TOWARD POTENTIAL DENSITY OF RATIONAL POINTS ON ELLIPTIC K3 SURFACES

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Abstract

Let X be a smooth projective variety over a number field K. We say rational points on X are *potentially dense* if $X(K')$ is Zariski dense for some finite field extension K' of K. The study of potential density is an important problem in arithmetic geometry. Common strategies found throughout the literature include studying algebraic varieties with large automorphism groups, elliptic or abelian fibrations, or certain conic bundles [HT01]. In this work we provide a brief introduction to the theory of *elliptic* and *K3 surfaces,* and describe in detail techniques originally by Bogomolov and Tschinkel [BT00] used to propagate rational points on K3 surfaces that admit elliptic fibrations.

Resumen

Sea X una variedad proyectiva suave definida sobre un cuerpo de números K. Decimos que los puntos racionales en X son potencialmente densos *si $X(K')$ es Zariski denso para cierta extensión de cuerpos finita K' de K. El estudio de la densidad potencial es un problema importante en geometría aritmética. Algunas estrategias comunes en la literatura son estudiar variedades algebraicas con grupos de automorfismos grandes, con fibraciones elípticas o abelianas, o con ciertos fibrados cónicos [HT01]. En este trabajo proveemos una breve introducción a la teoría de las* superficies elípticas *y* K3, *y describimos en detalle técnicas originalmente de Bogomolov y Tschinkel [BT00] usadas para propagar puntos racionales en superficies K3 que admiten fibraciones elípticas.*

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Introduction

An *elliptic surface* is a smooth projective surface S that admits a fibration such that the general fiber is a smooth projective curve of genus one. There is a natural way to associate a *Jacobian* elliptic surface $J(S)$ to S, and then the general fiber is an elliptic curve [Huy16]. Elliptic surfaces form a very important class of algebraic surfaces which possess a rich geometric and arithmetic structure. For example, their Mordell-Weil group may be studied from a lattice-theoretic point of view, and so they lend themselves to a very natural definition of *height* [SS10]. A subclass of special relevance is that of elliptic K3 surfaces, whose key distinctive properties are that they are not abelian and that the same surface may admit more than one distinct Jacobian elliptic fibration [SS10].

A *K3 surface* (the name was given in honour of Kähler, Kummer, and Kodaira) is a smooth projective surface S whose canonical bundle ω_S is trivial and $h^1(S, \mathcal{O}_S) = 0$. Complex K3 surfaces are *kähler*, simply connected, and all of them diffeomorphic and deformation equivalent to a quartic in $\mathbb{P}^3$ [Huy16]. Their second integral cohomology group has the structure of an even unimodular lattice. Furthermore, they admit very rich Hodge and moduli theories (cf. local and global Torelli theorems).

There are not many results in the current literature concerning rational points on K3 surfaces [Hua21], although results by Bogomolov, Hassett and Tschinkel from the beginning of the present century ascertain *potential density* in the case of elliptic K3 surfaces [BT00, HT01]. More generally, these authors describe techniques which may be used to propagate rational points on a large class of algebraic varieties. Other recent results include necessary and sufficient conditions for the density of rational points on the family of *surfaces of Cassels-Schinzel type*, whose elements arise as twisted Kummer surfaces associated to the product of two twisted elliptic curves [Hua21].

This work has two main objectives.

- The first is to provide a brief and systematic introduction to the theory of elliptic and K3 surfaces. The aim is to be as self-contained as reasonably possible, with an orientation toward the second objective, and somewhat light on proofs so as to not distract the reader. The only prerrequisite is a first graduate course in algebraic geometry, e. g. to the level of [Har77], Chapters I to III, although we will mostly deal with algebraic varieties, which we will define as separated geometrically integral schemes of finite type over a field. *This is the content of Chapters II and III.*

- A secondary objective is to provide basic preliminaries on lattices, surfaces, and Hodge theory. *This is the content of Chapter I.*
- Another secondary objective is to provide a brief introduction to rational and integral points, along with a short survey of the current state of the art. *This is the content of Chapter IV.*

- The second is to expand on the ideas originally in the article *Density of Rational Points on Elliptic K3 Surfaces*, by Bogomolov and Tschinkel, here [BT00], wherein they prove that if a K3 surface admits an elliptic fibration or has an infinite group of automorphisms, then it has potentially dense rational points, here Theorem 4.1.23. This allows them to characterise potential density of rational points on K3 surfaces in terms of their Picard number and the existence of a -2-curve (i. e., a curve of self-intersection equal to -2). We will closely follow their arguments and use the language we will have acquired in the previous chapters to give more complete explanations. *This is the content of Chapter VI.*

Chapter 1

Preliminaries

In this chapter we provide basic preliminaries on lattices (Section 1.1), surfaces (1.2), and Hodge theory (1.3).

In Section 1.1, we follow Debarre's notes *Hyperkähler Manifolds* [Deb22], Kondō's book *K3 Surfaces* [Kon20], and Shimada's talk *K3 Surfaces and Lattice Theory* [Shi14]. We try to mention synonyms found throughout the literature, and to quickly state the equivalence between bilinear and quadratic forms.

In Section 1.2, we mostly follow the first pages of Chapter V of Hartshorne's classic book *Algebraic Geometry* [Har77]. We elaborate on a remark on Bogomolov and Tschinkel's article *Density of Rational Points on Elliptic K3 Surfaces* [BT00].

In Section 1.3, we mostly follow Huybrechts's book *Lectures on K3 Surfaces* [Huy16] and Makarova's article *General Introduction to K3 Surfaces* [Mak16]. We give a short proof of the famous Lefschetz $(1,1)$ theorem found in Griffiths and Harris's book *Principles of Algebraic Geometry* [GH94], and state the Hodge-theoretic version of the Torelli theorem for curves.

1.1 Lattices

Definition 1.1.1

A **lattice** is a pair $(L, \langle\cdot,\cdot\rangle)$, where $L \cong \mathbb{Z}^r$ is a free abelian group of finite rank r, and $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}$ is an integral-valued form such that:

1. For all $x, y \in L$, $\langle x, y\rangle = \langle y, x\rangle$. *(Symmetry.)*
2. For all $x, y, z \in L, m, n \in \mathbb{Z}$, $\langle mx + ny, z\rangle = m\langle x, z\rangle + n\langle y, z\rangle$. *($\mathbb{Z}$-bilinearity.)*
3. For all $x \in L$, $\langle x, y\rangle = 0$ for all $y \in L$ implies $x = 0$. *(Non-degeneracy.)*

Equivalently, a lattice is a pair (L, q), where $q : L \to \mathbb{Z}$ is an non-degenerate integral-valued *quadratic form* (i. e., $q(nx) = n^2 q(x)$ for all $x \in L, n \in \mathbb{Z}$, the map $L \times L \to \mathbb{Z}, (x, y) \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$-bilinear, and the discriminant of q, which we will not presently define, is non-zero).

Definition 1.1.2

Let $(L, \langle\cdot,\cdot\rangle_L), (L', \langle\cdot,\cdot\rangle_{L'})$ be lattices such that $L' \subset L$. We say L' is a **sublattice** of L if both bilinear forms are compatible (i. e., if $\langle\cdot,\cdot\rangle_{L'} = \langle\cdot,\cdot\rangle_L|_{(L'\times L')}$). In this situation, we also say L is a **superlattice** of L'.

Definition 1.1.3

Let $(L, \langle\cdot,\cdot\rangle_L), (L', \langle\cdot,\cdot\rangle_{L'})$ be lattices. Then:

1. A **lattice homomorphism** or **embedding** is a group homomorphism $\varphi : L \to L'$ such that for all $x, y \in L$, $\langle x, y\rangle_L = \langle\varphi(x), \varphi(y)\rangle_{L'}$. In this situation, $\varphi(L')$ is a sublattice of L. An embedding φ is **primitive** if the quotient group $L/\varphi(L')$ is torsion-free.
2. An **isomorphism** or **isometry** is a homomorphism that is also a group isomorphism. An **automorphism** is an isomorphism from a lattice to itself.
3. The **orthogonal group** $O(L)$ is the group of automorphisms of L.

Remark 1.1.4

If a lattice is given by its bilinear form, we may obtain its quadratic form by defining

$$q(x) := \langle x, x \rangle.$$

Inversely, if a lattice is given by its quadratic form, we may recover its bilinear form via

$$\langle x, y \rangle := \frac{1}{2}(q(x+y) - q(x) - q(y)), \quad \text{(cf. “ } (x+y)^2 = x^2 + 2xy + y^2. \text{ ”)}.$$

Notation 1.1.5

We may write L instead of $(L, \langle \cdot, \cdot \rangle)$ or (L, q) and use both definitions as necessary.

Remark 1.1.6

We may define orthogonal direct sums of lattices as usual. If $L' \subset L$ is a sublattice, the orthogonal complement of L'

$$L'^{\perp} := \{x \in L \quad | \quad \forall y \in L', \langle x, y \rangle = 0\}$$

does *not* generally give an orthogonal direct sum decomposition of L. In any case, $L' \oplus L'^{\perp} \subset L$ is a sublattice of finite index.

Definition 1.1.7

Let L be a lattice, and let $\{e_i\}_{1 \le i \le r}$ be a basis of L as a free abelian group. Then:

1. The **Gram matrix** of L is the symmetric matrix

   $$A := (a_{ij})_{1 \le i,j \le r} := (\langle e_i, e_j \rangle)_{1 \le i,j \le r}.$$

   The determinant
   $$d(L) := \det(A)$$
   is well-defined (i. e., it is independent of the choice of basis of L).

2. L is **even** if all diagonal entries of A are even. Equivalently, if for all $x \in L$, $\langle x, x \rangle$ is even. L is **odd** if it is not even.

3. L is a **negative-definite root lattice** if all diagonal entries of A are equal to -2.

Remark 1.1.8

Root lattices are of particular importance in algebraic geometry; they arise naturally in the context of elliptic and K3 surfaces. They are also in correspondence with a class of graphs called *Dynkin diagrams* (cf. [Kon20], [SS10]).

Definition 1.1.9

Let L be a lattice, let $\{e_i\}_{1\leq i\leq r}$ be a basis of L as a free abelian group, and let $x = \sum_{i=1}^{r} x_i e_i \in L, x_i \in \mathbb{Z}$. Write $q(x) = xAx^t$. By Sylvester's law of inertia, we may diagonalise A and obtain p positive and q negative (integer) eigenvalues in the diagonal. Then:

1. The **signature** of L is the pair (p, q). The **index** of L is the integer $p - q$.
2. L is **positive definite** (resp., **negative definite**) if $(p, q) = (r, 0)$ (resp., $(p, q) = (0, r)$).
3. L is **definite** if it is positive or negative definite, and **indefinite** if it is not.
4. L is **hyperbolic** if $(p, q) = (1, r - 1)$.

Definition 1.1.10

Let L be a lattice. The **dual lattice** is the lattice

$$L^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} \quad | \quad \forall y \in L, \langle x, y\rangle \in \mathbb{Z}\} \subset L \otimes \mathbb{Q},$$

where we extend $\langle \cdot, \cdot\rangle$ to $L \otimes \mathbb{Q}$ bilinearly.

Proposition 1.1.11

There is a natural inclusion $i : L \hookrightarrow L^{\vee}, x \mapsto \langle x, \cdot\rangle$.

Proof. The map is injective as a consequence of the non-degeneracy. $\square$

Definition 1.1.12

Let L be a lattice. Then:

1. The **discriminant group** is the quotient group
$$D(L) := L^\vee/L,$$
where $L \subset L^\vee$ is the natural injection.

2. If L is even, the **discriminant quadratic form** is the induced map
$$\hat{q} : D(L) \to \mathbb{Q}/2\mathbb{Z}, \quad x + L \mapsto q(x) \mod 2\mathbb{Z},$$
where we extend $q(x) = \langle \cdot, \cdot \rangle$ to $L^\vee$ by using the extension of $\langle \cdot, \cdot \rangle$ to $L \otimes \mathbb{Q}$.

3. Let $O(D(L), \hat{q})$ be the group of automorphisms of the abelian group $D(L)$ that preserve $\hat{q}$ (i. e., $\varphi \in O(D(L), \hat{q})$ if and only if φ is an automorphism of the abelian group $D(L)$ and $\hat{q}(x) = \hat{q}(\varphi(x))$). It is easy to see that there is a canonical group homomorphism $f : O(L) \to O(D(L), \hat{q})$. The **stable orthogonal group** is the kernel
$$\hat{O}(L) := \ker(f) \subset O(L).$$

Definition 1.1.13

Let L be a lattice. L is **unimodular** if any of the following equivalent conditions are satisfied:

1. $d(L) = \pm 1$.

2. The natural inclusion $i : L \hookrightarrow L^\vee$ is an isomorphism.

3. $D(L) = \{0\}$ (in particular, $\hat{q} = 0$).

Remark 1.1.14

Indefinite odd and even unimodular lattices of a given signature are classified (modulo isomorphism). The proof is rather long, but not too difficult (cf. [Kon20] Theorem 1.22, 1.27). We will only use the particular case described in Important examples 1.1.15.

Important examples 1.1.15

1. The **hyperbolic plane** U is is the lattice defined by the Gram matrix
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
This lattice is even, unimodular, and of signature $(1,1)$.

2. The **negative-definite root lattice of type** E_8 E_8^- is the lattice defined by the Gram matrix
$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$
This lattice is even, unimodular, and of signature $(0,8)$.

3. The **K3 lattice** is the lattice
$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus (E_8^-)^{\oplus 2}.$$
This lattice is even, unimodular, and of signature $3(1,1) + 2(0,8) = (3,19)$. We will later state while studying *K3 surfaces* that, if X is such a surface, this lattice is surprisingly isomorphic to $H^2(X,\mathbb{Z})$ with the topological intersection form (cup product).

Definition 1.1.16

Let L be a lattice, and let $x \in L$. Then:

1. x is **isotropic** if $\langle x, x\rangle = 0$.

2. x is **primitive** if $x/m \in L$ and $m \in \mathbb{Z} \setminus \{0\}$ implies $m = \pm 1$.

The following result (the conclusion for the second item is by Meyer) will be very useful for determining when a K3 surface admits an elliptic fibration.

Lemma 1.1.17: ([Kon20], Proposition 1.23, 1.24.)

Let L be lattice such that:

1. L is indefinite and unimodular, or
2. L is indefinite and $\mathrm{rank}(L) \geq 5$.

Then, L has a non-zero isotropic element.

The following lemma is a consequence of a result often called Eichler's criterion (cf. [Deb22], Theorem 2.9) and will be used in the proof of a representation theorem for effective divisors on K3 surfaces (see Section 5.2).

Lemma 1.1.18: ([Kon20], Lemma 1.45.)

Let L be an even unimodular lattice such that L has an orthogonal decomposition $L = U^{\oplus 2} \oplus L'$. Then, for all primitive $x, y \in L$ such that $q(x) = q(y) = n \in \mathbb{Z}$, there exists $\varphi \in O(L)$ such that $\varphi(x) = y$. In other words, the orbit of a primitive element of L under the action of $O(L)$ is determined by its square.

