\Delta)}(E) - 1)E.$$

If Δ_E is the **different** of Δ_Y on E (that is, $K_E + \Delta_E = (K_Y + \Delta_Y + E)|_E$) then

$$\delta_Z(X,\Delta;V_\bullet) = \inf_{F,\,Z\subseteq c_X(F)}\frac{A_{(X,\Delta)}(F)}{S(V_\bullet;F)}$$

verifies the condition

$$\delta_Z(X,\Delta;V_\bullet) \geq \min\left\{\frac{A_{(X,\Delta)}(E)}{S(V_\bullet;E)}, \inf_{Z'}\delta_{Z'}(E,\Delta_E;\mathbf{W}^E_{\bullet,\bullet})\right\},$$

with $Z' \subset Y$ ranging over the subvarieties of Y such that $\pi(Z') = Z$, and where

$$\delta_{Z'}(E,\Delta_E;\mathbf{W}^E_{\bullet,\bullet}) = \inf_{F,\,Z'\subseteq c_E(F)}\frac{A_{(E,\Delta_E)}(F)}{S(\mathbf{W}^E_{\bullet,\bullet};F)}.$$

[10] Recall that (X,Δ) is plt (resp. klt) if $A_{X,\Delta}(E) > 0$ (resp. $A_{X,\Delta}(E) > 0$ and $\lfloor\Delta\rfloor \leq 0$) for every divisor E over X.

The term $S(\mathbf{W}^E_{\cdot,\cdot};F)$ is obtained analogously to $S(V;E)$ but considering the refined linear serie given by

$$W^E_{m,j} := \mathrm{Im}(\mathrm{H}^0(Y, m(K_X+\Delta) - jE) \to \mathrm{H}^0(E, m(K_E+\Delta_E) - jE|_E)).$$

In practice, this volume can be calculated or estimated using the notion of *restricted volume* defined by Lazarsfeld and collaborators, which in turn is shown to be calculable using *slices* of Newton-Okounkov bodies. The latter, in the case of surfaces, is calculated using the *Zariski decomposition*.

Remark III.6.8. By definition, $\delta(X;D;V) = \inf_{Z\subseteq X}\delta_Z(X;D;V)$. In particular, the condition $\delta_p(X;D;V)\geq 1$ for every $p\in X$ implies that X is K-semistable.

From now on, X will be a **surface** and $E\subset Y\to X$ will be a smooth curve[11] fixed on X. In this case, Z' (which is a subvariety in E) will be a point $Z'=p$ such that $p\in\overline{c_E(F)}$, i.e., $p=F$. We need to calculate

$$\delta_p(E;D_E;W^E_{\cdot,\cdot}) = \frac{A_{(E;D_E)}(p)}{S(W^E_{\cdot,\cdot};p)} = \frac{1-\mathrm{ord}_p(D_E)}{S(W^E_{\cdot,\cdot};p)}$$

where we have used that E is smooth. Let $\tau = \sup\{u\in\mathbb{R}_{\geq 0};\ (-K_X-D)-uE$ is pseudo-effective$\}$, and consider the Zariski decomposition

$$(-K_X-D)-uE = \underbrace{P(u)}_{\text{nef}} + \underbrace{N(u)}_{\text{negative}}.$$

We will assume that $\mathrm{Supp}(E)\not\subseteq N(u)$ for every u (for simplicity). In such a case, we have a *flag* $\{p\}\subset E\subset Y$, whose *Newton-Okounkov body* allows calculating the volume of the divisor[12]:

$$\begin{aligned}S(W^E_{\cdot,\cdot};p) &= \frac{\dim(X)}{\mathrm{vol}(L)}\int_0^{\tau}\int_0^{+\infty}\mathrm{vol}(P(u)|_E - vp)\,\mathrm{d}v\,\mathrm{d}u\\ &= \frac{2}{(-K_X-D)^2}\int_0^{\tau}\int_0^{t(u)}\max\{\mathrm{ord}_p(P(u)|_E)-v,0\}\,\mathrm{d}v\,\mathrm{d}u\end{aligned}$$

Example III.6.9. Using the Abban-Zhuang method, we will prove that every cubic surface is K-semistable. Let X be a cubic surface, $p\in X$ and $E\in|-K_X|$ an elliptic curve (smooth) such that $p\in E$ and $E|_E=3p$. Here, $D=0$ and due to smoothness $A_X(E)=1$. We calculate

$$S_X(E) = \frac{1}{(-K_X)^2}\int_0^{+\infty}\mathrm{vol}(-K_X-tE)\,\mathrm{d}t = \frac{1}{(-K_X)^2}\int_0^1(-K_X)^2(1-t)^2\,\mathrm{d}t = \frac{1}{3}.$$

[11] It suffices to consider smooth curves due to the plt hypothesis.

[12] See Corollary 1.109 in *The Calabi Problem for Fano Threefolds, Araujo et al., 2023*.

Now note that

$$-K_X - uE \sim_{\mathbb{R}} (1-u)(-K_X) \text{ is nef } \iff (1-u)(-K_X) \text{ is pseudo-effective } \iff 0 \leq u \leq 1.$$

In this case $P(u) = (1-u)(-K_X)$ and $N(u) = 0$, and so $P(u)|_E = 3(1-u)p; \operatorname{ord}_p(P(u)|_E) = 3(1-u)$. Then,

$$\begin{aligned} S(W^E_{\bullet,\bullet};p) &= \frac{2}{\operatorname{vol}(L)} \int_0^{\infty} \int_0^{t(u)} \max\{\operatorname{ord}_p(P(u)|_E) - v, 0\} \,\mathrm{d}v \,\mathrm{d}u \\ &= \frac{2}{3} \int_0^1 \int_0^{3(1-u)} (3(1-u) - v) \,\mathrm{d}v \,\mathrm{d}u = 1. \end{aligned}$$

We calculate that $\delta_p(E; D_E; W^E_{\bullet,\bullet}) = 1$, and then

$$\delta_p(X;V) \geq \min\left\{ \frac{A_X(E)}{S_X(E)}, \delta_p(E; \underbrace{\Delta_E}_{=0}; W^E_{\bullet,\bullet}) \right\} = \min\{3, 1\} = 1.$$

The previous calculation concludes that X is K-semistable.

Remark III.6.10. In fact, Abban-Zhuang verify that $\delta(X) \geq 3/2$, and every cubic surface is K-stable.

To finish this chapter, we present another example of application of the Abban-Zhuang estimate, in this case for a log pair.

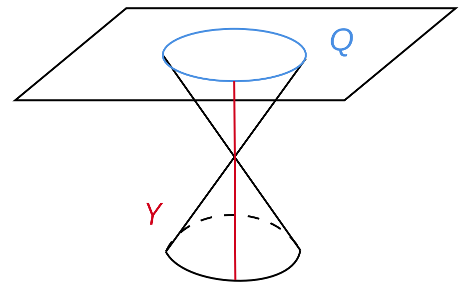
Example III.6.11. Consider the singular variety $X = \mathbb{P}(1,1,2)$. Bear in mind that this variety is constructed as follows: take the graduated ring $S = k[x,y,z]$ with the graduation $\deg(x) = \deg(y) = 1, \deg(z) = 2$, and define $X \stackrel{\text{def}}{=} \operatorname{Proj}(S)$. The shifted module $S(2)$ gives a line bundle $\mathcal{O}_X(2) \stackrel{\text{def}}{=} \widetilde{S(2)}$, whose global sections are $\Gamma(X, \mathcal{O}_X(2)) = S(2)_0 = \langle x^2, y^2, xy, z \rangle$ so we have an embedding $\varphi : X \hookrightarrow \mathbb{P}^3_{[x:y:z:w]}$ which gives $\mathbb{P}(1,1,2) = V(xy - z^2)$.
In this example we will consider $(X, \Delta) = (\mathbb{P}(1,1,2), \lambda Q)$ where $\lambda \in \mathbb{Q}^{>0}$ and $Q = X \cap \{w = 0\}$ is the hyperplane section at infinity. Precisely, we will show the following facts:

1. $(X, \lambda Q)$ is K-unstable for any $\lambda \neq 1/2$.

2. $(X, \Delta) = (X, \frac{1}{2}Q)$ is K-semistable.

First, consider $Y = \{x = z = 0\} \subset X$ as a ruling through the vertex, as the following image shows.

$X = \mathsf{P}(1;1;2)$



Note first that $Q = 2Y$, and in fact it can be proved that $K_X = \quad 4Y$. In first place, we will calculate $\ _{(X;\ )}(Y)$. By definition we note that the log discrepancy in this case corresponds to $A_{(X;\ Q)}(Q) = 1 \quad$. Moreover, we note that $(\ \ K_X \quad \Delta) \quad (2 \quad)Q$ and this shows that

$$K_X \quad \Delta \text{ is nef } (\ ) \quad K_X \quad \Delta \text{ is pseudoeffective } (\ ) \ 2 \qquad 0.$$

Using the previous facts we can compute

$$\begin{aligned} S_{(X;\ )}(Q) &= \frac{1}{\mathrm{vol}(\ \ K_X \quad \Delta)} \int_0^2 \ \mathrm{vol}(\ \ K_X \quad \Delta \quad tQ)dt \\ &= \frac{1}{(2 \quad)^2 Q^2} \int_0^2 \ (2 \qquad t)^2 Q^2 dt \\ &= \frac{1}{3}(2 \quad); \end{aligned}$$

and then $\ _{(X;\ )}(Q) = \frac{3(1 \quad)}{2}$. In particular, when we take $\quad = 0$ we obtain $\ _{(X;\ )}(Q) = \frac{2}{3}$, which shows that X is K-unstable by the valuative criterion. Moreover, note that

$$\ _{(X;\ )}(Q) = \frac{3(1 \quad)}{2} \quad 1 \quad (\ ) \qquad \frac{1}{2} \quad \_ \qquad > 2:$$

To obtain the conclusion 1: we will do the blow-up of the vertex $p\ 2\ X$ and we will calculate the $\ $-invariant for the exceptional divisor. The vertex $p\ 2\ X$ is a rational double point (also known as a A1 singularity), an its resolution of singularities correspondes exactly to the blow-up at the vertex

$$\mathsf{F}_2 := \mathsf{P}(O_{\mathsf{P}^1} \quad O_{\mathsf{P}^1}(2)) \ \overset{"}{!}\ X;$$

where F_2 is the Hirzebruch surface and we denote by E the exceptional divisor of this resolution. It is known that this is a crepant resolution, i.e., $"\ K_X = K_{\mathsf{F}_2}$, and we can calculate

$$!_{\mathsf{F}_2} = "\ (!_{\mathsf{P}^1} \quad O_{\mathsf{P}^1}(\ \ 2)) \quad O_{\mathsf{F}_2}(\ \ 2) = O_{\mathsf{F}_2}(\ \ 2):$$

In terms of divisors, we obtain that $K_{\mathsf{F}_2} = -2$ where is the class of a section of the projection $\mathsf{F}_2 \to \mathsf{P}^1$. We will calculate $_{(X;\ Q)}(E)$. For this, we review the intersection theory of F_2. We know that $\mathrm{Pic}(\mathsf{F}_2) = \mathsf{Z}[f] \oplus \mathsf{Z}[\ ]$ (see [Bea83, Proposition III.18]) where f is the class of a fiber of the projection $\mathsf{F}_2 \to \mathsf{P}^1$. The rules for intersection are

$$^2 = 2; \quad f^2 = 0; \quad f \cdot \ = 1;$$

and we can prove that the exceptional divisor E of $\pi : \mathsf{F}_2 \to X$ corresponds to $E = -2f$. We can write $\pi^* Q = af + b$ and as Q doesn't contain the vertex we have

$$0 = Q \cdot \pi_* E = \pi^* Q \cdot E = (af + b\ ) \cdot (\ -2f) = a$$

and similarly

$$2 = Q^2 = (\pi^* Q)^2 = b^2\ ^2 = 2b^2 \Rightarrow b = 1$$

so we obtain

$$\pi^*(-K_X - Q) - tE = (2 - - t) + 2tf.$$

As we know that $\mathrm{Pseff}(\mathsf{F}_2) = \mathrm{Nef}(\mathsf{F}_2) = \mathsf{R}_+[\ ] \oplus \mathsf{R}_+[f]$, we note the threshold is $= 2 -$ and then it follows

$$\begin{aligned} S_{(X;\ Q)}(E) &= \frac{1}{\mathrm{vol}(-K_X - Q)} \int_0^{2-} \mathrm{vol}(\pi^*(-K_X - Q) - tE)dt \\ &= \frac{2}{2(2-\ )^2} \int_0^{2-} ((2-\ )^2 - t^2)dt \\ &= \frac{2}{3}(2-\ ) \end{aligned}$$

Clearly, in this case $A_{(X;\ Q)}(E) = 1$, and then $_{(X;\ Q)}(E) = \frac{3}{2(2-\ )}$. Now we can observe

$$_{(X;\ )}(E) = \frac{3}{2(2-\ )} \geq 1 \iff \geq \frac{1}{2}, \quad < 2.$$

These calculations shows that the pair $(X;\ Q)$ is K-unstable when $\neq \frac{1}{2}$.

In the following we fix the notation $(X;\Delta) = (X;\frac{1}{2}Q)$, and to prove this pair is K-semistable using the Abban-Zhuang estimate, we will have to bound the number $_p(X;\Delta)$ for any point $p \in X$. First, we take $p \in X$ as any smooth point. To use the adjunction we have to choose the divisor E, which we choose as a ruling Y through the point p. Same calculations that we already did shows that $(X;\Delta;Y) = 1$. Now, note that

$$-K_X-\Delta-uY\sim(3-u)Y \text{ is nef} \iff \text{it is pseudo-effective} \iff 0<u<3.$$

By this observation, the Zariski decomposition is $P(u)=-K_X-\Delta-uY, N(u)=0$. Moreover, we note that

$$P(u)|_Y=(3-u)Y|_Y=\frac{3-u}{2}(2Y)|_Y=\frac{3-u}{2}p$$

and then $\mathrm{ord}_p(P(u)|_Y)=\frac{3-u}{2}$.

$$\begin{aligned} S(W^Y_{\bullet,\bullet};p) &= \frac{4}{9}\int_0\int_0^{t(u)}\max\{\mathrm{ord}_p(P(u)|_Y)-v,0\}dvdu \\ &= \frac{4}{9}\int_0^3\int_0^{\frac{3-u}{2}}\left(\frac{3-u}{2}-v\right)dvdu=\frac{1}{2}. \end{aligned}$$

Finally, as $Y=\mathbb{P}^1$, the different divisor is

$$K_Y+\Delta_Y=(K_X+Y)|_Y=-3Y|_Y=-\frac{3}{2}p \Rightarrow \Delta_Y=-\frac{3}{2}p+2p=\frac{p}{2},$$

so $A_{(Y,\Delta_Y)}(p)=\frac{1}{2}$ and we conclude $\delta_p(Y,\Delta;W^Y_{\bullet,\bullet})\geq 1$. This gives the bound $\delta_p(X,\Delta)\geq 1$ for any smooth point $p\in X$.