1.2 Surfaces

Let K be an arbitrary field. An algebraic variety X over K is a separated geometrically integral scheme X of finite type over K. An algebraic surface is an algebraic variety of dimension 2.

Let X be a smooth algebraic variety. Recall that the Picard group $\mathrm{Pic}(X)$ is isomorphic to the group of (Cartier, Weil) divisors modulo linear equivalence, i. e.,

$$\mathrm{Pic}(X) \cong \mathrm{Div}(X)/\sim_{\mathrm{lin}} \cong \mathrm{Div}(X)/\,\mathrm{PDiv}(X).$$

We will define other equivalence relations on the group of divisors, namely algebraic equivalence $\sim_{\mathrm{alg}}$ and numerical equivalence $\equiv$, and their respective quotient groups. Then, it is well-known that

$$\text{linear equivalence} \implies \text{algebraic equivalence} \implies \text{numerical equivalence}.$$

Definition 1.2.1

We define **algebraic equivalence** $\sim_{\text{alg}}$ on the group of divisors step by step:

1. Let $C, D \in \operatorname{Div}(X)$ be effective. Then, $C \sim'_{\text{alg}} D$ if and only if there exist a smooth curve $T \subset X$, $E \subset X \times T$ effective and flat over T, and points $0, 1 \in T$ such that $E_0 \cong C$ and $E_1 \cong D$.
2. Let $C, D \in \operatorname{Div}(X)$ be arbitrary. Then, $C \sim'_{\text{alg}} D$ if and only if there exist $C_1, C_2, D_1, D_2 \in \operatorname{Div}(X)$ effective such that $C_1 \sim'_{\text{alg}} D_1, C_2 \sim'_{\text{alg}} D_2$, and $C = C_1 - C_2, D = D_1 - D_2$.
3. Finally, define $\sim_{\text{alg}}$ as the transitive closure of the relation $\sim'_{\text{alg}}$.

Definition 1.2.2

The **Néron-Severi group** $\operatorname{NS}(X)$ is the group of divisors modulo algebraic equivalence, i. e.,

$$\operatorname{NS}(X) \cong \operatorname{Div}(X)/\sim_{\text{alg}} \cong \operatorname{Div}(X)/\operatorname{Div}^0(X) \cong \operatorname{Pic}(X)/\sim_{\text{alg}} \cong \operatorname{Pic}(X)/\operatorname{Pic}^0(X),$$

where $\operatorname{Div}^0(X)$ (resp., $\operatorname{Pic}^0(X)$) is the group of divisors (resp., line bundles modulo isomorphism) algebraically equivalent to zero.

Theorem 1.2.3: (Néron, Severi. Theorem of the base.)

The Néron-Severi group $\operatorname{NS}(X)$ is a finitely generated abelian group (i. e., its rank is finite).

Definition 1.2.4

The **Picard number** $\rho(X)$ is the rank of $\operatorname{NS}(X)$.

Now, let X be a smooth projective surface.

Theorem 1.2.5: ([Har77], Chapter V, Theorem 1.1.)

There is a unique pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$, $(C, D) \mapsto C.D$, the **intersection form**, such that:

1. For all smooth $C, D \in \mathrm{Div}(X)$ meeting transversally, $C.D = \#(C \cap D)$.
2. For all $C, D \in \mathrm{Div}(X)$, $C.D = D.C$. *(Symmetry.)*
3. For all $C_1, C_2, D \in \mathrm{Div}(X)$, $(C_1 + C_2).D = C_1.D + C_2.D$. *(Additivity.)*
4. For all $C_1, C_2 \in \mathrm{Div}(X)$ such that $C_1 \sim_{\mathrm{lin}} C_2$, then $C_1.D = C_2.D$. In other words, the intersection form induces a pairing on $\mathrm{Pic}(X)$.

Definition 1.2.6

We define **numerical equivalence** $\equiv$ on the group of divisors. Let $C, D \in \mathrm{Div}(X)$. Then, $C \equiv D$ if and only if for every irreducible curve $C' \subset X$, $C.C' = D.C'$.

Definition 1.2.7

The **numerical Néron-Severi group** $\mathrm{Num}(X)$ is the group of divisors modulo numerical equivalence, i. e.,

$$\mathrm{Num}(X) \cong \mathrm{Div}(X)/\equiv \cong \mathrm{Div}(X)/K \cong \mathrm{Pic}(X)/\equiv \cong \mathrm{Pic}(X)/(K/\sim_{\mathrm{lin}}),$$

where K is the kernel of the intersection form on $\mathrm{Div}(X)$ and the group of divisors numerically equivalent to zero.

Remark 1.2.8

By definition, the group that comes next in the construction is a quotient of the previous. Thus, there are natural surjections

$$\mathrm{Pic}(X) \to \mathrm{NS}(X) \to \mathrm{Num}(X).$$

In particular, $\mathrm{Num}(X)$ is also a finitely generated abelian group.

We will state the Riemann-Roch theorem for surfaces (a consequence, of course, of the famous Hirzebruch-Riemann-Roch theorem, but we will not define Chern or Todd classes), Noether's formula, give a few properties of ample divisors on surfaces, state the Hodge index theorem, and then obtain the signature of the numerical Néron-Severi group $\mathrm{Num}(X)$.

Theorem 1.2.9: (Riemann-Roch theorem for surfaces.)

Let $\mathcal{L} \in \mathrm{Pic}(X)$. Then, in multiplicative notation, the theorem is

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \frac{\mathcal{L}.(\mathcal{L} \otimes \omega_X^\vee)}{2},$$

where $\chi(\cdot)$ is the Euler characteristic, and $\omega_X^\vee$ is the dual of the canonical bundle ω_X.

In additive notation, the theorem is

$$\chi(D) = \chi(0_X) + \frac{D.(D - K_X)}{2},$$

where $D, 0_X, K_X \in \mathrm{Div}(X)$ represent $\mathcal{L}, \mathcal{O}_X, \omega_X \in \mathrm{Pic}(X)$, respectively.

Theorem 1.2.10: (Noether's formula.)

Let $\chi(\mathcal{O}_X)$ (resp., $\chi_{\mathrm{top}}(X)$) be the holomorphic (resp., topological) Euler characteristic of X, and let ω_X be the canonical bundle of X. Then,

$$\chi(\mathcal{O}_X) = \frac{(\omega_X.\omega_X) + \chi_{\mathrm{top}}(X)}{12}.$$

Proposition 1.2.11: (Some properties of ample divisors on surfaces.)

Let $D \in \mathrm{Div}(X)$ be ample. Then:

1. If D is very ample and $C \subset X$ is a curve, then $D^2 = \deg_i(X) > 0$ and $C.D = \deg_i(C) > 0$, where $i : X \hookrightarrow \mathbb{P}^n$ is the embedding induced by D.

2. If $E \in \mathrm{Div}(X)$ is such that $D \equiv E$, then E is ample, but if additionally D is very ample, then E is not necessarily very ample.

3. If $E \in \mathrm{Div}(X)$ is such that $D.E > 0$ and $E^2 > 0$, then for all $n \gg 0$ there exists $F_n \in \mathrm{Div}(X)$ effective such that $nE \sim_{\mathrm{lin}} F_n$.

Proof.

1. [Har77], Chapter V, Exercise 1.2.

2. [Har77], Chapter V, Exercise 1.12.

3. [Har77], Chapter V, Corollary 1.8.

□

Theorem 1.2.12: (Nakai-Moishezon criterion.)

Let $D \in \mathrm{Div}(X)$. Then, D is ample if and only if $D^2 > 0$ and for every irreducible curve $C \subset X$, $C.D > 0$.

Proof. If D is ample, the conclusion follows by Proposition 1.2.11.1 applied to a multiple of D. The other direction is much harder (cf. [Har77], Chapter V, Theorem 1.10). □

Theorem 1.2.13: (Hodge index theorem. [Har77], Chapter V, Theorem 1.9.)

Let $D \in \mathrm{Div}(X)$ be ample, and let $E \in \mathrm{Div}(X)$ such that $E \not\equiv 0$ and $D.E = 0$. Then, $E^2 < 0$.

Corollary 1.2.14

The numerical Néron-Severi group $\mathrm{Num}(X)$ is a free abelian group and a lattice with the induced intersection form of signature $(1, r-1)$, where $r \leq \rho(X)$. In particular, the index of $\mathrm{Num}(X)$ is $2-r$.

Proof. It is easy to see that $\mathrm{Num}(X)$ is a lattice of rank $r \leq \rho(X)$. Fix an ample divisor D on X. It is possible to take a real multiple $tD \in \mathrm{Num}(X)$ such that $\mathrm{Num}(X) = tD \oplus D^{\perp}$. By Sylverster's law of inertia, we may diagonalise the Gram matrix of $\mathrm{Num}(X)$. One eigenvalue is t^2D^2 and the others are of the form E^2, where $E \in D^{\perp}$. The conclusion follows by the (easy direction of the) Nakai-Moishezon criterion and the Hodge index theorem. □

Corollary 1.2.15

If $\mathrm{NS}(X) \cong \mathrm{Num}(X)$, $r = \rho(X)$.

Finally, we talk about immersed and embedded curves on surfaces.

Definition 1.2.16

Let C be a smooth curve, and let $f : C \to X$ be a regular map of degree 1. Then:

1. f is an **immersion** if, for all $x \in C$, the differential
$$d_x f : T_x C \hookrightarrow T_{f(x)} X$$
is injective (i. e., non-zero).
2. f is an **embedding** if f is an immersion and $f(C)$ is smooth.
3. The **normal bundle** of C on X is the quotient vector bundle on C
$$N_{C/X} := (T_X|_C)/T_C.$$

Proposition 1.2.17: ([BT00], Remark 2.1.)

Let C be a smooth curve, and let $f : C \to X$ be an embedding. Then, the normal bundle $N_{C/X}$ of C on X is a line bundle on C. If $\omega_X \cong \mathcal{O}_X$ (e. g., if X is (locally) a K3 surface), then

$$N_{C/X} \cong \omega_C.$$

Proof. The rank of a quotient vector bundle is the difference of the two ranks. Thus,

$$\operatorname{rank}(N_{C/X}) = \operatorname{rank}(T_X|_C) - \operatorname{rank}(T_C) = \operatorname{rank}(T_X) - \operatorname{rank}(T_C) = 2 - 1 = 1,$$

and $N_{C/X}$ is a line bundle. If $\omega_X \cong \mathcal{O}_X$, then, by the adjunction formula,

$$\omega_C \cong \omega_X|_C \otimes \det(N_{C/X}) \cong \mathcal{O}_X|_C \otimes \det(N_{C/X}) \cong \mathcal{O}_C \otimes \det(N_{C/X}) \cong \det(N_{C/X}) \cong N_{C/X}.$$

□

Finally, let X be a smooth complex surface.

Proposition 1.2.18: ([BT00], Remark 2.1.)

Let C be a smooth curve, and let $f : C \to X$ be an embedding. Then, f is a local isomorphism. In other words, there exists an (abstract) local neighbourhood U of C such that $\dim_{\mathbb{C}}(U) = 2$ and f extends to a biholomorphic map $g : U \to g(U) \subset X$.

1.3 Hodge theory

Let V be a free abelian group of finite rank or a finite-dimensional $\mathbb{Q}$-vector space. Define $V_{\mathbb{R}}$ and $V_{\mathbb{C}}$ by scalar extension.

Definition 1.3.1

A **Hodge structure of weight** $k \in \mathbb{Z}$ on V is a direct sum decomposition as $\mathbb{C}$-vector spaces

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q},$$

where $V^{q,p} = \overline{V^{p,q}}$ and $\overline{\cdot}$ denotes complex conjugation.

Notation 1.3.2

We may write V instead of a Hodge structure on V if the context is clear.

Definition 1.3.3

Let $V' \subset V$ be a subgroup or subspace, and suppose we have Hodge structures of weight $k \in \mathbb{Z}$ on V and V'. We say V' is a **sub-Hodge structure** of V if both Hodge structures are compatible (i. e., if $V'^{p,q} = V'_{\mathbb{C}} \cap V^{p,q}$ for all $p + q = k$).

Remark 1.3.4

We may do linear algebra on Hodge structures by defining morphisms, direct sums, tensor products, duals, etc., in a similar way. If V, V' be Hodge structures of weight $k, k' \in \mathbb{Z}$ are such that $k \neq k'$, the definition of a morphism requires some care (cf. [Huy16], Chapter 3, iv. for more details).

Example 1.3.5: ([Huy16], Chapter 3, Example 1.3, 1.4.)

The *integral Tate Hodge structure* $\mathbb{Z}(1)$ is the Hodge structure of weight -2 such that $V = (2\pi i)\mathbb{Z} \subset \mathbb{C}$ and

$$V_{\mathbb{C}} = V^{-1,-1} = \mathbb{C}.$$

We define twists

$$\mathbb{Z}(k) := \mathbb{Z}(1)^{\otimes k}, k \in \mathbb{Z}^{+}, \quad \mathbb{Z}(-1) := \mathbb{Z}(1)^{\vee}, \quad \mathbb{Z}(k) := \mathbb{Z}^{\otimes -k}, k \in \mathbb{Z}^{-},$$

and $\mathbb{Z}(0)$ is the Hodge structure of weight 0 such that $V = \mathbb{Z} \subset \mathbb{C}$ and

$$V_{\mathbb{C}} = V^{0,0} = \mathbb{C}.$$

We also define *rational Tate Hodge structures* $\mathbb{Q}(k), k \in \mathbb{Z}$ analogously.

Important example 1.3.6

A Hodge structure V of weight 2 is **of K3 type** if $\dim_{\mathbb{C}}(V^{2,0}) = \dim_{\mathbb{C}}(V^{0,2}) = 1$, $\dim_{\mathbb{C}}(V^{1,1}) = 20$, and $\dim_{\mathbb{C}}(V^{p,q}) = 0$ for all other p, q such that $p + q = 2$. We will later state while studying K3 surfaces that, if X is such a surface, $H^2(X, \mathbb{Z})$ has a Hodge structure of weight 2 of K3 type.

Let X be a smooth projective variety over $\mathbb{C}$ or a compact Kähler manifold such that $\dim_{\mathbb{C}}(X) = n$.

Remark 1.3.7

Recall that we may identify the $\mathbb{C}$-vector space $H^{p,q}(X)$ of harmonic forms of type (p, q) on X and the Dolbeault cohomology $H^q(X, \Omega_X^p)$.

We will now talk about the most important example of a Hodge structure, the Hodge decomposition, and state some of its properties.

Theorem 1.3.8: (Hodge decomposition.)

Let $k \in \{0, \dots, n\}$. Then, there is a natural decomposition

$$H^k(X, \mathbb{Z}) \otimes \mathbb{C} = H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

of the torsion-free part of the k-th singular cohomology of X. This decomposition is a Hodge structure of weight k on $H^k(X, \mathbb{Z})$.

Remark 1.3.9

The Hodge decomposition is remarkable as the left hand side depends only on the topological structure of X, whereas the right hand side does depend on the complex structure of X (but not on the choice of Kähler metric). In other words, the numbers $\dim_{\mathbb{C}}(H^{p,q}(X))$ are "complex invariants" (in a certain sense) but not "topological invariants".