To conclude we will do the calculation for the singular point $p\in X$. For this we will do the Abban-Zhuang estimation with E as the exceptional divisor of the resolution $\pi:\mathbb{F}_2\to X$. As we already calculated $\pi^*(-K_X-\frac{1}{2}Q)-tE=(\frac{3}{2}-t)\sigma+2tf$ and $A(X,\Delta,E)=1$, it only remains to calculate $\delta_p(E,\Delta_E;W^E_{\bullet,\bullet})$. We see that

$$\pi^*(-K_X-\tfrac{1}{2}Q)-tE \text{ is nef} \iff \text{it is pseudo-effective} \iff t\leq 3/2.$$

and then we have the Zariski decomposition $P(u)=(\frac{3}{2}-t)\sigma+2tf, N(u)=0$. By adjunction, and the fact that $E=\mathbb{P}^1$, we have

$$\begin{aligned} K_E=(K_{\mathbb{F}_2}+\sigma)|_E &\Rightarrow \sigma|_E=2q \\ K_E=(K_{\mathbb{F}_2}+f)|_E &\Rightarrow f|_E=2q \end{aligned}$$

where $q\in E$ is any point. Also we have $E|_E=(\sigma-2f)|_E=-2q$ and then

$$P(u)|_E=(3+2u)q \Rightarrow \mathrm{ord}_p(P(u)|_E)=(3+2u)$$

We also need to calculate

$$\mathrm{vol}(\pi^*(-K_X-\Delta))=\mathrm{vol}\left(-K_{\mathbb{F}_2}-\frac{1}{2}\pi^*Q\right)=\mathrm{vol}\left(\frac{3}{2}\sigma\right)=\frac{9}{2}.$$

We can finally calculate

$$S(W^{E}_{;\,};q)=\frac{2}{\mathrm{vol}(''\ (\quad K_X \quad \Delta))}\int_0^{3=2}\int_0^{3+2u}(3+2u \quad v)^2 dv du=\frac{1}{3}$$

and

$$2p+\Delta_E=(K_{\mathrm{F}_2}+\widetilde{Q}+E)j_E \quad) \quad \Delta_E= \quad 3q:$$

$A_{(E; \quad E)}(q)=1 \quad (\quad 3)=4.$

This shows that

$$\inf_{q2E} \ {}_q(E;\Delta_E;\mathbf{W}^{E}_{;\,})=\frac{4}{1=3}=12$$

and then we have proven that ${}_p(X;\Delta) \quad 1$ for every point $p\ 2\ X$. We conclude $(X;\frac{1}{2}Q)$ is K-semistable.

Chapter IV

K-stability of Fano fourfolds

In this final chapter we will present a brief review about K-stability of del Pezzo fourfolds, and then the idea will be to study a family of Fano fourfolds and to present some evidence towards its K-stability.

More precisely, this chapter represents a work in progress on the K-stability of Fano fourfolds of genus 9, part of which was developed during a research stay at the University of Poitiers in October 2024. The author is deeply grateful to Adrien Dubouloz for the opportunity to present partial results at the workshop "K-stability, Geometry and Group Actions" and warmly thanks Takashi Kishimoto and Kento Fujita for their valuable comments during the stay.

IV.1 Review of K-stability of del Pezzo manifolds

As we already said, in [ACC+23] all K-stable Fano threefolds were classified, so the next question is about K-stability of Fano fourfolds. In this sense, we have the following theorem which is a result of several works of different people.

Theorem IV.1.1 ([ST24, AGP06, Fuj17, ACC+23])**.** *K-stable del Pezzo manifolds of* $d \neq 3$ *are completely classified in terms of its degree. Specifically, if* X *is a del Pezzo manifold:*

1. *If* $d = 1, 2$*,* X *is* K*-stable.*
2. *If* $d = 4$*,* X *is* K*-polystable.*

3. *If $d = 5$, in terms of the dimension we have:*
   - *If $\dim(X) = 3$, X is K-polystable.*
   - *If $\dim(X) = 4, 5$, X is not K-semistable.*
   - *If $\dim(X) = 6$, X is K-polystable.*
4. *If $d = 6$, X is K-polystable.*
5. *If $d = 7$, X is not K-polystable.*
6. *If $d = 8$, X is K-stable.*

In the case of degree $d = 3$ we only have the answer up to dimension 4.

Theorem IV.1.2 ([Liu22, Corollary 1.2])**.** *All smooth cubic fourfolds are K-stable.*

The conclusion of this section is then that we have a complete understanding of K-stability of del Pezzo fourfolds.

IV.2 Birational geometry of Fano-Mukai manifolds of genus 9

We already studied del Pezzo varieties and we developed in depth some examples of such varieties in dimension 4. Now we jump one step forward again in the index.

Definition IV.2.1 (Fano-Mukai variety)**.** A Fano-Mukai variety is a Fano variety X with index $\iota(X) = \dim(X) - 2$.

The name of these varieties was given in honor of S. Mukai, which completely classified prime (i.e., with Picard rank 1) Fano varieties of that index in arbitrary dimension (announced in [Muk89]). In this sense, we have the first important observation.

Lemma IV.2.2. *Let X be a Fano-Mukai variety of dimension $\dim(X) = n$ and $\rho(X) = 1$. Then X has genus $g \leq 12$ and $g \neq 11$ and it verifies the relation $d = 2g - 2$ where d is the degree of V. Moreover, if $g = 12$ then $\dim(X) = 3$.*

IV.2.1 Construction of Fano-Mukai manifolds of genus 9

Because we will be interested uniquely in the case of genus 9, we only give a description of these case. Here below we give an explicit construction that we will comment in the next sections. The complete description of prime Fano-Mukai varieties can be found in [AC13, Theorem 2.3].

Here below we discuss the definition of the Lagrangian Grassmannian, drawing primarily on [IR05].

Construction IV.2.3. Let $V = \mathbb{C}^6$ be a 6-dimensional complex vector space, and let $\omega : V \times V \to \mathbb{C}$ be a symplectic form on V, i.e., a skew-symmetric, bilinear and non-degenerate map. Fixing a basis $\{e_1, \dots, e_6\}$ of V we can assume that ω is given by the matrix

$$J = (\omega(e_i, e_j))_{i,j} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix},$$

where I_3 denotes the 3×3 identity matrix. A subspace $U \subset V$ is called isotropic if $\omega|_{U \times U} = 0$, and it can be proven that the maximal dimension of such subspace is 3. A maximal isotropic subspace is called a *Lagrangian subspace*, and the set of all Lagrangian subspaces in $V = \mathbb{C}^6$ is denoted $\mathbf{LG}(3, V) \subset \mathrm{Gr}(3, V)$. The symplectic form ω permits to define a *contraction map*

$$\Upsilon : \bigwedge^3 V \to V, \quad v_1 \wedge v_2 \wedge v_3 \mapsto \omega(v_1 \wedge v_2)v_3 + \omega(v_2 \wedge v_3)v_1 + \omega(v_3 \wedge v_1)v_2$$

which is surjective and $\bigwedge^{\langle 3 \rangle} V \overset{\text{def}}{=} \ker(\Upsilon)$ is of dimension $\dim(\bigwedge^{\langle 3 \rangle} V) = 14$. If $\varphi : \mathrm{Gr}(3, V) \hookrightarrow \mathbb{P}^{19}$ is the Plücker embedding, it can be showed that $\varphi(\mathbf{LG}(3, V)) = \mathbb{P}(\bigwedge^{\langle 3 \rangle} V) \cap \varphi(\mathrm{Gr}(3, V)) \subset \mathbb{P}(\bigwedge^{\langle 3 \rangle} V) = \mathbb{P}^{13}$, so it is a projective variety, and moreover, the symplectic group

$$\mathrm{Sp}_6(\mathbb{C}) = \{Z \in \mathrm{SL}(6, \mathbb{C}) : {}^t Z J Z = J\}$$

acts transitively in $\mathbf{LG}(3, V)$. Hence, the Lagrangian Grassmannian is a homogeneous space.

The reason to discuss the previous construction is the following characterization of prime Fano-Mukai n-folds of genus 9.

Proposition IV.2.4. *Let V be an n-dimensional Fano-Mukai variety with $\iota(V) = 1$ and $g(V) = 9$. Then V is a linear section of the Lagrangian Grassmanian $\mathbf{LG}(3, 6) \subset \mathbb{P}^{13}$ under the Plücker embedding.*

A final comment about the Lagrangian Grassmannian is the non-existence of planes.

Lemma IV.2.5. *The Lagrangian Grassmanian $\Sigma := \mathbf{LG}(3,6)$ does not contain planes.*

Proof. By Remark II.3.13 there is a universal exact sequence of vector bundles in $G = G(3,6)$, which we can restrict to Σ and obtain

$$0 \to S \to V_\Sigma \to Q \to 0$$

where V_Σ is the trivial bundle on Σ, and S is the universal subbundle, i.e., the fiber of S at a point $U \in \Sigma$ is the subspace U itself. Moreover, note that the linear map $L : V \to V^\vee, v \mapsto \omega(v, \cdot)$ induces an isomorphism $U^\vee = V^\vee/U^\perp = V/U$ for an isotropic subspace $U \subset V$. This implies that $Q = S^\vee$.

Suppose there is a plane $\Pi = \mathbb{P}^2 \hookrightarrow \Sigma$, i.e., Π is linearly embedded into Σ, meaning that $c_1(S|_\Pi) = -\ell$ where ℓ is the class of a line in $\mathbb{P}^2$. Then, the total Chern class of $S|_\Pi$ is

$$c(S|_\Pi) = 1 - \ell + x\ell^2$$

for some $x \in \mathbb{Z}$. By duality

$$c(S^\vee|_\Pi) = 1 + \ell + x\ell^2,$$

and then we compute

$$1 = c(V_\Sigma|_\Pi) = c(S|_\Pi) \cdot c(S^\vee|_\Pi) = 1 + (2x - 1)\ell^2.$$

The fact that $2x = 1$ gives a contradiction. $\square$

IV.2.2 Sarkisov link between W_5 and V_{16} after Prokhorov and Zaidenberg

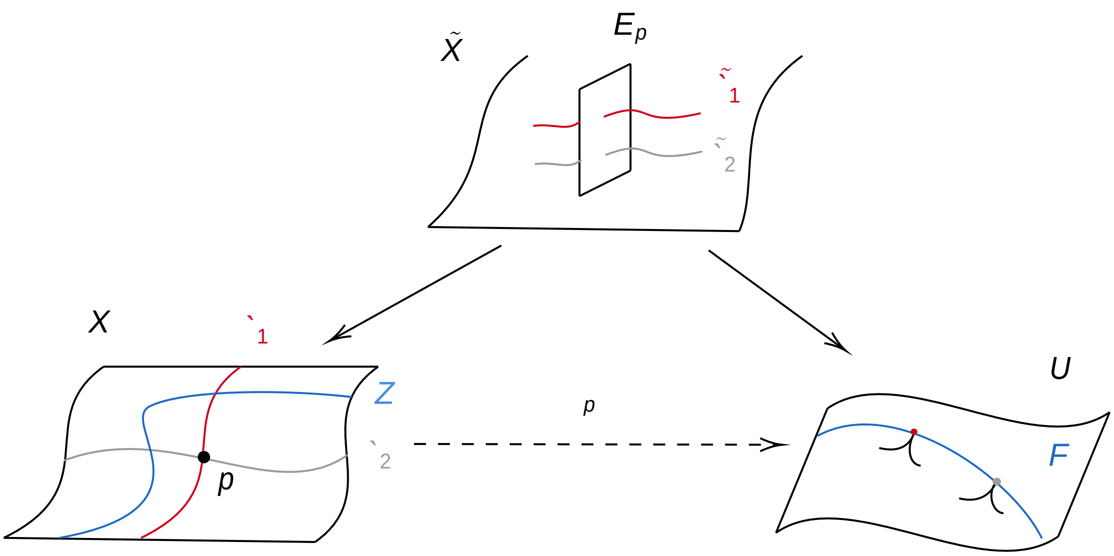
In this section we follow the articles by Prokhorov and Zaideberg [PZ17, PZ18]. Using the quintic del Pezzo fourfold $W = W_5$, we will be able to construct a Fano-Mukai fourfold $V = V_{16} \subset \mathbb{P}^{11}$, i.e., such that $-K_V = 2H$ where $\mathrm{Pic}(V) = \mathbb{Z}[H]$. In the case of V we will have $\deg(V) = 16$ and $g = g(V) = 9$.

Proposition IV.2.6 ([PZ17, Proposition 4.1])**.** *The quintic fourfold $W \subset \mathbb{P}^7$ has an hyperplane section containing an anticanonically embedded sextic del Pezzo surface $F_6 \subset \mathbb{P}^6$ such that none of the planes contained in W intersects F_6 in a conic.*

Proof. Let $X = X_6 \subset \mathbb{P}^7$ be a smooth del Pezzo 3-fold of degree 6. By Fujita's classification there exists a unique X and $\rho(X) = 2$, and by Prokhorov there exists a commutative diagram

$$\begin{tikzcd} & \widetilde{X} = \mathrm{Bl}_p(X) \arrow[dl, "\sigma"'] \arrow[dr, "\varphi"] & \\ p \in X \arrow[rr, dashed, "\pi_p"] & & U \subset \mathbb{P}^6 \end{tikzcd}$$

where π_p is the projection from a general point $p \in X$ and $U = U_5 \subset \mathbb{P}^6$ is a quintic del Pezzo 3-fold with **2 node singularities**. Explicitly, X can be realized as a smooth member of the linear system $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)|$ in $\mathbb{P}^2 \times \mathbb{P}^2$, so we have natural projections $\mathrm{pr}_i\colon X \to \mathbb{P}^2$ which are $\mathbb{P}^1$-bundles. We have the following geometric picture.



Take $D \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)|$ another smooth general divisor and let $Z := X \cap D$. By adjunction we have

$$K_Z = (K_{\mathbb{P}^2 \times \mathbb{P}^2} + X + D)|_Z = \mathcal{O}_Z(-3,-3) \otimes N_{Z/X} \otimes N_{X/D} = \mathcal{O}_Z(-3+2,-3+2) = \mathcal{O}_Z(-1,-1)$$

and then

$$\begin{aligned} (-K_Z)^2 = \mathcal{O}_Z(1,1) \cdot \mathcal{O}_Z(1,1) &= \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)^4 \\ &= (\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1) + \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(1))^4 \\ &= 6\, \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1) \cdot \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(1) \\ &= 6 \end{aligned}$$

i.e., Z is a smooth del Pezzo surface of degree 6.

On the other hand, we know that U_5 can be constructed as a linear section of codimension 3 of $G(2,5) \subset \mathbb{P}^9$. Then there exists a general hyperplane $H \subset \mathbb{P}^9$ and T_1, T_2 general **tangent** hyperplanes to $G(2,5)$. If we consider T as a general linear combination of T_1 and T_2, then $W := G(2,5) \cap H \cap T$ is a quntic del Pezzo fourfold which contains U, so it contains the image of $Z \subset X$ under π_p. □

The key for the construction is to use a surface like in the previous proposition, blow-up, and then obtain a new variety as a result of performing the 2-ray game. Here we only ennounce the main result. For the complete proof consult [PZ17, §5.1].