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & & \vdots & & & \\
h^{n,0} & & & \cdots & & & h^{0,n} \\
 & & & \vdots & & & \\
 & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & &
\end{array}$$

Table 1.1: The Hodge diamond of X.

Definition 1.3.10

The **Hodge number** $h^{p,q}(X)$ is the dimension of the $\mathbb{C}$-vector space $H^{p,q}(X)$. The **Hodge diamond** of X is an array as in Table 1.1.

Definition 1.3.11

Let $k \in \mathbb{Z}_0^+$. The k-th **Betti number** $b_k(X)$ is the rank of the abelian group $H_k(X)$, the k-th homology group of X.

Proposition 1.3.12: (Properties and symmetries of the Hodge diamond.)

The Hodge diamond of X has the following non-trivial properties and symmetries:

1. $b_k(X) = \sum_{p+q=k} h^{p,q}(X)$, for all $k \in \{0, \dots, 2n\}$.
2. $h^{k,k} \geq 1$, for all $k \in \{0, \dots, n\}$.
3. Hodge symmetry, i. e. $h^{p,q} = h^{q,p}(X)$. *(Reflection along the vertical axis.)*
4. By Serre duality, $h^{p,q}(X) = h^{n-p,n-q}(X)$. *(Point reflection through the middle.)*
5. As X is Hyperkähler, $h^{p,q} = h^{n-p,q}(X)$. *(Reflection along the horizontal axis.)*

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Table 1.2: The Hodge diamond of a smooth projective curve.

Example 1.3.13

Let C be a smooth projective curve of genus g. Then, its Hodge diamond is as in Table 1.2.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Table 1.3: The Hodge diamond of a K3 surface.

Important example 1.3.14

We will later state while studying K3 surfaces that, if S is such a surface, its Hodge diamond is as in Table 1.3.

We will now set the stage to state the famous Hodge conjecture and Lefschetz $(1,1)$ theorem.

Definition 1.3.15

Let V be a Hodge structure of even weight $2k, k \in \mathbb{Z}^+$. The **Hodge classes** in V are the elements of $V \cap V^{k,k}$, where $V \subset V_{\mathbb{C}}$ is the natural inclusion.

Definition 1.3.16

An **algebraic cycle** Y on X is a finite formal linear combination

$$Y = \sum_i n_i Y_i$$

of subvarieties (in the algebraic case) or submanifolds (in the complex case) Y_i with integral or rational coefficients.

Lemma 1.3.17

Let Y be an algebraic cycle on X of codimension k with coefficients in $R = \mathbb{Z}$ or $\mathbb{Q}$. Then, Y defines a Hodge class in $H^{2k}(X, R)$ (i. e., there is a class $[Y] \in H^{2k}(X, R) \cap H^{k,k}(X)$).

Definition 1.3.18

An **algebraic class** is a Hodge class that is the image of an algebraic cycle by the map described in Lemma 1.3.17.

The Hodge conjecture predicts the converse to Lemma 1.3.17 in the case of a smooth projective variety and $R = \mathbb{Q}$ (there are counterexamples assuming the negation of any of these hypotheses, cf. [Huy16]).

Conjecture 1.3.19: (Hodge conjecture.)

Let X be a smooth projective variety over $\mathbb{C}$. Then, every Hodge class in $H^{2k}(X, \mathbb{Q})$ is algebraic.

Remark 1.3.20

Let X be a smooth projective variety. An algebraic cycle with integral coefficients of pure codimension 1 (i. e., all subvarieties Y_i are of codimension 1) is, by definition, a (Weil) divisor on X. If Y is a divisor on X, the map described in Lemma 1.3.17 is exactly (modulo linear equivalence) the first Chern class $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. A posteriori (see the proof of the Lefschetz $(1,1)$ theorem), it will turn out that all algebraic classes in $H^2(X, \mathbb{Z})$ are represented by divisors.

The integral version of the Hodge conjecture is known for $k = 1$.

Theorem 1.3.21: (Lefschetz $(1,1)$ theorem.)

Let X be a smooth projective variety over $\mathbb{C}$ or a compact Kähler manifold. Then, the first Chern class $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is surjective in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. In particular, every Hodge class in $H^2(X, \mathbb{Z})$ is algebraic.

The following short proof may be found in e. g. [GH94].

Proof. X is a complex manifold, thus X admits an exponential sequence

$$0 \to \underline{\mathbb{Z}} \xhookrightarrow{2\pi i\cdot} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

which induces a long exact sequence in cohomology

$$\cdots \to H^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{H^2(2\pi i\cdot)} H^2(X, \mathcal{O}_X) \cong H^{0,2}(X) \to \cdots .$$

It is sufficient that the map $H^2(2\pi i\cdot)$ is zero on $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. Indeed, the map $H^2(2\pi i\cdot)$ equals

$$H^2(X,\mathbb{Z}) \xhookrightarrow{i} H^2(X,\mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X) \xrightarrow{\pi_{0,2}} H^{0,2}(X) \cong H^2(X, \mathcal{O}_X).$$

□

Corollary 1.3.22

Let X be a smooth projective variety over $\mathbb{C}$ or a compact Kähler manifold. Then, the following bound on the Picard number of X holds:

$$0 \le \rho(X) \le h^{1,1}(X).$$

We will now work toward stating the Hodge-theoretic version of the famous *Torelli theorem* for curves. The idea, as that of all Torelli-type theorems, is that a curve is determined up to isomorphism by its Hodge structure. We start by defining a few abstract notions (the details are left to the reader).

Definition 1.3.23: ([Huy16], Chapter 3, Definition 1.6.)

Let V, V' be a rational Hodge structures of weight $k \in \mathbb{Z}$.

1. The **Weil operator** C is the $\mathbb{C}$-linear map such that, for each p, q such that $p+q = n$,
$$C : V^{p,q} \to V^{p,q}, \quad x \mapsto i^{p-q}x.$$
In particular,
$$C((V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}}) = (V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}}.$$
2. A **polarisation** of V is a morphism of Hodge structures
$$\psi : V \otimes V \to \mathbb{Q}(-k),$$
such that, for all p, q such that $p + q = k$,
$$\psi_{\mathbb{R}} : (V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}} \to \mathbb{R}, \quad (x, y) \mapsto \psi_{\mathbb{R}}(x, Cy)$$
is a positive-definite symmetric form. Here, $\mathbb{Q}(-k)$ is a twist of the rational Tate Hodge structure $\mathbb{Q}(-1)$ (see Example 1.3.5).
3. If there exists a polarisation ψ of V, we say the pair (V, ψ) is a **polarised Hodge structure** and V is **polarisable**.
4. If there exist polarisations ψ, ψ' of V, V', a **Hodge isometry** is an isomorphism of Hodge structures $V \xrightarrow{\sim} V'$ that is compatible with the polarisations ψ, ψ'.

Remark 1.3.24

The notion of Hodge isometry is usually relaxed to mean an isomorphism of Hodge structures that is compatible with the intersection pairing in $H^n(X, \mathbb{Z})$.

Remark 1.3.25: ([Huy16], 2.1.)

In the case of Hodge structures of weight 1, it can be shown that there exist the following natural bijective correspondences:

$$\{\text{ Complex tori } X^n\} \leftrightarrow \left\{ \begin{array}{c} \text{Integral Hodge structures} \\ \text{of weight 1 } H^n(X, \mathbb{Z}) \end{array} \right\},$$
$$\{\text{ Abelian varieties } X^n\} \leftrightarrow \left\{ \begin{array}{c} \text{Polarisable integral Hodge structures} \\ \text{of weight 1 } H^n(X, \mathbb{Z}) \end{array} \right\},$$
$$\{\text{ Polarised abelian varieties } (X^n, D)\ \} \leftrightarrow \left\{ \begin{array}{c} \text{Polarised integral Hodge structures} \\ \text{of weight 1 } (H^n(X, \mathbb{Z}), \psi) \end{array} \right\}.$$

Here, X^n means that $\dim_{\mathbb{C}}(X) = n$.

Theorem 1.3.26: (Torelli theorem for curves. [Huy16], Theorem 2.2.)

Let C and C' be smooth compact complex curves. Then, C is isomorphic to C' if and only if there exists an isomorphism of integral Hodge structures from $H^1(C, \mathbb{Z})$ to $H^1(C', \mathbb{Z})$ that is compatible with the intersection pairing.

Remark 1.3.27

Using a more explicit form of Remark 1.3.25, we may recover the more classical statement involving *principally polarised Jacobian varieties.*

Chapter 2

Elliptic surfaces

In this chapter we provide a brief introduction to the theory of elliptic surfaces. We follow Schütt and Shioda's survey *Elliptic Surfaces* [SS10], Miranda's notes *The Basic Theory of Elliptic Surfaces* [Mir89], and Huybrechts's book *Lectures on K3 Surfaces* [Huy16].

In Section 2.1, we give a short reminder about elliptic curves, define algebraic and complex elliptic surfaces, give examples, and describe the Kodaira-Néron model.

In Section 2.2, we state the existence of the Jacobian elliptic surface and give its most important properties.

In Section 2.3, we go through Kodaira's classification of singular fibers, Tate's algorithm, talk about base change, and define the j-map.

In Section 2.4, we talk about the interplay between (multi)sections and points on the generic fiber, define the Mordell-Weil group $E(K(C))$, and state the relationship between $E(K(C))$ and $\mathrm{NS}(S)$ and the Shioda-Tate formula.

In Section 2.5, we give properties of line bundles on elliptic surfaces, Euler characteristic, and the Hodge diamond.

Let K be an arbitrary field. As before, an algebraic variety X over K is a separated geometrically integral scheme X of finite type over K. An algebraic curve (resp., surface) is an algebraic variety of dimension 1 (resp., 2).

2.1 First definitions, properties, and examples

We begin with the most important prerrequisites to the chapter, the definition and properties of *elliptic curves*.

Definition 2.1.1

An **elliptic curve** is a pair consisting of a smooth projective curve E over K of genus $g(E) = 1$, and a choice of base point $O \in E(K)$.

Hence, the *Abel-Jacobi map*

$$\varphi : E \xrightarrow{\sim} \mathrm{Pic}^0(E), \quad x \mapsto \mathcal{O}_E(x - O),$$

is an isomorphism between the elliptic curve E and the group

$$\mathrm{Pic}^0(E) := \ker(\deg : \mathrm{Pic}(E) \to \mathbb{Z}),$$

which in turn is isomorphic to the *Jacobian variety* $J(E)$ of E (e. g., if $K = \mathbb{C}$). This naturally induces a group structure G on E, with $0_G = \varphi^{-1}(\mathcal{O}_E(O))$.

There are two more well-known descriptions of an elliptic curve which are worth mentioning.

On one hand, an elliptic curve E over $K = \mathbb{C}$ is a *complex torus*

$$E \cong \mathbb{C}/L,$$

where $L \cong \mathbb{Z}^2$ is a lattice of rank 2. The induced (additive) group structure coincides with the group structure previously described. Two complex tori $\mathbb{C}/L_1$ and $\mathbb{C}/L_2$ are isomorphic if and only if the lattices L_1 and L_2 are isomorphic in the sense introduced in Section 1.1 (with the induced inner product from $\mathbb{C}$ to L_1 and L_2), and if $\mathbb{C}/L$ is a complex torus, there exists $\tau \in \mathbb{C}$ such that $\mathrm{im}(\tau) > 0$ and $\mathbb{C}/L \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

On the other hand, an elliptic curve E over an arbitrary field K has a model as a smooth cubic in $\mathbb{P}^2$. Bezout's theorem (and a relatively hard argument that proves associativity) implies that we can define the group structure G on E geometrically, with $0_G := O$ being the point of inflection. We can also find a linear transformation that maps 0_G to $[0, 1, 0]$ and the tangent line of E at 0_G to

$$\{[x, y, z] \in \mathbb{P}^2 : \quad z = 0\}.$$

The image of $E \setminus \{0_G\}$ is contained in the affine chart

$$\{[x, y, z] \in \mathbb{P}^2 : \quad z = 1\} \cong \mathbb{A}^2,$$

and the image of 0_G is a point at infinity. If $\mathrm{char}(K) \neq 2, 3$, this linear transformation allows us to write the equation defining E in *Weierstrass form*

$$E : \quad y^2 = x^3 + a_4 x + a_6.$$

If E is an elliptic curve over $K = \mathbb{C}$ given as a complex torus $\mathbb{C}/L$, the Weierstrass $\wp$ function

$$\wp : \mathbb{C} \setminus L \to \mathbb{C}, \quad z \mapsto \frac{1}{z^2} + \sum_{\ell \in L \setminus \{(0,0)\}} \left(\frac{1}{(z - \ell)^2} - \frac{1}{\ell^2} \right)$$

induces a map $(\hat{\wp}, \hat{\wp}')$ between E as a complex torus and E as a smooth cubic in $\mathbb{P}^2$ in Weierstrass form, with coefficients a_4, a_6 in terms of *Eisenstein series*, series that are defined in terms of powers of elements of L.

Let E be an elliptic curve over a field K such that $\mathrm{char}(K) \neq 2, 3$.

Definition 2.1.2

The **discriminant** of E is the number

$$\Delta := -16(4a_4^3 + 27a_6^2).$$

Proposition 2.1.3: ([SS10], Lemma 2.3.)

The curve E defined in Weierstrass form

$$E : y^2 = x^3 + a_4x + a_6$$

is smooth if and only if $\Delta \neq 0$.

Definition 2.1.4

The **j-invariant** of E is the number

$$j := -1728\frac{(4a_4)^3}{\Delta}.$$

Proposition 2.1.5: ([SS10], Theorem 2.4.)

Let E, E' be elliptic curves over a field K such that $\mathrm{char}(K) \neq 2, 3$, and let j, j' be their j-invariants. If $E \cong E'$, $j = j'$. If K is algebraically closed and $j = j'$, $E \cong E'$.

We now give the most important definitions of the chapter.

Definition 2.1.6

An *elliptic surface* may refer to any of the following objects:

1. An **algebraic elliptic surface** is a smooth projective surface S over K, a smooth projective curve C over K, and a regular map
$$p : S \to C$$
such that, for a general point $x \in C$, the fiber $F_x = p^{-1}(x)$ is a smooth projective curve of genus $g(F_x) = 1$ *(not necessarily an elliptic curve!)*

2. A **complex elliptic surface** is a complex surface S, a smooth complex curve C, and a holomorphic map
$$p : S \to C$$
such that, for a general point $x \in C$, the fiber $F_x = p^{-1}(x)$ is a smooth connected complex curve of genus $g(F_x) = 1$.

Proposition 2.1.7

We may identify algebraic elliptic surfaces S over $\mathbb{C}$ and *projective* complex elliptic surfaces S^{an}. Even more, the abelian categories of coherent sheaves on S and S^{an} are equivalent.

Proof. This is a consequence of Serre's GAGA principles and Chow's theorem. □

Remark 2.1.8

Due to Proposition 2.1.7, we will be intentionally vague with language and say *elliptic surface* to refer to an algebraic or a complex elliptic surface if the statement is true in either case.

Definition 2.1.9

Let $p : S \to C$ be an elliptic surface. Then:

1. $p : S \to C$ is **smooth** if S is smooth.