Theorem IV.2.7 ([PZ17, Theorem 2.1])**.** *Let $W = W_5 \subset \mathbb{P}^7$ be a del Pezzo fourfold of degree 5, and let $F \subset W \cap \mathbb{P}^6$ be an anticanonically embedded sextic del Pezzo surface such that $c_2(W) \cdot F = 26$ and such that F does not intersect any plane in W along a conic. Then there exists a commutative diagram*

$$\begin{tikzcd} & D \arrow[dl] & & \widetilde{W} \arrow[dl, "\sigma"'] \arrow[dr, "\varphi'"] & & E \arrow[dr] & \\ F & & W \arrow[rr, dashed, "\varphi"] & & V & & S \end{tikzcd}$$

where

- *$V = V_{16} \subset \mathbb{P}^{11}$ is a Mukai fourfold of genus $g = 9$.*
- *the map $\varphi : W \dashrightarrow V \subset \mathbb{P}^{11}$ is given by the linear system of conics through F*
- *$\sigma : \widetilde{W} \to W$ is the blow-up of F with exceptional divisor D, and $\varphi' : \widetilde{W} \to V$ is the blow-up of a smooth quadric surface $S \subset \mathbb{P}^3 \subset \mathbb{P}^{11}$ with exceptional divisor E and such that $c_2(V) \cdot S = 5$.*
- *if H is an ample generator of $\mathrm{Pic}(W)$ and L is an ample generator of $\mathrm{Pic}(V)$, then*

$$\sigma^* H \sim \varphi'^* L - E; \qquad D \sim \varphi'^* L - 2E;$$
$$\varphi'^* L \sim 2\sigma^* H - D; \qquad E \sim \sigma^* H - D.$$

IV.3 K-stablity of V_{16}

In this section we will present a calculation concerning K-stability of V_{16} based on the tools developed in [Fuj16]. In that article, K. Fujita introduces the notion of *divisorial* *stability*, which consists in the following idea. For a $\mathbb{Q}$-Fano variety X and a nonzero effective Weil divisor $D \in \mathrm{WDiv}(X)$ he constructs a test configuration of $(X, -rK_X)$, called the *basic semi-test configuration via* D, and he gave an explicit formula for the Donaldson-Futaki invariant of those configurations. Later, in [Fuj17] he would use these tools to prove that quintic del Pezzo fourfolds W_5 are K-unstable, and the proof consists of calculating the Donaldson–Futaki invariant associated with the semi-basic test configuration via the exceptional divisor of the Sarkisov link studied in §II.3.6.

First, we will introduce some tools from the first reference, and afterwards we will do the same calculation K. Fujita made for W_5, but for V_{16} and using the exceptional divisor of its corresponding Sarkisov link.

We start introducing some terminology from [KKL16], which is concerning to *geography of models*.

Definition IV.3.1. Let X be a normal projective variety and let D_X be an $\mathbb{R}$-Cartier divisor on X. Let $\varphi : X \dashrightarrow Y$ be a birational map to a normal projective variety such that $D_Y := \varphi_* D_Y$ is $\mathbb{R}$-Cartier, we say that:

1. φ is D_X**-nonpositive** if for a common resolution $(p, q) : W \to X \times Y$ we can write $p^* D_X = q^* D_Y + E$, with E effective and q-exceptional.
2. φ is a **semiample model** of D_X if φ is D_X-nonpositive and D_Y is semiample.
3. φ is the **ample model** of D_X if exists $f : X \dashrightarrow Z$ a semiample model of D_X and a morphism with connected fibers $g : Z \to Y$ such that $\varphi = g \circ f$ and such that $f_* D_X = g^* A_Y$ with A_Y an ample $\mathbb{R}$-divisor.

Assume now that X is a smooth Fano manifold of dimension n and that $S \subset X$ is a smooth subvariety of $\mathrm{codim}_X(S) = d$ and with associated ideal sheaf $I_S \subset O_X$. We denote by $\sigma : \widetilde{X} \to X$ the blow-up of X along S, with exceptional divisor $E_S = \mathbb{P}(N_{\overline{S}/W})$. We will assume that E_S is a **dreamy divisor** [1]. We recall that the **pseudo-effective threshold** of S with respect to $(X, -K_X)$ is defined as

$$\tau(S) := \max\{\tau \in \mathbb{R}_{\geq 0} ; \ \sigma^*(-K_X) - \tau E_S \text{ is pseudo-effective}\}.$$

Under the previous assumptions, the following is a particular case of [KKL16, Theorem 4.2].

[1] This occurs, thanks to [BCHM10], when $\widetilde{X}$ is a Fano variety.

Theorem IV.3.2. *There is a unique sequence* $\{(\lambda_i; X_i)\}_{1 \le i \le m}$ *with*

- $0 =: \lambda_0 < \lambda_1 < \cdots < \lambda_m = \tau(S)$ *are positive rational numbers,*
- $X_1, \dots, X_m$ *are normal projective varieties with* $X_1 = \widetilde{X}$*, and*
- *there are mutually distinct birational contractions* $\varphi_i : \widetilde{X} \dashrightarrow X_i$ *for* $i \in \{1, \dots, m\}$ *with* $\varphi_1 = \mathrm{Id}_{\widetilde{X}}$*,*

and such that

1. *For any* $x \in [\lambda_{i-1}, \lambda_i]$*,* φ_i *is a semiample model of* $\sigma^*(-K_X) - xE_S$*.*
2. *If* $x \in]\lambda_{i-1}, \lambda_i[$*,* φ_i *is the ample model of* $\sigma^*(-K_X) - xE_S$*.*

We say that $\{(\lambda_i; X_i)\}_{1 \le i \le m}$ *is the* **ample model sequence** *of* $(X; -K_X; I_S)$*, and we set* $E_i := (\varphi_i)_* E_S$*.*

Construction IV.3.3 (K. Fujita)**.** Let $r \in \mathbb{N}^{\ge 1}$ be a fixed and sufficiently divisible positive integer such that $r \cdot \tau(S) \in \mathbb{N}^{\ge 1}$ and such that the graded $\mathbb{C}$-algebra $\bigoplus_{k \ge 0; 0 \le j \le r\tau(S)} \mathrm{H}^0(X; \mathcal{O}_X(-krK_X) \cdot I_S^j)$ is generated as a $\mathbb{C}$-algebra by $\bigoplus_{0 \le j \le r\tau(S)} \mathrm{H}^0(X; \mathcal{O}_X(-rK_X) \cdot I_S^j)$. In particular, the divisor $-rK_X$ is very ample.

We define the coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{X \times \mathbb{A}^1}$ associated to $(X; -K_X; I_S)$ to be

$$\mathcal{I} := J_{r\tau(S)} + J_{r\tau(S)-1}\langle t\rangle + \cdots + J_1\langle t^{r\tau(S)-1}\rangle + \langle t^{r\tau(S)}\rangle \subset \mathcal{O}_{X \times \mathbb{A}^1},$$

where $J_j \subset \mathcal{O}_X$ is the image of $\mathrm{H}^0(X; \mathcal{O}_X(-rK_X) \cdot I_S^j) \otimes \mathcal{O}_X(rK_X) \to \mathcal{O}_X$.

We denote by $\Pi : \mathcal{X} \to X \times \mathbb{A}^1$ the blow-up of $X \times \mathbb{A}^1$ along $\mathcal{I} \subset \mathcal{O}_{X \times \mathbb{A}^1}$ and, if $E_{\mathcal{X}} \subset \mathcal{X}$ is the Cartier divisor defined by $\mathcal{O}_{\mathcal{X}}(-E_{\mathcal{X}}) = \mathcal{I} \cdot \mathcal{O}_{\mathcal{X}}$, we set $\mathcal{L} := \Pi^*\mathrm{pr}_1^*(-rK_X) \otimes \mathcal{O}_{\mathcal{X}}(-E_{\mathcal{X}})$.

The following result by K. Fujita in [Fuj16] can be seen as a more precise version of the valuative criterion of K-stability in this particular situation.