2. $p : S \to C$ is **relatively minimal** if, for all $x \in C$, the fiber $F_x = p^{-1}(x)$ does not contain any -1-curve (i. e., any curve of self-intersection -1).

3. $p : S \to C$ is **Jacobian** or **with section** if p has a *section* (i. e., a regular or holomorphic right inverse) $s : C \to S$.

Remark 2.1.10

In this *thesis* we will assume all elliptic surfaces to be smooth and without multiple fibers. Furthermore, if $p : S \to C$ is a smooth elliptic surface, we can blow-down -1-curves contained in the fibers to obtain a unique smooth relatively minimal elliptic surface $p' : S' \to C$ and a birational map $f : S \to S'$ such that $p = p' \circ f$. Therefore, in this *chapter* we will also assume, unless explicitly said otherwise, K to be algebraically closed and all elliptic surfaces to be relatively minimal.

We now give (a sketch of) the most important family of examples.

Example 2.1.11: (Cubic pencils. [Mir89], I.5.1.)

Let C_1 and C_2 be cubics in $\mathbb{P}^3$ defined by homogeneous polynomials p_1 and p_2, respectively. The *cubic pencil* generated by C_1 and C_2 is the elliptic surface

$$p : S = \{(x, [y, y']) \in \mathbb{P}^2 \times \mathbb{P}^1 : \quad yp_1(x) + y'p_2(x) = 0\} \xrightarrow{\pi_2} \mathbb{P}^1.$$

Blowing up $\mathbb{P}^2$ at the nine base points of the cubic pencil (i. e., the points in $C_1 \cap C_2 \subset \mathbb{P}^2$) gives a rational minimal Jacobian elliptic surface (exceptional divisors are sections) which is birational to the cubic pencil.

Cubic pencils illustrate most of the behaviour of elliptic surfaces and their singular fibers. The interested reader may find a thorough discussion in [Mir89], I.5.

Remark 2.1.12: (Kodaira-Néron model. [SS10], 3.5.)

The generic fiber of an elliptic surface is a smooth projective curve of genus one over $K(C)$. If the elliptic surface is Jacobian, the generic fiber is an elliptic curve over $K(C)$. If E is an elliptic curve over $K(C)$, the *Kodaira-Néron model* allows us to associate a Jacobian elliptic surface to E such that its generic fiber is E. The idea is to write E in extended Weierstrass form, and for each $t \in C$ such that $\Delta(t) \neq 0$ (see the discussion in Section 2.3 for more details), read the fiber F_t (i. e., the extended Weierstrass form of F_t is the evaluation of the extended Weierstrass form of E at t). This results in a quasi-projective surface with a fibration $p^* : S^* \to C^*$ which is missing the singular fibers. Then, by Tate's algorithm (see Section 2.3), the singular fibers are already determined (this may require a desingularisation of the surface). This gives a Jacobian elliptic surface which is unique up to isomorphism.

2.2 Jacobian elliptic surfaces

Definitions 2.1.9 imply the following.

Proposition 2.2.1

Let $\hat{p} : J \to C$ be a Jacobian elliptic surface. Then:

1. $\hat{p}$ is surjective and s is injective.

2. For all $x \in C$ such that the fiber $\hat{F}_x = \hat{p}^{-1}(x)$ is a smooth connected projective curve of genus 1, $\hat{F}_x$ is an elliptic curve. In particular, $\hat{F}_x$ is isomorphic to its Jacobian $J(\hat{F}_x)$.

3. J can be viewed as a group scheme over C and as a sheaf over C.

We now state the existence of the Jacobian elliptic surface and give its most important properties. The construction is very natural, although technical. An excellent exposition is found in [Huy16], Chapter 11, Section 4.

Proposition 2.2.2: ([Huy16], Chapter 11, Section 4. [BHPVdV04], Chapter V, Section 9, 11.)

If K is algebraically closed, $p : S \to C$ is an elliptic surface, and E is its generic fiber, we can associate an algebraic Jacobian elliptic surface $\hat{p} : J(S) \to C$ to $p : S \to C$ whose fibers satisfy

$$\hat{F}_x \cong \hat{p}^{-1}(x) = J(F_x) \cong \mathrm{Pic}^0(F_x), \quad \forall x \in C \text{ closed},$$

and its generic fiber $\hat{E}$ is isomorphic to $J(E) \cong \mathrm{Pic}^0(E)$.

Furthermore, if $p : S \to C$ does not admit multiple fibers, the elliptic surfaces $p : S \to C$ and $\hat{p} : J(S) \to C$ are locally isomorphic (i. e., for all $x \in C$ there exists a neighbourhood $U \subset C$ of x such that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is Jacobian and $p^{-1}(U) \cong \hat{p}^{-1}(U)$).

This construction will prove to be central to the arguments in Chapter 5, so we will return to it then. In particular, we will talk about the *Tate-Shafarevich group*, which parametrises all elliptic surfaces with a given Jacobian.

2.3 Classification of singular fibers

Let $p : S \to C$ be an elliptic surface over K, and let E be its generic fiber over $K(C)$.

Definition 2.3.1

Let $F_x = p^{-1}(x), x \in C$ be a singular fiber. The **extended Dynkin diagram associated to** F_x is the graph constructed by drawing a vertex (ignoring multiplicity) for each irreducible component of F_x, and drawing an edge (ignoring multiplicity) for each intersection point between two irreducible components.

Remark 2.3.2

Extended Dynkin diagrams are classified in families $\tilde{A}_n, n \geq 1$, $\tilde{D}_n, n \geq 4$, and $\tilde{E}_n, n \geq 6$. Although they are very easy to draw, we will omit their explicit description. The interested reader may find them in [SS10], Figure 6.

Reduction	Type	Description
Multiplicative	I_0	A smooth elliptic curve (i. e., a general fiber).
Multiplicative	I_1	A nodal rational curve.
Multiplicative	I_n, $n \geq 2$	n smooth rational curves meeting in a cycle.
Additive	II	A cuspidal rational curve.
Additive	III	Two smooth rational curves meeting at one point of multiplicity two.
Additive	IV	Three smooth rational curves all meeting at one point of multiplicity one.
Additive	I_n^*, $n \geq 0$	n smooth rational curves of varying multiplicities meeting as in the extended Dynkin diagram $\tilde{D}_{n+4}$.
Additive	II^*	Nine smooth rational curves of varying multiplicities meeting as in the extended Dynkin diagram $\tilde{E}_8$.
Additive	III^*	Eight smooth rational curves of varying multiplicities meeting as in the extended Dynkin diagram $\tilde{E}_7$.
Additive	IV^*	Seven smooth rational curves of varying multiplicities meeting as in the extended Dynkin diagram $\tilde{E}_6$.

Table 2.1: Classification of singular fibers ([Mir89], Table 1.4.1, [SS10], Figure 4.)

The possible *types* of singular fibers of $p : S \to C$ are as in Table 2.1. They are the same as those of its associated Jacobian elliptic surface $\hat{p} : J(S) \to C$, as they are determined by *local monodromy*, which we will introduce later in this thesis (see Section 5.4). Therefore, we may assume $p : S \to C$ is Jacobian.

The original classification was done by Kodaira in the case $K = \mathbb{C}$ [Kod60, Kod63]. An algorithm was described by Tate for the case K a perfect field [Tat75]. We will only give the first steps of Tate's algorithm in order to introduce relevant language. We will follow a more detailed sketch which may be found in [SS10], 4.2.

We also assume $\mathrm{char}(K) \neq 2$. In this case, the generic fiber E over $K(C)$ may be written in *extended Weierstrass form*

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

and the discriminant of E is

$$\Delta(t) := -4a_2(t)^3a_6(t) - 4a_4(t)^3 + a_2(t)^2a_4(t)^2 - 27a_6(t)^2 + 18a_2(t)a_4(t)a_6(t),$$

where t is a local parameter on C with normalised valuation v_t. By the Kodaira-Néron model (Remark 2.1.12), bases of singular fibers F_t are characterised by $\Delta(t) = 0$ or, equivalently, $v_t(\Delta) \geq 1$. In this case, a linear transformation maps the singular point of F_t to $(x, y) = (0, 0)$, the extended Weierstrass form to

$$y^2 = x^3 + a_2(t)x^2 + ta_4'(t)x + ta_6'(t),$$

and the discriminant to

$$\Delta'(t) := -4ta_2(t)^3a_6'(t) - 4t^3a_4'(t)^3 + t^2a_2(t)^2a_4'(t)^2 - 27t^2a_6'(t)^2 + 18t^2a_2(t)a_4'(t)a_6'(t). \quad (2.1)$$

If $v_t(a_2) = 0$ (resp., if $v_t(a_2) \geq 1$), letting $t = 0$ gives a nodal (resp., cuspidal) rational curve.

Definition 2.3.3

The fiber F_t is **multiplicative**, **semi-stable**, or has **multiplicative** or **semi-stable reduction** if $v_t(a_2) = 0$, and **additive** or has **additive reduction** if $v_t(a_2) \geq 1$.

Remark 2.3.4: ([SS10], 7.2.)

The smooth points of any fiber F_t form an algebraic group over C. The smooth points of the identity component (i. e., the component intersecting the *zero section* O, see Notation 2.4.4) form an algebraic (sub)group isomorphic to $\mathbb{G}_m$, if F_t is multiplicative, or to $\mathbb{G}_a$, if F_t is additive. This explains Definition 2.3.3.

It is easy to see that $(0, 0)$ is also a singular point of S if and only if $v_t(a_6') \geq 1$.

If F_t is multiplicative, then by definition $v_t(\Delta') \geq 1$ and $v_t(a_2) = 0$. If $v_t(\Delta') = 1$, then by the first term of Equation 2.1, $v_t(a_6') = 0$, and $(0, 0)$ is a singular point of F_t but not of S. This gives type I_1 in Table 2.1. If $v_t(\Delta') \geq 2$, then $(0, 0)$ is a singular point of F_t and S. A series of blow-ups gives type I_n, where $n = v_t(\Delta')$. Note that

$$\#\{\text{components of } F_t\} = v_t(\Delta').$$

If F_t is additive, the situation is much more complicated, and results in the rest of the types in Table 2.1. If $\mathrm{char}(K) \neq 2, 3$, we have

$$\#\{\text{components of } F_t\} = v_t(\Delta') - 1.$$

If $\mathrm{char}(K) = 2$ or 3, divisibility phenomena already visible in Equation 2.1 may imply that

$$v_t(\Delta') - 1 \geq \#\{\text{components of } F_t\}.$$

Definition 2.3.5

If F_t is additive, the **index of wild ramification** of F_t is the number

$$\delta_t := v_t(\Delta') - 1 - \#\{\text{components of } F_t\}.$$

Remark 2.3.6

If F_t is additive and $\mathrm{char}(K) \neq 2, 3$,
$$\delta_t = 0.$$

If $\mathrm{char}(K) \neq 2, 3$, the fiber type is determined completely by the values of $v_t(a_4)$ and $v_t(a_6)$ (cf. [SS10], Table 1). More specifically, for each fiber type there is a range of possible values of $v_t(a_4)$ and $v_t(a_6)$, and we have $v_t(a_4) < 4$ or $v_t(a_6) < 6$. If the extended Weierstrass form does not satisfy these restrictions, there is a process called *minimalising* (a change of variables involving t) which allows us to obtain a local (at t) *minimal Weierstrass form* that does (cf. [SS10], 4.8). If $K[C]$ is a principal ideal domain and $K[C] = K[t]$ (e. g., if $C = \mathbb{P}^1$), then after minimalising at all places this local form is global. If not, there may not exist a global minimal Weierstrass form.

n	**Kodaira dimension** $\kappa(S)$	**Type**
1	$-\infty$	Rational
2	0	K3
≥ 3	1	Honestly elliptic surface

Table 2.2: Classification of Jacobian elliptic surfaces over $\mathbb{P}^1$ as projective surfaces ([SS10], 4.10.)

Proposition 2.3.7: ([SS10], 4.10.)

Suppose $C = \mathbb{P}^1$ and the extended Weierstrass form is minimal at all finite places. Let $n \in \mathbb{Z}^+$ be the smallest positive integer such that, for all $i \in \{2, 4, 6\}$, $\deg(a_i) \leq ni$. Then, $n = \chi(\mathcal{O}_S)$, and S fits into the classification of projective surfaces as in Table 2.2.

We now talk about two of the most fundamental operations on elliptic surfaces.

Remark 2.3.8: (Base change, quadratic twists. [SS10], 5.)

If $f : B \to C$ is a surjective regular map, the *base change* of S from C to B is the elliptic surface defined as the usual base change or fibered product $p_B : S \times_C B \to B$. If we write the generic fiber E of $p : S \to C$ in extended Weierstrass form, the effect of base change is applying the pullback by f on each coefficient. The effect on smooth fibers is that they remain smooth. If there is not any wild ramification, the effect on singular fibers is determined by the ramification index of f at the base points modulo a small integer (cf. [SS10], Table 3).

If $\mathrm{char}(K) \neq 2$, and E is the generic fiber of $p : S \to C$ in extended Weierstrass form, we may also apply a *quadratic twist*. In other words, let $d \in K^*$, and consider the extended Weierstrass form

$$S_d : \quad y^2 = x^3 + da_2(t)x^2 + d^2a_4(t)x + d^3a_6(t).$$

The effect on singular fibers is also classified (cf. [SS10], (16)).

We finish the section by stating two important definitions.

Definition 2.3.9

The ***j*-map** of $p : S \to C$ is the birational map

$$j : C \to \mathbb{P}^1, \quad t \mapsto j(F_t),$$

where $j(\cdot)$ is the j-invariant, regular at the base points of smooth fibers. If j is constant, $p : S \to C$ is **isotrivial**.

2.4 Mordell-Weil group

Let $p : S \to C$ be a Jacobian elliptic surface over K, and let E be its generic fiber over $K(C)$.

Remark 2.4.1: ([SS10], 3.4.)

A point $P \in E(K(C))$ is equivalent to a section $s_P : C \to S$. Indeed, if $P \in E(K(C))$, we take the closure $\overline{P}$ of P in S, and, by Zariski's main theorem, the restriction $p|_{\overline{P}} : \overline{P} \to C$ is an isomorphism. Then, the section $s_P : C \to S$ is the inverse $p|_{\overline{P}}^{-1}$. Conversely, if $s : C \to S$ is a section, then the point $P_s \in E(K(C))$ is the restriction $s(C) \cap E$. This construction will hold true much later in this thesis, when we talk about *multisections* (see Section 5.3).

Definition 2.4.2

The **Mordell-Weil group** $E(K(C))$ of S is the set of $K(C)$-rational points of E.

Theorem 2.4.3: (Mordell-Weil theorem for Jacobian elliptic surfaces.)

The Mordell-Weil group $E(K(C))$ is a finitely generated abelian group.

By Remark 2.4.1, a point $P \in E(K(C))$ is equivalent to a section $s_P : C \to S$, which defines a divisor $D_P = s_P(C) \in \mathrm{Div}(S)$.

Notation 2.4.4

If $D \in \mathrm{Div}(S)$ is a divisor, we will also write D for its image in the Néron-Severi group $\mathrm{NS}(S)$. Choose a zero rational point $0_{E(K(C))} \in E(K(C))$. We will write O for the divisor $D_{0_{E(K(C))}}$ defined by the zero section $s_{0_{E(K(C))}} : C \to S$.

It is not too surprising, then, that there exists a geometric proof of the Mordell-Weil theorem for Jacobian elliptic surfaces which uses ideas from lattice and intersection theory, and also sheds some light on the relationship between $E(K(C))$ and $\mathrm{NS}(S)$. We will only state the most important results on this relationship, as they will prove to be very useful. The interested reader is referred to [SS10], 6 for more details.

Theorem 2.4.5: ([SS10], Theorem 6.5.)

On S, algebraic equivalence $\sim_{\mathrm{alg}}$ and numerical equivalence $\equiv$ are equivalent. In particular, $\mathrm{NS}(S) \cong \mathrm{Num}(S)$, a free abelian group and a lattice with the induced intersection form of signature $(1, 1 - \rho(X))$ (see Corollary 1.2.14, 1.2.15).

Proposition 2.4.6

Let $x, x' \in C$. Then, $F_x \sim_{\mathrm{alg}} F_{x'}$ and $F_x.F_x = 0$. In particular, if F is a smooth fiber, $F.F = 0$.

Definition 2.4.7

A divisor $D \in \mathrm{Div}(S)$ is **vertical** if it is contained in a fiber $F_x, x \in C$. It is **horizontal** if it is not vertical.

Notation 2.4.8

We will use the notation in Table 2.3.

Notation	Description
F	A smooth fiber.
F_x	The fiber above $x \in C$ (i. e., $F_x = p^{-1}(x)$).
n_x	The number of components of the fiber F_x.
$C_{x,0}$	The component of the fiber F_x intersecting the zero section O.
$C_{x,i}$	The other components of a reducible fiber F_x, indexed by $\{1, \ldots, n_x - 1\}$.
T_x	The sublattice of $\mathrm{NS}(S)$ generated by the components $C_{x,i}$ of a reducible fiber not intersecting the zero section O.

Table 2.3: Notation for divisors on the fibers of an elliptic surface (similar to [SS10], 6.4.)

Definition 2.4.9

The **trivial lattice** T is the sublattice of $\mathrm{NS}(S)$ generated by the zero section O and the fiber components $F, C_{x,0}, C_{x,i}, i \in \{1, \ldots, n_x - 1\}$.

Proposition 2.4.10: ([SS10], Proposition 6.6.)

The trivial lattice

$$T = \mathbb{Z}O \oplus \mathbb{Z}F \oplus \bigoplus_{x \in C, F_x \text{ reducible}} T_x.$$

In particular,

$$\mathrm{rank}(T) = 2 + \sum_{x \in C, F_x \text{ reducible}} (n_x - 1).$$

We may now state the relationship between $E(K(C))$ and $\mathrm{NS}(S)$. In short, $\mathrm{NS}(S)$ is generated by vertical divisors and sections.

Theorem 2.4.11: ([SS10], Theorem 6.3.)

There exists a group isomorphism

$$\varphi : E(K(C)) \xrightarrow{\sim} \mathrm{NS}(S)/T, \quad P \mapsto D_P \mod T.$$

Corollary 2.4.12: (Shioda-Tate formula.)

The Picard number

$$\rho(S) = \mathrm{rank}(T) + \mathrm{rank}(E(K(C))) = 2 + \sum_{x \in C, F_x \text{ reducible}} (n_x - 1) + \mathrm{rank}(E(K(C)).$$

2.5 Invariants

Let $p : S \to C$ be a Jacobian elliptic surface over K.

The canonical divisor K_S is vertical and algebraically equivalent to a multiple of a smooth fiber F. The proof uses Zariski's lemma, spectral sequences, and the Riemann-Roch theorem.

Theorem 2.5.1: [SS10], Theorem 6.8.

There exists a line bundle $\mathcal{L} \in \mathrm{Pic}(C)$ such that $\deg(\mathcal{L}) = -\chi(\mathcal{O}_S)$ and the canonical bundle

$$\omega_S = p^*(\omega_C \otimes \mathcal{L}^\vee).$$

Furthermore,

$$K_S \sim_{\text{alg}} (2g(C) - 2 + \chi(\mathcal{O}_S))F.$$

Corollary 2.5.2

$$K_S.K_S = 0.$$

Proof. This follows from Proposition 2.4.6 and Theorem 2.5.1. □

Corollary 2.5.3: ([SS10], Corollary 6.9.)

Let $P \in E(K(C))$. Then,

$$D_P.D_P = -\chi(\mathcal{O}_S).$$

Proof. This follows from the adjunction formula. □

Theorem 2.5.4: ([SS10], Theorem 6.10.)

The topological Euler characteristic

$$\chi_{\text{top}}(F_x) = \begin{cases} 0, & \text{if } F_x \text{ is smooth }, \\ n_x, & \text{if } F_x \text{ is multiplicative }, \\ n_x + 1, & \text{if } F_x \text{ is additive }, \end{cases} \quad \forall x \in C,$$

and

$$\chi_{\text{top}}(S) = \sum_{x \in C} (\chi_{\text{top}}(F_x) + \delta_x),$$

where the sum is actually finite and runs over the singular fibers.

Corollary 2.5.5: ([SS10], Corollary 6.11.)

The Euler characteristic
$$\chi(\mathcal{O}_S) = \frac{\chi_{\text{top}}(S)}{12}.$$
In particular, if $p : S \to C$ has a singular fiber,
$$\chi(\mathcal{O}_S) > 0.$$

Proof. This follows from Noether's formula (Theorem 1.2.10), Corollary 2.5.2, and Theorem 2.5.4. $\square$

$$\begin{array}{ccccc}
 & & 1 & & \\
 & g(C) & & g(C) & \\
g(S) = H^0(S, \omega_S) & & 10\chi(\mathcal{O}_S) + 2g(C) & & g(S) = H^0(S, \omega_S) \\
 & g(C) & & g(C) & \\
 & & 1 & &
\end{array}$$

Table 2.4: The Hodge diamond of a Jacobian complex elliptic surface.

Proposition 2.5.6: ([SS10], 6.10.)

If $p : S \to C$ is complex, its Hodge diamond is as in Table 2.4.

Chapter 3

K3 surfaces

Dans la seconde partie de mon rapport, il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire.

- André Weil.

In this chapter we provide a brief introduction to the theory of K3 surfaces. We mostly follow Huybrechts's book *Lectures on K3 Surfaces* [Huy16], Makarova's article *General Introduction to K3 Surfaces* [Mak16], and Schütt and Shioda's survey *Elliptic surfaces* [SS10].

In Section 3.1, we define algebraic and complex K3 surfaces, give properties and examples, prove that the natural surjections $\mathrm{Pic}(X) \to \mathrm{NS}(X) \to \mathrm{Num}(X)$ are isomorphisms, and talk about $H^2(X, \mathbb{Z})$ and Λ_{K3}.

In Section 3.2, we introduce the period domain, deformations, the period map, and talk about the local and global Torelli theorems for K3 surfaces.

In Section 3.3, we give examples of elliptic K3 surfaces, and necessary and sufficient conditions for a K3 surface to admit an elliptic fibration.

Let K be an arbitrary field. As before, an algebraic variety X over K is a separated geometrically integral scheme X of finite type over K. An algebraic surface (resp., complex surface) is an algebraic variety (resp., complex manifold) of dimension 2.

3.1 First definitions, properties, and examples

Notation 3.1.1

Let X be an algebraic variety over K of dimension n. Recall the different notations and identities for the canonical bundle of X:

$$\omega_X := \det(\Omega_X) = \wedge_{i=1}^n \Omega_X =: \Omega_X^n.$$

Definition 3.1.2

A *K3 surface* may refer to any of the following objects:

1. An **algebraic K3 surface** over K is a complete smooth surface X such that $\omega_X \cong \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$.
2. A **complex K3 surface** is a compact connected complex surface such that $\Omega_X^2 \cong \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$.

Proposition 3.1.3: (Basic properties of algebraic K3 surfaces.)

Let X be an algebraic K3 surface.

1. There is a non-canonical isomorphism $\Omega_X \cong \mathcal{T}_x$.
2. $h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X)$.
3. $\chi(\mathcal{O}_X) = 2$.
4. $\chi_{\text{top}}(X) = 24$.

Proof. 1. The exterior product $\Omega_X \times \Omega_X \to \omega_X \cong \mathcal{O}_X$ induces a non-canonical isomorphism $\Omega_X \cong \Omega_X^\vee =: \mathcal{T}_X$.

2. By Serre duality, a choice of trace map induces an isomorphism $H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X^\vee \otimes \omega_X)^\vee \cong H^0(X, \mathcal{O}_X)^\vee$.

3. $\chi(\mathcal{O}_X) := h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1 - 0 + 1$.

4. By Noether's formula (Theorem 1.2.10), $2 = \frac{0+\chi_{\text{top}}(X)}{12}$.

□

The proofs of the following facts are outside the scope of this thesis, so they may as well be stated now (cf. [Deb22], [Huy16]).

Proposition 3.1.4: (More properties of complex K3 surfaces.)

1. If X is a complex K3 surface, then the defining conditions $\Omega_X^2 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$ are equivalent to the existence of a non-degenerate nowhere vanishing holomorphic $(2,0)$ form ω on X.

2. All complex K3 surfaces are Kähler. In particular, if X is a complex K3 surface, there exists a real closed $(1,1)$ form on X associated to a Hermitian metric, the *Kähler form*.

3. The local Torelli theorem (which we will later state, see Theorem 3.2.8) implies that all complex K3 surfaces are deformation equivalent. In particular, they are diffeomorphic.

4. The theory of analytic elliptic surfaces implies that all complex K3 surfaces are simply connected.

Remark 3.1.5

By Proposition 3.1.4, complex K3 surfaces are the two-dimensional *Calabi Yau manifolds*, n-dimensional Kähler manifolds with trivial canonical bundle. They are also *Hyperkähler*.

The following proposition is a consequence of a result sometimes referred to as the Zariski-Goodman theorem (cf. [Mak16]).

Proposition 3.1.6

Let K be an algebraically closed field. Then, a complete smooth surface over K is projective.

Corollary 3.1.7

Let K be an algebraically closed field. Then, an algebraic $K3$ surface is projective.

Proposition 3.1.8

We may identify algebraic K3 surfaces X over $\mathbb{C}$ and *projective* complex K3 surfaces X^{an}. Even more, the abelian categories of coherent sheaves on X and X^{an} are equivalent.

Proof. This is a consequence of Serre's GAGA principles and Chow's theorem. □

The following are classical examples of K3 surfaces. In fact, it was André Weil (see the epigraph at the beginning of the chapter) who provided the original definition, while studying all smooth surfaces diffeomorphic to a smooth quartic in $\mathbb{P}^3$ [SS10].

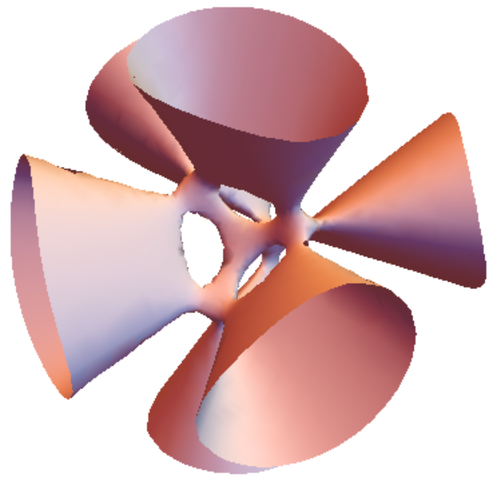

Figure 3.1: A smooth quartic surface. The figure shows part of the real locus ([BTo19]).
Author: BTotaro.
URL: `https://commons.wikimedia.org/wiki/File:K3_surface.png`.

Examples 3.1.9

1. A smooth quartic X in $\mathbb{P}^3$ (see Figure 3.1). The degree of this K3 surface is 4.
2. A smooth complete intersection of type $(2,3)$ in $\mathbb{P}^4$ and $(2,2,2)$ in $\mathbb{P}^5$. The degree of these K3 surfaces is 6 and 8, respectively.
3. A double covering of $\mathbb{P}^2$ branched on a sextic curve, all over a field of characteristic not equal to two.
4. A Kummer surface (i. e., the desingularisation of the quotient A/i, where A is an abelian surface over an algebraically closed field of characteristic not equal to two, and i is the natural involution
$$i : A \to A, \quad x \mapsto -x,$$
which has 16 fixed points).

Proof. We will only prove item 1, and give an idea for item 2. The other proofs may be found in e. g. [Mak16], Example 1.1.5, 1.1.4.

1. By the adjunction formula,
$$\omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X \cong (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X \cong \mathcal{O}_{\mathbb{P}^3}|_X \cong \mathcal{O}_X.$$
The ideal sheaf $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-4)$ induces a short exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \hookrightarrow \mathcal{O}_{\mathbb{P}^3} \twoheadrightarrow \mathcal{O}_X \to 0,$$

which induces a long exact sequence in cohomology

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots$$

As $h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \neq 0$ implies $i = 0$ or $i = n$, $H^1(X, \mathcal{O}_X)$ is between two trivial terms, thus it is also trivial.

2. The idea is to (inductively) use a similar argument to prove that a smooth complete intersection of type $(d_1, \ldots, d_n)$ in $\mathbb{P}^{n+2}$ is a K3 surface if and only if $\sum_{i=1}^n d_i = n + 3$. Then, to (naturally) assume that all $d_i > 1$, and to finally obtain the finite possibilities by basic combinatorics.

□

Theorem 3.1.10: (Riemann-Roch theorem for K3 surfaces.)

Let X be an algebraic K3 surface, and let $\mathcal{L} \in \mathrm{Pic}(X)$. Then,

$$\chi(\mathcal{L}) = 2 + \frac{\mathcal{L}.\mathcal{L}}{2},$$

where $\chi(\cdot)$ is the Euler characteristic.

Proof. Start with the Riemann-Roch theorem for surfaces (Theorem 1.2.9) and use the fact that ω_X is trivial (by definition) and that $\chi(\mathcal{O}_X) = 2$ (Proposition 3.1.3.3). □

Proposition 3.1.11: ([Mak16], Proposition 1.2.9.)

Let X be an algebraic K3 surface. Then, the natural surjections

$$\mathrm{Pic}(X) \to \mathrm{NS}(X) \to \mathrm{Num}(X)$$

(see Remark 1.2.8) are isomorphisms. In particular, $\mathrm{Pic}(X)$ is an even lattice with the induced intersection form of signature $(1, \rho(X) - 1)$.