Theorem IV.3.4 ([Fuj16, Theorem 5.1])**.** *With the above notation and assumptions,* $(\mathcal{X}; \mathcal{L}) \to \mathbb{A}^1$ *is a test configuration of* $(X; -rK_X)$*, called the* **basic test configuration** *of* $(X; -rK_X)$ *via* S*, and we have*

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) = \frac{r^{2n}(-K_X)^n}{2 \cdot (n!)^2} \cdot \beta(S)$$

$$\text{where} \quad \beta(S) = n\sum_{i=1}^{m}\int_{\lambda_{i-1}}^{\lambda_i}(d-x)((-K_{X_i}+(d-1-x)E_i)^{n-1}\cdot E_i)\,\mathrm{d}x.$$

Moreover, $\beta(S) = (-K_X)^n(A_X(E_S) - S_X(E_S)) \overset{\text{def}}{=} (-K_X)^n \beta_X(E_S)$.

A first thing that we will need to carry out the computation will be the Chern classes of the conormal bundle of the smooth quadric $S \subset V_{16}$ appearing in Theorem IV.2.7.

Lemma IV.3.5. *Let* $V = V_{16} \subset \mathbb{P}^{11}$ *be a prime Fano-Mukai* 4*-fold of genus* $g = 9$ *and let* $S \subset \mathbb{P}^3 \subset \mathbb{P}^{11}$ *be a smooth quadric such that* $S \subset V$ *and* $c_2(V)\cdot S = 5$*. Then*

$$c_1(N^{\vee}_{S/V}) = \mathcal{O}_S \quad\text{and}\quad c_2(N^{\vee}_{S/V}) = 1.$$

Proof. We know that $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\omega^{\vee}_S = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$. By adjunction we have that

$$\mathcal{O}_V(-K_V)|_S = \mathcal{O}_S(-K_S) \otimes \det(N_{S/V})$$

and $\mathcal{O}_V(-K_V) = (\mathcal{L})^{\otimes 2}$ where $\operatorname{Pic}(V) = \mathbb{Z}[\mathcal{L}]$. We can write

$$\mathcal{O}_V(\mathcal{L})|_S = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b) \text{ for some } a,b \geq 1 \text{ by ampleness,}$$

and since $S \hookrightarrow \mathbb{P}^3$ we deduce $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)) = 4$. and the Künneth formula implies $a = b = 1$. It follows that $c_1(N^{\vee}_{S/V}) \overset{\text{def}}{=} \det(N^{\vee}_{S/V}) = \mathcal{O}_S$.

Taking total Chern classes in the exact sequence

$$0 \longrightarrow T_S \longrightarrow T_V|_S \longrightarrow N_{S/V} \longrightarrow 0$$

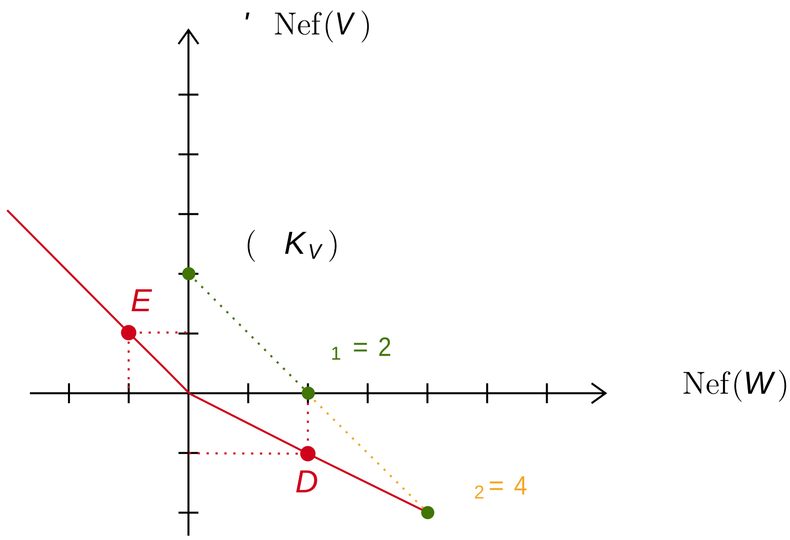
we obtain that

$$c_2(V)\cdot S = c_2(S) + c_1(S)\cdot\underbrace{c_1(N_{S/V})}_{=0} + c_2(N_{S/V})$$

and we know $c_2(S) = \chi_{\text{top}}(\mathbb{P}^1\times\mathbb{P}^1) = \chi_{\text{top}}(\mathbb{P}^1)^2 = 4$. □

Proposition IV.3.6. *With the notation of IV.2.7, we have* $\beta_V(E) = \frac{7}{20}$.

Proof. The following picture



shows that the ample model sequence of $(V; \quad K_V)$ corresponds to $X_1 = \widetilde{W}$ with $_1 = 2$, and $X_2 = W$ with $_2 = 4$. According to Theorem IV.3.4 this implies that we have the following formula for the -invariant:

$$
_V(E) = \frac{4}{(\quad K_V)^4}\Bigg(\underbrace{\int_0^2 (2 \quad t)(\quad K_{\hat{W}} + (1 \quad t)E)^3 \quad E\,dt}_{=:I_1}
+ \underbrace{\int_2^4 (2 \quad t)(\quad K_W + (1 \quad t) \quad E)^3 \quad E\,dt}_{=:I_2}\Bigg)
$$

We already know $(\quad K_V)^4 = 16\deg(V) = 256$, and in the following we will calculate the intersection number $(\quad K_{\hat{W}} + (1 \quad t)E)^3 \quad E$. To do this we have a list of observations.

1. If $\quad : E = \mathrm{P}(N_{S=V}) \;!\; S$ is the natural projection, we have that

$$
O_E(K_E) = \quad \mathrm{rk}(N_{S=V})O_E(1) + \quad (O_S(K_S) \quad \det(N_{S=V}))
$$
$$
) \; K_E = \quad 2 \;_E + \quad (K_S):
$$

and by adjunction

$$
K_E = K_{\hat{W}}j_E + Ej_E \quad) \quad (\quad K_{\hat{W}})j_E = \quad _E \quad K_E:
$$

Then

$$
O_{\hat{W}}(\quad K_{\hat{W}} + (1 \quad t)E)j_E = O_E(x \;_E \quad (K_S))
$$

2. By definition of Chern classes (see [Har77, §A.3]) it is verified that

$$
\begin{aligned}
{}_{E}^{2} \quad c_1(N_{\bar{S}/V}) \quad {}_{E} + \quad c_2(N_{\bar{S}/V}) = 0 \\
\Rightarrow \ {}_{E}^{2} = \quad c_2(N_{\bar{S}/V}) = \quad f
\end{aligned}
$$

where f corresponds to the class of a fiber of (here we use the previous lemma which says $c_2(N_{\bar{S}/V}) = 1$), and then ${}_{E}^{3} = \quad {}_{E} \quad f = \quad 1$.

3. By projection formula

$$
{}_{E}^{2} \quad (K_S) = \quad (f\{\mathrm{pt}\}) \quad (K_S) = f\{\mathrm{pt}\} \quad K_S = 0.
$$

4. Since S is a del Pezzo surface of degree 6

$$
{}_{E} \quad (K_S)^2 = {}_{E} \quad (K_S^2) = 8 \ {}_{E} \quad f = 8
$$

Using the previous calculations we can compute

$$
\begin{aligned}
I_1 &= \int_0^2 (2 \quad x)(\quad K_{\tilde{W}} + (1 \quad x)E)^3 \quad E\,dx \\
&= \int_0^2 (2 \quad x)(x \ {}_{E} \quad K_S)^3 dx \\
&= \int_0^2 (2 \quad x)(x^3 \ {}_{E}^{3} \quad 3x^2 \ {}_{E}^{2} \quad (K_S) + 3x \ {}_{E} \quad (K_S)^2 \quad (K_S)^3) dx \\
&= \int_0^2 (2 \quad x)(\quad x^3 + 24x) dx = \frac{152}{5}
\end{aligned}
$$

Besides, since (E) is a hyperplane section of W, we have $E = H$ and we reach

$$
\begin{aligned}
I_2 &= \int_2^4 (2 \quad x)(\quad K_W + (1 \quad x) \quad E)^3 \quad E\,dx \\
&= \int_2^4 (2 \quad x)(4 \quad x)^3 H^4 dx = \quad 8
\end{aligned}
$$

and finally we obtain

$$
{}_{V}(E) = \frac{4}{256}\left(\frac{152}{5} \quad 8\right) = \frac{7}{20} > 0
$$

□

Remark IV.3.7. The above calculation does not permit to decide about the K-stability of V. Therefore, we will require more precise tools such as Abban-Zhuang estimates, and moreover equivariant versions of such estimates.

IV.4 Automorphisms group of V_{16} and invariant subvarieties

In this section, we analyze the structure of the automorphism group of V_{16}, following closely the framework developed in [DM22]. This study leverages the explicit characterization of these varieties as linear sections of the Lagrangian Grassmannian, a topic we explored in detail earlier in this chapter.

Our interest on the automorphisms group is based in the following criterion.

Theorem IV.4.1 ([Zhu21, Theorem 1.1]). *Let X be a klt Fano variety and $G \subseteq \mathrm{Aut}(X)$ a reductive subgroup. Then X is*

1. *K-semistable if and only if $\beta(F) \geq 0$ for every G-invariant dreamy divisor F over X.*

2. *K-polystable if $\beta(F) > 0$ for every G-invariant dreamy divisor F over X.*

In view of the previous theorem, a very useful approach to prove K-semistability is to determine all the G-invariant subvarieties under a suitable subgroup of automorphisms, and then to compute their associated invariants (see for example the approach used in [CS23]). In the case of V_{16} we have to look for invariant subvarieties of dimension $0, 1, 2, 3$. Neverthless, the following theorem eliminates the case of invariant divisors.

Theorem IV.4.2 ([Fuj16, Corollary 9.3]). *Let X be a smooth Fano variety of $\rho(X) = 1$ such that $X \not\cong \mathbb{P}^n$. Then X is divisorially stable, i.e., $\beta(E) > 0$ for all divisor E on X.*

In the following, we explicitly describe the automorphisms group of V_{16} and we prove that its action has no fixed points. We finish with some comments about invariant curves in V_{16}.

For the remainder of this section, we adopt the notation established in §IV.2.1. Specially, we fix $V_{14} = \wedge^{\langle 3\rangle}\mathbb{C}^6, G = \mathrm{Sp}_6(\mathbb{C})$, $L \subset V_{14}$ as a general codimension 2 linear subspace, and $V = \mathbf{LG}(3,6) \cap \mathbb{P}(L)$ as a general Mukai fourfold of $g(V) = 9$. he following theorem, central to our analysis, establishes a precise relationship between isomorphisms of these varieties and elements of the group G.

Theorem IV.4.3 ([DM22, Proposition 4]). *Let $V = \mathbf{LG}(3,6) \cap \mathbb{P}(L), V' = \mathbf{LG}(3,6) \cap \mathbb{P}(L')$ be two Fano-Mukai fourfolds of genus 9, and suppose $\varphi : V \to V'$ is an isomorphism. Then there exists $g \in G$ such that $g(L) = L'$ and $\varphi = g|_V$.*

This theorem reduces the problem to study automorphisms of V to a problem at the level of linear algebra, because it is sufficient to look for symplectic matrices that stabilizes the subspace L. Using this result as the starting point, it is possible to find the automorphisms group of V. The strategy adopted in [DM22] consists of the following main steps.

1. Note that if $g \in G$ stabilizes L, and $g = g_s g_n$ is its Jordan decomposition (i.e., g_s is semisimple and $g_n$is unipotent) then g_s, g_n both preserves L. In conclusion, it is enough to analyze these two types of elements separately.

2. Prove there is no unipotent elements preserving L.

3. A semisimple element has a positive dimensional family of stable subspaces if and only if has eigenvalues of mutiplicity greater than 1. Dividing semisimple elements in terms of degenerations of its eigenvalues, a dimension count permits to detect elements that stabilized a general linear subspace of V_{16}.

Using these ideas systematically the complete description of the automorphisms can be given. We summarize this description in the following theorem.

Theorem IV.4.4 ([DM22])**.** *For a general codimension* 2 *linear subspace* $L \quad V_{14}$ *there exists*

(I) *a unique triple* (A_1, A_2, A_3) *of non-dgenerate, pairwise orthogonal planes in* $\mathbb{C}^6$*, such that* $L^\perp \quad A_1 \quad A_2 \quad A_3$*. Moreover, the involutions* $\quad \mathrm{Id}_{A_1} \quad \mathrm{Id}_{A_2} \quad \mathrm{Id}_{A_3}$.

(II) *twelve pairs* (E, F) *of transverse Lagragian subspaces of* $\mathbb{C}_6$ *such that* E *and* F *boh meets each* A_i *non trivially and* $L^\perp$ *intersects non trivially* $\wedge^3 E \quad (E \quad \wedge^2 F)$ *and* $(\wedge^2 E \quad F) \quad \wedge^3 F$*. These pairs of subspaces defines twelve anti-involutions of the form* $i(\mathrm{Id}_E \quad \mathrm{Id}_F)$.

Type (I) *involutions defines* 3 *different involutions in* $\mathrm{PSp}_6(\mathbb{C})$ *stabilizing* L*, and type* (II) *anti-involutions defines* 9 *different involutions in* $\mathrm{PSp}_6(\mathbb{C})$*. Moreover, these constitute all the automorphisms of* V*, and they are mutually commutative. In conclusion, the group of automorphisms of* V *corresponds to*

$$\mathrm{Aut}(V) = (\mathbb{Z}/2\mathbb{Z})^4 \quad \mathrm{PSp}_6(\mathbb{C}).$$

Another interesting description that is given in [DM22] are the fixed loci of each type of involution.

Theorem IV.4.5 ([DM22, Proposition 18]). *The fixed locus in V of a type I involution is a del Pezzo surface of degree 4. The fixed locus of type II involution is a disjoint union of two Veronese surfaces.*

We are particularly interested in studying the invariant subvarieties of V_{16} under the action of its automorphism group, and in this direction, we may begin by writting down all the automorphisms explicitly in coordinates.

More precisely, for a basis $\mathbf{C}^6 = he_1; e_2; e_3; e\ _3; e\ _2; e\ _1 i$, the six vectors

$$g\ _1 = e\ _1 \wedge (e_2 \wedge e\ _2 \quad e_3 \wedge e\ _3); \quad g\ _2 = e\ _2 \wedge (e_3 \wedge e\ _3 \quad e_1 \wedge e\ _1)$$
$$g\ _3 = e\ _3 \wedge (e_1 \wedge e\ _1 \quad e_2 \wedge e\ _2)$$

joint with the eight vectors $f \qquad = e\ _1 \wedge e\ _2 \wedge e\ _3$ forms a basis of V_{14}.