Proof. We prove the map $\mathrm{Pic}(X) \to \mathrm{Num}(X)$ is injective. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be non-trivial. If for all ample divisors $\mathcal{L}'$, $\mathcal{L}.\mathcal{L}' \neq 0$, then $\mathcal{L}$ is not numerically trivial. If there exists $\mathcal{L}' \in \mathrm{Pic}(X)$ ample such that $\mathcal{L}.\mathcal{L}' = 0$, then $\mathcal{L}^\vee.\mathcal{L}' = 0$. By Proposition 1.2.11.1 applied to a very ample multiple of $\mathcal{L}'$, $\mathcal{L}$ and $\mathcal{L}^\vee$ are not represented by an effective divisor. In particular,

$$H^0(X, \mathcal{L}) = H^0(X, \mathcal{L}^\vee) = 0.$$

By Serre duality,

$$H^2(X, \mathcal{L}) = H^0(X, \mathcal{L}^\vee \otimes \omega_X)^\vee = H^0(X, \mathcal{L}^\vee \otimes \mathcal{O}_X)^\vee = H^0(X, \mathcal{L}^\vee)^\vee.$$

In particular,

$$\chi(\mathcal{L}) := h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) + h^2(X, \mathcal{L}) = -h^1(X, \mathcal{L}) \leq 0.$$

By the Riemann-Roch theorem for K3 surfaces (Theorem 3.1.10),

$$\chi(\mathcal{L}) = 2 + \frac{\mathcal{L}.\mathcal{L}}{2} \leq 0$$

and $\mathcal{L}.\mathcal{L} < 0$, then $\mathcal{L}$ is not numerically trivial. Therefore, $\mathrm{Pic}(X) \to \mathrm{Num}(X)$ is injective, and an isomorphism.

By Corollary 1.2.14, 1.2.15, $\mathrm{Num}(X)$ is a lattice with the induced intersection form of signature $(1, \rho(X) - 1)$, thus $\mathrm{Pic}(X)$ too. By the Riemann-Roch theorem for K3 surfaces,

$$\mathcal{L}.\mathcal{L} = 2(\chi(\mathcal{L}) - 2),$$

thus the lattice is even. □

Corollary 3.1.12

Let X be an algebraic K3 surface. Then, the intersection form on $\mathrm{Pic}(X)$ is non-degenerate, and $\mathrm{Pic}(X)$ is torsion-free.

Proof. This follows by Proposition 3.1.11 and our definition of lattice. □

We give the following important fact about complex K3 surfaces and give an idea of the proof.

Proposition 3.1.13: ([Mak16], Fact 2.1.2. [SS10], 12.2.)

Let X be a complex K3 surface. Then, the integral cohomology $H^2(X, \mathbb{Z})$ is a lattice with the topological intersection form (cup product). It is isomorphic to the K3 lattice

$$\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus (E_8^-)^{\oplus 2}$$

(see Important examples 1.1.15). It is even, unimodular, and of signature $(3, 19)$. Furthermore, $H^2(X, \mathbb{Z})$ is an integral Hodge structure of weight 2 of K3 type (see Important example 1.3.6). In particular, any non-zero $(2, 0)$ form ω on X generates $H^{2,0}(X)$ as a $\mathbb{C}$-vector space, and $\overline{\omega}$ generates $H^{0,2}(X)$.

Proof. (Idea.) For the first part, it is sufficient to prove that $H^2(X, \mathbb{Z})$ is even, unimodular, and of signature $(3, 19)$, as then the classification of indefinite even unimodular lattices (see Remark 1.1.14) implies that it is isomorphic to Λ_{K3}. To show that it is even, one uses Wu's formula. To show that it is unimodular, one uses Poincaré duality. To obtain the signature, one uses the topological index theorem. □

Remark 3.1.14: ([Mak16], Remark 2.1.3.)

If X is a complex K3 surface that is not projective, it can be shown that the first Chern class $c_1 : \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$ is an embedding of lattices (i. e., the intersection forms are compatible), and that this implies that the natural surjection $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X)$ is an isomorphism. *However,* $\mathrm{NS}(X)$ *and* $\mathrm{Num}(X)$ *may not be isomorphic!*

Remark 3.1.15

The Hodge diamond of an algebraic or complex K3 surface is as in Table 1.3. Therefore, if X is a complex K3 surface, then, by Corollary 1.3.22,

$$0 \leq \rho(X) \leq h^{1,1}(X) = 20.$$

In general, if X is an algebraic K3 surface over an arbitrary field, then we only have

$$0 \leq \rho(X) \leq b_2(X) = 22.$$

3.2 Deformations and Torelli theorem

We start by defining the *period domain* and giving its main properties. Recall the definition of the *K3 lattice* Λ_{K3} (see Important examples 1.1.15).

Definition 3.2.1: ([Mak16], Definition 2.3.1.)

The **period domain** is the set

$$D := \{x \in \mathbb{P}(\Lambda_{\mathrm{K3},\mathbb{C}}) : \quad (x, x) = 0, (x, \overline{x}) > 0)\}.$$

Proposition 3.2.2

The period domain is an (analytic) open subset of a smooth quadric in $\mathbb{P}^{21}_{\mathbb{C}}$. In particular, $\dim_{\mathbb{C}}(D) = 20$.

Proof. By definition, the rank of Λ_{K3} is 22, so $\mathbb{P}(\Lambda_{\mathrm{K3},\mathbb{C}}) \cong \mathbb{P}^{21}_{\mathbb{C}}$. The bilinear form in Λ_{K3} is equivalent to a non-degenerate quadratic form, which defines a smooth quadric of dimension 20. The additional condition $(x, \overline{x})$ is open in the analytic topology. □

We now define *deformations*, *markings*, and the *period map*.

Definition 3.2.3: ([Mak16], Definition 2.3.3. [Huy16], Chapter 6, 2.3. [Vak00].)

Let
$$F(\mathcal{X}_{\mathcal{T}}, \mathcal{T}) : \mathcal{X}_{\mathcal{T}} \to \mathcal{T}$$
be a smooth proper family of K3 surfaces with a distinguished point $0_{\mathcal{T}} \in \mathcal{T}$ and fibers $X_t := \mathcal{X}_{\mathcal{T},t}, t \in \mathcal{T}$ such that $\mathcal{T}$ is connected and simply connected.

1. We say $F(\mathcal{X}_{\mathcal{T}}, \mathcal{T})$ is a **local deformation** of X_0.
2. The family $F(\mathcal{X}_{\mathcal{T}}, \mathcal{T})$ is **complete** or a **versal deformation** of X_0 if, for all smooth proper families $F(\mathcal{X}'_{\mathcal{T}'}, \mathcal{T}') : \mathcal{X}'_{\mathcal{T}'} \to \mathcal{T}'$ such that $X_0 \cong X'_0$, there exists a morphism $\mathcal{T}' \to \mathcal{T}$ such that
$$\mathcal{X}'_{\mathcal{T}'} \cong \mathcal{X}_{\mathcal{T}} \times_{\mathcal{T}} \mathcal{T}'.$$
3. The family $F(\mathcal{X}_{\mathcal{T}}, \mathcal{T})$ of X_0 is the **universal deformation** if it is a versal deformation and the morphism $\mathcal{T}' \to \mathcal{T}$ is unique.

Proposition 3.2.4: ([Mak16], Fact 2.3.3.)

Let X_0 be a complex K3 surface. Then, there exist local deformations of X_0. Furthermore, the universal deformation $F(\mathcal{X}_{\mathcal{T}}, \mathcal{T}) : \mathcal{X}_{\mathcal{T}} \to \mathcal{T}$ exists and $\dim_{\mathbb{C}}(\mathcal{T}) = 20$.

Proposition 3.2.5

Let X_0 be a complex K3 surface, and let $F(\mathcal{X}_{\mathcal{T}}, \mathcal{T}) : \mathcal{X}_{\mathcal{T}} \to \mathcal{T}$ be a local deformation of X_0. The fact that $\mathcal{T}$ is simply connected implies that for all $t \in \mathcal{T}$ there exists a canonical isomorphism $\varphi_t : H^2(X_t, \mathbb{Z}) \xrightarrow{\sim} H^2(X_0, \mathbb{Z})$.

Let X_0 be a complex K3 surface, and let $F(\mathcal{X}_{\mathcal{T}}, \mathcal{T}) : \mathcal{X}_{\mathcal{T}} \to \mathcal{T}$ be a local deformation of X_0.

Definition 3.2.6

1. A **marking** is an isomorphism $\varphi : H^2(X_0, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ (see Proposition 3.1.13).
2. The **period map** is the map
$$P : \mathcal{T} \to \mathbb{P}(\Lambda_{\mathrm{K3},\mathbb{C}}), \quad t \mapsto [\varphi_{\mathbb{C}} \circ \varphi_{t,\mathbb{C}}(H^{2,0}(X_t))],$$
(see Proposition 3.1.13, 3.2.5).

Proposition 3.2.7: ([Mak16], Fact 2.3.2.)

The period map is holomorphic and its range is contained in $D \subset \mathbb{P}(\Lambda_{\mathrm{K3}})$.

We may now state the local deformation-theoretic version of the Torelli theorem for complex

K3 surfaces.

Theorem 3.2.8: (Local Torelli theorem for complex K3 surfaces. [Mak16], Theorem 2.3.4.)

The period map is a local isomorphism.

Remark 3.2.9: ([Mak16], Fact 2.3.5, Theorem 2.3.6.)

The universal deformations of K3 surfaces in Proposition 3.2.4 may be glued to obtain a global universal family

$$F(\mathcal{X}'_{\mathcal{T}'}, \mathcal{T}') : \mathcal{X}'_{\mathcal{T}'} \to \mathcal{T}'$$

such that $\mathcal{T}'$ is not Haussdorff and $\dim_{\mathbb{C}}(\mathcal{T}') = 20$. The base $\mathcal{T}'$ is the *moduli space of K3 surfaces* with a fixed marking

$$\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}.$$

The global period map

$$P : \mathcal{T}' \to \mathbb{P}(\Lambda_{\mathrm{K3},\mathbb{C}})$$

is surjective on D and a local isomorphism. It can be shown that there exists a bijective correspondence between the period domain D and certain Hodge structures of K3 type on Λ_{K3} that are related to polarised Hodge structures. Therefore, although we have not described how a polarisation of a complex K3 surface X induces a polarisation of the Hodge structure $H^2(X, \mathbb{Z})$, it makes sense that the global period map is injective if we also take into account polarisations on the domain. Clearly, this would imply a Hodge-theoretic version of the Torelli theorem for complex K3 surfaces.

Definition 3.2.10: ([Mak16], Definition 2.3.4.)

A **polarised** complex K3 surface is a pair $(X, \mathcal{L})$, where X is a complex K3 surface, and $\mathcal{L}$ is an ample line bundle.

Remark 3.2.11

Not all complex K3 surfaces are polarisable, as not all complex K3 surfaces are projective. Let X be a complex K3 surface, and let ω be a $(2,0)$ form on X. By the Nakai-Moishezon criterion (Theorem 1.2.12) and the Lefschetz $(1,1)$ theorem (Theorem 1.3.21), if X is algebraic, then there exists $x \in (\varphi \circ c_1)(\mathrm{Pic}(X)) \subset \Lambda_{\mathrm{K3}}$ such that $(x, x) > 0$ and $(\omega, x) = 0$. Furthermore, X determines a hyperplane

$$H_X := \{x \in \Lambda_{\mathrm{K3},\mathbb{C}} : \quad (\omega, x) = 0\}.$$

For a general point $x \in H_X$ such that $(x, x) > 0$, either x or $-x$ is a polarisation of X.

We now state the Hodge-theoretic version of the Torelli theorem for complex K3 surfaces.

Theorem 3.2.12: (Global Torelli theorem for complex K3 surfaces. [Mak16], Corollary 2.3.8.)

Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be polarised complex K3 surfaces. Then, $(X, \mathcal{L})$ is isomorphic to $(X', \mathcal{L}')$ if and only if there exists a Hodge isometry $H^2(X, \mathbb{Z})$ to $H^2(X', \mathbb{Z})$ such that $c_1(\mathcal{L})$ maps to $c_1(\mathcal{L}')$.

3.3 Elliptic K3 surfaces

We start by stating a fact and giving examples.

Proposition 3.3.1: (Cf. [SS10], 12.7.)

Let $p : S \to C$ be an elliptic K3 surface. Then, $C = \mathbb{P}^1$.

Examples 3.3.2

1. ([SS10], Example 12.5.) A family of elliptic K3 surfaces S may be obtained by applying an adequate quadratic base change or an adequate quadratic twist to a rational elliptic surface over $\mathbb{P}^1$. The Picard number satisfies

$$\rho(S) \geq 10.$$

2. ([SS10], Example 12.6.) If $\mathrm{char}(K) \neq 2$, and E and E' are two elliptic curves over given in Weierstrass form

$$E : y^2 = f(x), \quad E' : y^2 = g(x'),$$

then the desingularisation of the double sextic

$$S : y^2 = f(x)g(x')$$

is a model for the Kummer surface $E \times E'$. The projections to E and E' give two isotrivial elliptic fibrations with four singular fibers of type I_0^* at ∞ and the three roots of f and g, respectively. The Picard number depends on the number of isogenies from E to E':

$$\rho(S) = \begin{cases} 18, & \text{if } E \text{ and } E' \text{ are not isogenous,} \\ 19, & \text{if } E \text{ and } E' \text{ are isogenous and do not have complex multiplication,} \\ 20, & \text{if } E \text{ and } E' \text{ are isogenous and have complex multiplication.} \end{cases}$$

3. ([Has03], Example 6.12.) If S is a quartic in $\mathbb{P}^3$ containing a line

$$D = \{[u, v, x, y] \in \mathbb{P}^3 : \quad u = v = 0\},$$

the projection p from S to the first two coordinates u, v gives an elliptic fibration. Indeed, each element $[\alpha, 1] \in \mathbb{P}^1$ gives a hyperplane in $\mathbb{P}^3$ containing D:

$$H_\alpha = \{[u, v, x, y] \in \mathbb{P}^3 : \quad u + \alpha v = 0\}.$$

There exists a cubic E_α on S such that

$$H_\alpha \cap S = D \cup E_\alpha$$

and $p^{-1}([\alpha, 1]) = E_\alpha$. This covers S by cubics E_α, and E_α is a smooth projective curve of genus one for a general $[\alpha, 1] \in \mathbb{P}^1$, thus this gives an elliptic fibration.

4. ([Has03], Example 6.12.) An instance of the example in the previous item is the Fermat surface

$$S = \{[u, v, x, y] \in \mathbb{P}^3 : \quad u^4 + v^4 - x^4 - y^4 = 0\}$$

containing the line

$$D = \{[u, v, x, y] \in \mathbb{P}^3 : \quad u - x = v - y = 0\}.$$

We give a characterisation of elliptic K3 surfaces which gives a converse to Proposition 2.4.6.

Proposition 3.3.3: ([SS10], Proposition 12.8, 12.10.)

Let S be an algebraic K3 surface. Then, S admits an elliptic fibration if and only if there exists a non-trivial divisor $D \in \mathrm{NS}(S)$ such that $D.D = 0$. If the class $D \in \mathrm{NS}(S)$ is represented by a singular rational projective curve D_{rational}, S admits a unique elliptic fibration with D_{rational} contained in its singular fibers. Furthermore, any projective curve D' on X such that $D.D' = 1$ induces a section of the elliptic fibration.

Corollary 3.3.4

Let S be an algebraic K3 surface such that $\rho(S) = 1$. Then, S does not admit an elliptic fibration.

Proof. By Corollary 3.1.12, $\mathrm{NS}(S)$ is a free abelian group. Let D be a generator of $\mathrm{NS}(S)$. If there exists a non-trivial divisor $nD \in \mathrm{NS}(S), n \in \mathbb{Z}^+$ such that $n^2 D.D = 0$, then $mD.D = 0$ for all $m \in \mathbb{Z}^+$, and $\mathrm{NS}(S)$ is trivial, a contradiction.

Alternatively, this is an easy consequence of the Shioda-Tate formula (Corollary 2.4.12). □

Corollary 3.3.5: ([SS10], Corollary 12.9.)

Let S be an algebraic K3 surface such that $\rho(S) \geq 5$. Then, S admits an elliptic fibration.

Proof. (Idea.) Any indefinite lattice of rank greater than or equal to 5 has a non-trivial element of square zero. Then, apply Proposition 3.3.3. □

The following follows from a result of Nikulin.

Proposition 3.3.6: ([SS10], Lemma 12.22.)

Let S be an algebraic K3 surface over $\mathbb{C}$ such that $\rho(S) \geq 13$. Then S admits a Jacobian elliptic fibration.

The situation for $2 \leq \rho(S) \leq 4$ depends on S.

Finally, we give results about the singular fibers.

Remark 3.3.7

If $\mathrm{char}(K) \neq 2, 3$ and $p : S \to \mathbb{P}^1$ is an elliptic K3 surface, by Theorem 2.5.4 and Proposition 3.1.3, we have

$$\chi_{\mathrm{top}}(S) = 24 = \sum_{x \in \mathbb{P}^1} \chi_{\mathrm{top}}(F_x),$$

where the sum is actually finite and runs over the singular fibers. By the classification of singular fibers (see Table 2.1), this restricts the possible number of fibers of each type.

Proposition 3.3.8: ([Huy16], Remark 1.12.)

A general elliptic K3 surface has exactly 24 singular fibers all of type I_1.

Proposition 3.3.9: ([BT00], Lemma 3.26.)

Let $p : S \to \mathbb{P}^1$ be a complex elliptic K3 surface such that $\rho(S) \leq 19$. Then, $p : S \to \mathbb{P}^1$ has at least four singular fibers, including at least one multiplicative fiber.

Chapter 4

Integral and rational points

In this chapter we provide a brief introduction to rational and integral points, along with a short survey of the current state of the art.

In Section 4.1, we mostly follow Hassett's article *Potential Density of Rational Points on Algebraic Varieties* [Has03] and then survey results in Bogomolov and Tschinkel's article *Density of Rational Points on Elliptic K3 Surfaces* [BT00], Huang's article *Rational Points on Elliptic K3 Surfaces of Quadratic Twist Type* [Hua21], and Huybrechts's book *Lectures on K3 Surfaces* [Huy16].

In Section 4.2, we follow the first chapter of Corvaja's book *Integral Points on Algebraic Varieties, An Introduction to Diophantine Geometry* [Cor16] and Hassett and Tschinkel's article *Density of Integral Points on Algebraic Varieties* [HT01].

Let K be a number field. A variety X over K is a separated geometrically integral scheme X of finite type over K. Let X be a variety over K. $X_{\overline{K}}$ is the base change of X to an algebraically closed field extension $\overline{K}$ of K.

4.1 Rational points

4.1.1 First definitions, properties, and examples

Definition 4.1.1

A K-**rational point** x of X is a point $x \in X_{\overline{K}}$ such that its coordinates lie in K. Equivalently, it is a section $s_x : \mathrm{Spec}(K) \to X$ of the structural morphism $X \to \mathrm{Spec}(K)$. The set of K-rational points of X is denoted $X(K)$.

Remark 4.1.2

If K is algebraically closed, $X(K)$ determines much of the structure of the variety X. If not, clearly, $X(K)$ may even be empty. The philosophy behind the second definition is that of identifying a scheme up to isomorphism with its *functor of points*

$$\mathrm{Hom}(\cdot, X) : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Set},$$

where AffSch and Set are the categories of affine schemes and sets, respectively. This philosophy will be useful later, when we define *integral points*.

Notation 4.1.3

If the ground field K is clear and K' is a field extension of K, we will write $X(K')$ instead of $X_{K'}(K')$.

Definition 4.1.4

The set of K-rational points of X $X(K)$ is **dense** if it is not contained in any Zariski closed subset of X. It is **potentially dense** if there exists a finite field extension K' of K such that $X(K')$ is dense.

The following example motivates the notion of density, as we are interested in rational points that do not satisfy any equation that is not a multiple of the equations defining the variety.

Example 4.1.5: (Cf. [Has03], 1.)

If

$$X = \{[x, y] \in \mathbb{P}^1_{\mathbb{Q}} : \quad x^3 + y^3 = 1\},$$

then $X(\mathbb{Q})$ is also contained in

$$X' = \{[x, y] \in \mathbb{P}^1_{\mathbb{Q}} : \quad xy = 0\},$$

thus $X(\mathbb{Q})$ is not dense.

Note that

$$(-1)^3 + (\sqrt[3]{2})^3 = 1,$$

thus $X(\mathbb{Q}(\sqrt[3]{2}))$ is not contained in $X(\mathbb{Q}(\sqrt[3]{2})) \cap X'(\mathbb{Q}(\sqrt[3]{2}))$.

A motivation for the notion of potential density is that we are interested in *geometric* properties of X, instead of properties that depend on K.

Examples 4.1.6: ([Has03], Example 2.2, 2.3.)

1. $\mathbb{P}^n(\mathbb{Q})$ and $\mathbb{P}^n(K)$ are dense.
2. If
$$X = \{[x, y, z] \in \mathbb{P}^2_{\mathbb{Q}} : \quad x^2 + y^2 + z^2 = 0\},$$
then $X(\mathbb{Q})$ is empty, but if
$$X' = \{[x, y, z] \in \mathbb{P}^2_{\mathbb{Q}} : \quad x^2 + y^2 - z^2 = 0\},$$
then there are isomorphisms
$$X_{\mathbb{Q}(i)} \xrightarrow{\sim} X'_{\mathbb{Q}(i)}, \quad [x, y, z] \mapsto [x, y, iz]$$
and
$$\mathbb{P}^1_{\mathbb{Q}(i)} \xrightarrow{\sim} X'_{\mathbb{Q}(i)}, \quad [x, y] \mapsto [2xy, x^2 - y^2, x^2 + y^2].$$
Therefore, as $\mathbb{P}^1(\mathbb{Q}(i))$ is dense, $X(\mathbb{Q}(i))$ too, and $X(\mathbb{Q})$ is potentially dense (see Corollary 4.1.10).

The second example motivates the following definitions.

Definition 4.1.7

X is:

1. **Brauer-Severi**, if there exists $n \in \mathbb{Z}^+$ such that $X_{\overline{K}} \cong \mathbb{P}^n_{\overline{K}}$.
2. **Unirational**, if there exists a finite field extension K' of K, $n \in \mathbb{Z}^+$, and a dominant rational map $\mathbb{P}^n \dashrightarrow X_{K'}$.

Proposition 4.1.8

If X is Brauer-Severi, then there exists a finite field extension K' of K, $n \in \mathbb{Z}^+$ and an isomorphism $\mathbb{P}^n \xrightarrow{\sim} X_{K'}$. In particular, X is unirational.

Proposition 4.1.9: ([Has03], Proposition 3.1.)

If X and Y are projective varieties over K, $f : Y \dashrightarrow X$ is a dominant rational map, and $Y(K)$ is potentially dense, then $X(K)$ is potentially dense.

Proof. Let K' be a field extension of K such that $Y(K')$ is dense, and let $U \subset Y_{K'}$ be the dense open domain of definition of $f_{K'} : Y_{K'} \dashrightarrow X_{K'}$. Identify $U = U(K')$, and note that $f_{K'}$ is dominant and the image of a dense set under a dominant rational map is dense. □

Corollary 4.1.10: ([Has03], Corollary 3.3.)

If X is unirational, then $X(K)$ is potentially dense. In particular, if X is Brauer-Severi, then $X(K)$ is potentially dense.

Corollary 4.1.11

Potential density is a birational invariant.

We give the following generalisation of Proposition 4.1.9 without proof. The interested reader may find a sketch on [Has03], Proposition 3.4.

Theorem 4.1.12: (Chevalley-Weil.)

If X, Y are proper varieties over K, $f : Y \to X$ is an étale morphism, and $Y(K)$ is potentially dense, then $X(K)$ is potentially dense.

4.1.2 Some known results

Proposition 4.1.13

If $X(\mathbb{Q})$ is non-empty, then for all valuations v on $\mathbb{Q}$, $X(\mathbb{Q}_v)$ is non-empty. In particular, $X(\mathbb{A})$, where $\mathbb{A}$ is the adele ring, is non-empty.

The *Hasse principle* (cf. [SS10], 13.17) is the converse, i. e., if $X(\mathbb{A})$ is non-empty, then $X(\mathbb{Q})$ is non-empty. The Hasse principle is true for conics (i. e., smooth projective curves of genus zero and degree two) in $\mathbb{P}^2_{\mathbb{Q}}$, and is the basis for an algorithm that determines whether a conic in $\mathbb{P}^2_{\mathbb{Q}}$ has a $\mathbb{Q}$-rational point. The Hasse principle is also true for smooth projective curves of genus zero over $\mathbb{Q}$, as every one of them admits a model as a conic in $\mathbb{P}^2_{\mathbb{Q}}$. Furthermore, if X is a smooth projective curve of genus zero over K and $X(K)$ is non-empty, then $X(K) \cong \mathbb{P}^1_K$, so $X(K)$ is dense.

In general, however, the Hasse principle is not true. In 1970, Manin (cf. [Man96]) found that this could be explained by the *Brauer-Manin obstruction*. In short, he considered the *Brauer group* $\mathrm{Br}(X)$ (which is often isomorphic to the *Tate-Shafarevich group*, which we will use extensively in Section 5.3), and defined an action on X such that the subsets

$$X(\mathbb{Q}) \subset X(\mathbb{A})^{\mathrm{Br}(X)} \subset X(\mathbb{A})$$

are proper. Whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle is a difficult question.

The Hasse principle is not true for smooth projective curves of genus one over $\mathbb{Q}$. A counterexample due to Ernst Selmer (see [Sel51]) is the following curve

$$\{[x, y, z] \in \mathbb{P}^2_{\mathbb{Q}} : \quad 3x^3 + 4y^3 + 5z^3 = 0\}.$$

If X is a smooth projective curve of genus one over K and $X(K)$ is non-empty, then X is an *elliptic curve* and we have the well-known *Mordell-Weil theorem* and *Birch-Swinnerton-Dyer conjecture*. If X is an elliptic curve and the Mordell-Weil rank of X is positive, $X(K)$ is potentially dense (see [SS10], 13.16).

Theorem 4.1.14: (Mordell-Weil.)

Let A be an abelian variety over K. Then, the Mordell-Weil group $A(K)$ of K-rational points of A is abelian and finitely-generated. In particular, if X is an elliptic curve over K, $X(K)$ is abelian and finitely-generated.

If the genus is greater than or equal to 2, we have *Falting's theorem*.

Theorem 4.1.15: (Falting.)

If X is a smooth projective curve of genus $g(X) \geq 2$, then $X(K)$ is finite.

Corollary 4.1.16

If X is a smooth projective curve of genus $g(X) \geq 2$, then $X(K)$ is not potentially dense.

Colliot-Thélène, Skorobogatov, and Swinnerton-Dyer (cf. [Has03], Example 3.7) constructed an example of a variety such that its set of K-rational points is not potentially dense by deriving a contradiction with the Chevalley-Weil theorem (Theorem 4.1.12) and Falting's theorem.

A generalisation of Falting's theorem is the *Bombieri-Lang conjecture*, which is true for subvarieties of abelian varieties of general type (see [Has03], Conjecture 3.8). Recall that X is of *general type* if the canonical bundle ω_X is big, i. e., if the Kodaira dimension $\kappa(X)$ is maximal (equal to $\dim(X)$).

Conjecture 4.1.17: (Bombieri-Lang.)

If X is projective and of general type, then $X(K)$ is not potentially dense.

In the case of *Fano varieties* (cf. [Has03], 5), it is known that they do not admit any non-trivial étale covers, and that they do not dominate varieties of general type. It is also known that *Del Pezzo surfaces* (i. e., Fano varieties of dimension two) are birational to $\mathbb{P}^2$, thus, by Corollary 4.1.11, they have (potentially) dense K-rational points. Smooth cubic hypersurfaces of dimension greater than or equal to two are Fano and unirational, thus, by Corollary 4.1.10, they have potentially dense K-rational points. Smooth Fano threefolds (i. e., varieties of dimension three) are unirational except in three cases, two of which admit *generalised elliptic fibrations* over $\mathbb{P}^2$.

The answer to the question of potential density is affirmative for *abelian varieties*.

Definition 4.1.18

If X is an abelian variety, a rational point $x \in X(K)$ is **non-torsion** if the orbit $\mathbb{Z}x$ is infinite, and **non-degenerate** if $\mathbb{Z}x$ is dense.

Theorem 4.1.19: ([Has03], Proposition 4.2.)

If X is an abelian variety, there exists a field extension K' of K such that $X(K')$ contains a non-degenerate point.

Corollary 4.1.20

If X is an abelian variety, $X(K)$ is potentially dense.

There are many approaches which can be used to *propagate* rational points on algebraic varieties. In [Has03], Hassett elaborates on three approaches involving additional geometric structures, such as abelian fibrations, and using subvarieties where it is easier to find rational points, namely non-degenerate, non-torsion, or elliptic multisections. Other methods use the group of automorphisms. In [BT00], Bogomolov and Tschinkel use elliptic fibrations and the group of automorphisms to propagate rational points on K3 surfaces. The main aim of this thesis is to discuss their work in detail, and we will dedicate Chapter 5 to this endeavour. We now give a flavour of the ideas involved in a generalisation of these results.

Definition 4.1.21

Let B be a variety over K. An **abelian fibration** $p : X \to B$ over K is a projective morphism such that the generic fiber F is an abelian variety over the function field $K(B)$. A **generalised elliptic fibration** is an abelian fibration of relative dimension one.

In Section 5.3 we will define *multisections*. For now, it is sufficient to know that a multisection induces a point in the generic fiber of the abelian fibration. In the case of a generalised elliptic fibrations, a multisection is non-torsion (i. e., its induced point in the generic is non-torsion) if and only if it is non-degenerate.

Theorem 4.1.22: ([Has03], Proposition 4.13.)

If $p : X \to B$ be an abelian fibration over K with a non-degenerate multisection M such that $M(K)$ is dense, then $X(K)$ is dense.

We now state the main results in [BT00], which include a specialisation of Theorem 4.1.22.

Theorem 4.1.23: ([BT00], Theorem 1.1.)

If X is a K3 surface such that X admits an elliptic fibration $p : X \to C$ or the group of automorphisms $\mathrm{Aut}(X)$ is infinite, then $X(K)$ is potentially dense.

Corollary 4.1.24: (Cf. [BT00].)