Modulo change of coordinates, we can suppose $L^? = hf_{+++}; f \qquad i$ in order to give the explicit list of involutions. In this case the Type I involutions corresponds to

$$_1 = \mathrm{Id}_{A_1} \quad \mathrm{Id}_{A_2} \quad \mathrm{Id}_{A_3}; \qquad _2 = \mathrm{Id}_{A_2} \quad \mathrm{Id}_{A_3} \quad \mathrm{Id}_{A_1}; \qquad _3 = \mathrm{Id}_{A_3} \quad \mathrm{Id}_{A_1} \quad \mathrm{Id}_{A_2};$$

where $A_i = he_i; e\ _i i$, and it verifies the relation $\ _3 = \ _2 \ \ _1$. Besides, we have three fundamental type II involutions given by

$$s = i(\mathrm{Id}_E \quad \mathrm{Id}_F); \quad t = i(\mathrm{Id}_{E^0} \quad \mathrm{Id}_{F^0}); \quad u = st = i(\mathrm{Id}_{E^{00}} \quad \mathrm{Id}_{F^{00}})$$

for the following pairs of Lagragian subspaces

$$(E; F) = (he_1; e_2; e_3 i; he\ _3; e\ _2; e\ _1 i)$$
$$(E^0; F^0) = (he_1 + ie\ _1; e_2 + ie\ _2; e_3 + ie\ _3 i; he_1 \quad ie\ _1; e_2 \quad ie\ _2; e_3 \quad ie\ _3 i)$$
$$(E^{00}; F^{00}) = (he_1 + e\ _1; e_2 + e\ _2; e_3 + e\ _3 i; he_1 \quad e\ _1; e_2 \quad e\ _2; e_3 \quad e\ _3 i)$$

The remaining 9 type II involutions arise from composing the involutions $s; t; u$ with the $\ _i$ involutions. The action of each $\ _i$ merely permuts the eigenvectors of $s; t; u$. In particular, we have the identities

$$t(f \qquad) = (\quad 1)^{"} f \qquad \text{and} \quad t(g\ _k) = \quad g\ _k.$$

where $"$ corresponds to the number of positive signs in the subscript of $f \qquad$.

The previous description of automorphisms allows us to conclude the following.

Proposition IV.4.6. *There are no fixed points in V_{16} under the action of* $\mathrm{Aut}(V_{16})$.

Proof. Note that a fixed point of V_{16} corresponds to an invariant line in the vector space $\bigwedge^{\langle 3 \rangle}(V)$, following the notation used in §IV.2.1.. The linear isomorphisms s, t are anti-involutions of the space $\wedge^{\langle 3 \rangle}\mathbb{C}^6$, so their uniques eigenvalues are $\pm i$. The conclusion follows verifying that the conditions $s(v) = \lambda v$ and $t(v) = \mu v$ for $\lambda, \mu = \pm i$ implies $v = 0$. $\square$

Remark IV.4.7. The proof of the previous proposition shows that it is enough to consider a subgroup $G = (\mathbb{Z}/2\mathbb{Z})^2 \subset \mathrm{Aut}(V)$ in order to have an action without fixed points.

Now, following the idea commented at the beginning of this section, the next step would be to analyze G-invariant curves. So, a first approach would be to bound the degree of destabilizing curves (e.g. [Fuj23b]).

An alternative approach would be trying to prove there is no destabilizing curves in V. The idea is as follows. Suppose there is a G-invariant divisor E over V such that its center $C := c_V(E)$ is a curve and $\beta_V(E) \leq 0$. Denote by $\eta \in V$ the generic point of C. By [Fuj23a, Proposition 2.9], we assume $\delta_{G,\eta}(V) < \frac{4}{5}$, where

$$\delta_{G,\eta}(V) = \sup\left\{ \lambda \in \mathbb{Q} \,\middle|\, \begin{array}{l} (V, \lambda\Delta) \text{ is lc at } \eta \text{ for every effective} \\ \mathbb{Q}\text{-divisor } \Delta \text{ on } V \text{ such that } \Delta \sim_{\mathbb{Q}} (-K_V) \end{array} \right\}$$

This implies that there exists $\lambda \in [\frac{4}{5}, \frac{5}{6}) \cap \mathbb{Q}$, an effective $\mathbb{Q}$-divisor and a G-invariant $\Delta \sim_{\mathbb{Q}} (-K_V)$ such that $(V, \lambda\Delta)$ is lc but not klt at the point $\eta \in V$. By construction, there exists an effective, irreducible, and G-invariant $\mathbb{Z}$-divisor D_0 such that

$$C \subset D_0 \subset \mathrm{Nklt}(V, \lambda D),$$

where $\mathrm{Nklt}(V, \lambda D)$ denotes the locus of points where the pair $(V, \lambda D)$ is not klt. Moreover, we have $D_0 \leq \lambda D$, and since $\lambda < \frac{5}{6}$, we observe that

$$D_0 \leq \lambda D \leq \frac{5}{6}D \sim \frac{5}{3}H.$$

Since $\mathrm{Pic}(V) = \mathbb{Z}[H]$, it necessarily follows that $D_0 \sim H$. We then obtain the existence of a G-invariant hyperplane section of V which contains C.

This section is also a prime Fano variety of genus 9 and degree 16, and a general variety of this type does not have non-trivial automorphisms (Dedieu-Manivel). However, since the action of $\mathrm{Aut}(V)$ has no fixed points, restricting the automorphisms we obtain at least one non-trivial automorphism in the hyperplane section. Thus, it remains to answer if there exists such threefolds with non-trivial automorphisms.

Conclusion

In this work, we have studied many techniques from different authors and papers that together build the foundation of K-stability theory as presented here. The main concepts in this field were collected and explained with various examples to show their meaning and applications.

After exploring and summarizing these techniques, the goal outlined in Chapter IV was to prepare the groundwork for proving that a general Fano-Mukai fourfold of genus 9 is K-stable. To this end, we applied methods similar to those in [Fuj17], where the idea of *Geography of Models* was used to show that a general del Pezzo fourfold of degree 5 is K-unstable. This was done by finding a divisor associated with a Sarkisov link, whose beta invariant turned out to be negative.

In Chapter II, we studied del Pezzo fourfolds of degree 5 in detail, and in Chapter IV, we observed a Sarkisov link connecting these varieties to Fano-Mukai fourfolds of genus 9. Using the approach in [Fuj17], we computed the beta invariant of the divisor linked to V_{16} and W_5. The result was positive, meaning this calculation alone does not prove the K-stability of V_{16}.

Because of this, we need more advanced strategies to reach a conclusion about V_{16}. To help with this, the final part of Chapter IV includes a summary of [DM22], which describes the automorphism group of V_{16} and its fixed subvarieties. This is important because, as noted in [ACC+23], it is often enough to check the numerical conditions for divisors that are invariant under the automorphism group to prove K-stability.

For V_{16}, this means we need to study whether there are fixed points, curves, surfaces, or divisors under the automorphism group. Additionally, as it is mentioned in §IV.4, [Fuj16, Corollary 9.3] shows that any smooth $\mathbb{Q}$-Fano variety, different than the projective space, with Picard number $(X) = 1$ is *divisorially stable*, so it is enough to focus on points, curves, and surfaces.

A useful tool for limiting the possibilities of curves in a Fano variety is [Fuj23b, Corollary 4.2], which gives a bound on the degree of a curve. The next step would be to use this bound to identify invariant curves explicitly and perform detailed calculations. For surfaces, a similar approach could be taken by generalizing this result to subvarieties of higher dimension.

In summary, a list of the next steps for this work, as well as future projects, is presented below.

- Complete the case of G-invariant curves by investigating the automorphisms of hyperplane sections (family No. 1.8 of Fano threefolds).
- Describe G-invariant surfaces in V_{16}.
  Idea: describe two involutions in W_5 such that the PZ link is equivariant.
- **Alternative approach:** determine an efficient subvariety to apply the Abban-Zhuang adjunction method.
  Idea: perform the blow-up of a line (in the work [Han10], the Hilbert scheme of lines in V_{16} is characterized as a divisor of degree $(1,1,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).
- Describe the moduli space of V_{16} varieties and identify special members. Determine the K-stability of these special varieties.

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