If X is a K3 surface, the following conditions imply that $X(K)$ is potentially dense:

1. $\rho(X_{\mathbb{C}}) = 1$, and X is a double cover of $\mathbb{P}^2$ ramified in a singular curve of degree six.
2. $\rho(X_{\mathbb{C}}) = 2$ and X does not contain a -2-curve (i. e., a curve of self-intersection equal to 2).
3. $\rho(X_{\mathbb{C}}) = 3$, except possibly for 6 isomorphism classes of lattices $\mathrm{Pic}(X_{\mathbb{C}})$.
4. $\rho(X_{\mathbb{C}}) = 4$, except possibly for 2 isomorphism classes of lattices $\mathrm{Pic}(X_{\mathbb{C}})$.
5. $\rho(X_{\mathbb{C}}) \geq 5$.

Recent results by Huang (cf. [Hua21]) concern a family of K3 surfaces over $\mathbb{Q}$ called of *Cassels-Schinzel type*. They arise as isotrivial quadratic twists or, alternatively, as twisted Kummer surfaces associated to the product of two twisted elliptic curves. If $d \in \mathbb{Z}^+$, the affine model is

$$S^{d,1} = \{(x, y, t) \in \mathbb{A}^3_{\mathbb{Q}} : \quad d(1 + t^4)y^2 - x^3 + x = 0\}.$$

The approach in [Hua21] relates density in the analytic and Zariski topology and is guided by conjectures of Mazur, Corvaja and Zannier. Their main theorem is:

Theorem 4.1.25: ([Hua21], Theorem 1.4.)

There exists an infinite set of square-free integers $d \in \mathbb{Z}^+$ such that $S^{d,1}(\mathbb{Q})$ is (Zariski) dense.

Another interesting topic is the relationship between rational points and *rational curves*. This topic is not well-understood, but there is a conjecture:

Conjecture 4.1.26: (Bogomolov's logical possibility. Cf. [Huy16], Chapter 13, 0.3.)

If X is a K3 surface and $x \in X(\overline{K})$, then $X_{\overline{K}}$ contains a rational curve C such that $x \in C$.

Corollary 4.1.27

Suppose Bogomolov's logical possibility is true. If X is a K3 surface, then X contains infinitely many rational curves.

In [BT00], Bogomolov and Tschinkel prove that, if X admits an algebraic elliptic fibration $p : X \to C$ and $\rho(X_{\mathbb{C}}) \leq 19$, then there exist infinitely many rational multisections on X. They also restate a result by Bogomolov and Mumford that implies that every polarised K3 surface contains at least one rational curve. The ideas by Bogomolov and Tschinkel have led to a stronger theorem of Chen and Lewis (cf. [Huy16], Chapter 13, Theorem 5.1.) on analytical density of rational curves on polarised K3 surfaces. Finally, we mention a strong conjecture, which is true for unirational K3 surfaces (equivalent to supersingular K3 surfaces), and which has been a strong motivation for recent research:

Conjecture 4.1.28: ([Huy16], Chapter 13, Conjecture 0.2.)

If $(X_{\overline{K}}, H)$ is a polarized K3 surface, then $X_{\overline{K}}$ contains infinitely many rational curves C such that there exists $n_C \in \mathbb{Z}^+$ with $C \sim_{\text{lin}} n_C H$.

4.2 Integral points

As in the case of rational points, there are two approaches to defining integral points on a variety over K. Both require an extension of the usual definition of the ring of K-integers $\mathcal{O}_K$, as suggested by the following example.

Example 4.2.1

Let $p \in \mathbb{Z}^+$ be a prime, and consider the line of equation $px + py = 1$ in $\mathbb{A}^2_{\mathbb{Q}}$. It does not have any integral points (i. e., points with coordinates in $\mathbb{Z}$), although it is isomorphic to the line of equation $x + y = 1$ in $\mathbb{A}^2_{\mathbb{Q}}$, which has infinitely many integral points. Both lines have infinitely many points with coordinates in $\mathbb{Z}[1/p]$.

Recall that a *place* of K is an equivalence class of absolute values over K, where two absolute values are equivalent if they induce the same topology. By Ostrowski's theorem, a place of K is either *non-archimedean* (also *finite*, *p-adic*, *ultrametric*) or *archimedean*. Non-archimedean places are in bijective correspondence to non-zero prime ideals of $\mathcal{O}_K$. If v is a non-archimedean place corresponding to a prime ideal $\mathfrak{p}$, an absolute value is

$$|\cdot|_v : K \to \mathbb{Q}, \quad x \mapsto |x|_v = (1/2)^{\operatorname{ord}_{\mathfrak{p}}(x)},$$

where $\operatorname{ord}_{\mathfrak{p}}(x)$ is the maximal exponent of $\mathfrak{p}$ in the prime factorisation of $x\mathcal{O}_K$. Archimedean places are in bijective correspondence to embeddings of K in $\mathbb{C}$ up to conjugation.

Definition 4.2.2

Let v is a non-archimedean place. The **ring of v-integers** is the local valuation ring

$$\mathcal{O}_v := \{x \in K : \quad |x|_v \leq 1\}$$

with unique maximal ideal

$$\mathfrak{m}_v := \{x \in K : \quad |x|_v < 1\}$$

and finite residue field

$$\kappa_v := \mathcal{O}_v/\mathfrak{m}_v.$$

Let S be a finite set of places containing all archimedean places. The **ring of S-integers** is the ring

$$\mathcal{O}_S := \{x \in K : \quad \forall v \notin S, |x|_v \leq 1\}.$$

Remark 4.2.3

If v is a non-archimedean place corresponding to a prime ideal $\mathfrak{p}$, $|x|_v < 1$ if and only if $\mathfrak{p}|x\mathcal{O}_K$ ("$\mathfrak{p}$ is in the numerator of x"), $|x|_v = 1$ if and only if $\mathfrak{p} \not| x\mathcal{O}_K$ and $p \not| x^{-1}\mathcal{O}_K$, and $|x|_v > 1$ if and only if $\mathfrak{p}|x^{-1}\mathcal{O}_K$ ("$\mathfrak{p}$ is in the denominator of x"). Therefore, $x \in \mathcal{O}_S$ if and only if all prime ideals $\mathfrak{p}$ in the denominator of x are in S. If S contains only the archimedean places, $\mathcal{O}_S = \mathcal{O}_K$.

Recall also that if v is a non-archimedean place, there exists a *reduction modulo v map*

$$\cdot_v : \mathbb{P}^n_K \to \mathbb{P}^n_{\kappa_v}, \quad x = [x_0, \ldots, x_n] \mapsto x_v = [\lambda_x x_0 + \mathfrak{m}_v, \ldots, \lambda_x x_n + \mathfrak{m}_v],$$

where $\lambda_x \in K$ is such that, for all $i \in \{0, \ldots, n\}$, $\lambda_x x_i \in \mathcal{O}_v$, and not all $\lambda_x x_i \in \mathfrak{m}_v$. If $X \subset \mathbb{P}^n_K$ is a projective variety defined by a homogeneous ideal I_X of $K[x_0, \ldots, x_n]$, the reduction modulo v of X is the projective variety $X_v \subset \mathbb{P}^n_{\kappa_v}$ defined by the homogeneous ideal

$$(I_X \cap \mathcal{O}_v[x_0, \ldots, x_n])/\mathfrak{m}_v$$

of $\kappa_v[x_0, \ldots, x_n]$.

Definition 4.2.4

Let S a finite set of places containing all archimedean places, let $X \subset \mathbb{P}^n_K$ be a projective variety, let $D \subset X$ be a reduced effective Weil divisor, and let $x \in X$. Then:

1. x **reduces modulo** v **to** D if $x_v \in D_v$.
2. x is an **integral point of** (X, D) or **integral with respect to** D if, for all non-archimedean places v of K, x_v does not reduce modulo v to D. In particular, $x \notin D$.
3. x is an S**-integral point of** (X, D) if, for all places $v \notin S$, x_v does not reduce modulo v to D.

Remark 4.2.5: (Cf. [Cor16], 1.)

Let $X \subset \mathbb{A}^n_K$ be an affine variety, and let $i : \mathbb{A}^n_K \hookrightarrow \mathbb{P}^n_K$ be the canonical embedding. Then, the S-integral points of $(\overline{i(X)}, \overline{i(X)} \setminus i(X))$ are exactly the points of X with coordinates in $\mathcal{O}_S$. This recovers the intuitive definition of integral points discussed in Example 4.2.1.

Example 4.2.6: (Cf. [Cor16], 1.)

Let $X = \mathbb{A}^1_{\mathbb{Q}}$, and let

$$i : \mathbb{A}^1_{\mathbb{Q}} \hookrightarrow \mathbb{P}^1_{\mathbb{Q}}, \quad x \mapsto [x, 1].$$

If S contains only the archimedean places and $x = a/b \in \mathbb{Q}, a, b \in \mathbb{Z}, \gcd(a, b) = 1, b \neq 0$ is such that $i(x) = [x, 1]$ is an S-integral point of $(\overline{i(X)}, \overline{i(X)} \setminus i(X))$, then $[a, b] \not\equiv [1, 0]$ mod p for all $p \in \mathbb{Z}^+$ prime. In particular, $p \nmid b$ for all $p \in \mathbb{Z}^+$ prime, thus $b = \pm 1$ and $x \in \mathbb{Z} = \mathcal{O}_S$.

The second approach to defining integral points is a generalisation to schemes over K. It rests upon the notion of a *model*.

Definition 4.2.7

Let S be a finite set of places of K containing all archimedean places, let X be a variety over K, let $D \subset X$ be a reduced effective Weil divisor, and let $U = X \setminus D$.

1. A **model** of X over $\mathcal{O}_S$ is a flat scheme $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ such that the generic fiber
$$\mathcal{X}_K := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_S)} K \cong X.$$
2. Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a model of X over $\mathcal{O}_S$. An **S-integral point of $\mathcal{X}$** is a section $s : \mathrm{Spec}(\mathcal{O}_S) \to \mathcal{X}$ of the structural morphism $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$. The set of S-integral points of X is denoted $\mathcal{X}(\mathcal{O}_S)$.
3. Let $\mathcal{X}, \mathcal{D} \to \mathrm{Spec}(\mathcal{O}_S)$ be (compatible) models of X, D over $\mathcal{O}_S$, respectively, such that $\mathcal{X}$ is a normal proper scheme. An **S-integral point of** $(\mathcal{X}, \mathcal{D})$ is an S-integral point $s : \mathrm{Spec}(\mathcal{O}_S) \to \mathcal{X}$ of X such that s does not intersect $\mathcal{D}$, i. e., such that, for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_S)$, $s(\mathfrak{p}) \notin \mathcal{D}_{\mathfrak{p}}$.

We give a few claims in [HT01] as a proposition, and we give an idea of the proof in the projective case.

Proposition 4.2.8: ([HT01], 2.1.)

Consider the setup in Definition 4.2.7.

1. Let $x \in X(K)$. Then, there exists a finite set S of places of K containing all archimedean places and a model of X over $\mathcal{O}_S$ such that $x \in \mathcal{X}(\mathcal{O}_S)$.
2. Let $x \in X(K)$ such that $x \notin D$. Then, there exists a (large) finite set S of places of K containing all archimedean places and a model of X over $\mathcal{O}_S$ such that x is an S-integral point of $(\mathcal{X}, \mathcal{D})$.
3. There exists a finite field extension K' of K, a finite set S' of places of K' containing all archimedean places, and a model $\mathcal{X}$ of X such that the set of S'-integral points of $(\mathcal{X}_{\mathrm{Spec}(\mathcal{O}_{S'})}, \mathcal{D}_{\mathrm{Spec}(\mathcal{O}_{S'})})$ is non-empty.

Proof. (Idea in the projective case.)

1. Enlarge S to include the set of prime ideals in the denominator of x.
2. Let $f_1, \dots, f_k$ be regular functions on X defining (the irreducible components of) D. As $x \notin D$, $f_i(x)$ is non-zero for all $i \in \{1, \dots, k\}$, so we may enlarge S to include the prime ideals in the numerators of the numbers $f_i(x)$.
3. As $X(\overline{K}) \setminus D(\overline{K})$ is non-empty, there exists a finite field extension K' of K such that $X(K') \setminus D(K')$ is non-empty. Then, the claim follows from Item 2.

□

Remark 4.2.9: (Cf. [Cor16], 1.)

Definition 4.2.4 depends on the embedding $X \hookrightarrow \mathbb{P}^n_K$. In fact, to specialise Definition 4.2.7 to Definition 4.2.4, note that, for every ring $\mathcal{O}_S$, there exists a canonical integral model of $\mathbb{P}^n_K$ over $\mathcal{O}_S$. This induces an integral model of X over $\mathcal{O}_S$ via the embedding $X \hookrightarrow \mathbb{P}^n_K$.

Definition 4.2.10: (Cf. [HT01], 1.)

We say integral points on (X, D) are **potentially dense** if there exists a finite field extension K' of K, a finite set S' of places of K' containing all archimedean places, and a model $\mathcal{X}$ of X over $\mathcal{O}_S$ such that the set of S'-integral points of $(\mathcal{X}_{\mathrm{Spec}(\mathcal{O}_{S'})}, \mathcal{D}_{\mathrm{Spec}(\mathcal{O}_{S'})})$ is Zariski dense in $\mathcal{X}_{\mathrm{Spec}(\mathcal{O}_{S'})}$.

Remark 4.2.11: (Cf. [HT01], 1.)

Potential density does not depend on the choices of S' or $\mathcal{X}$. Similarly to the case of rational points, Hassett and Tschinkel study potential density of integral points by using large automorphism groups and additional geometric structures. In particular, they say "the analogs of elliptic fibrations in log geometry are conic bundles with a bisection removed" [HT01].

Definition 4.2.12: ([HT01], Definition 2.1.)

1. A **pair** is an ordered pair (X, D), where X is a normal proper variety, and $D \subset X$ is a reduced effective Weil divisor.

2. A **morphism of pairs** $f : (X, D) \to (X', D')$ is a regular map $f : X \to X'$ such that $f^{-1}(D') \subset D$. In particular, $f|_{X \setminus D} : X \setminus D \to X' \setminus D'$.

3. A morphism of pairs $f : (X, D) \to (X', D')$ is **dominant** if $f : X \to X'$ is dominant.

4. A morphism of pairs $f : (X, D) \to (X', D')$ is **proper** if $f : X \to X'$ and $f|_{X \setminus D} : X \setminus D \to X' \setminus D'$ are proper.

5. A **resolution** of a pair (X', D') is a proper morphism of pairs $f : (X, D) \to (X', D')$ such that X is smooth, D is normal crossings, and $f : X \to X'$ is birational.

6. Finally, a pair (X', D') is of **log general type** if there exists a resolution $f : (X, D) \to (X', D')$ such that $\omega_X(D) := \omega_X \otimes \mathcal{O}_X(D)$ is big.

Proposition 4.2.13: ([HT01], Definition 2.1.)

1. If (X, D) dominates (X', D') and integral points on (X, D) are dense, then integral points on (X', D') are dense (after a choice of model).

2. If (X', D') is of log general type, then for all resolutions $f : (X, D) \to (X', D')$, $\omega_X(D)$ is big.

We may now state *Vojta's conjecture*, which is an important generalisation of the Bombieri-Lang conjecture (Conjecture 4.1.17).

Conjecture 4.2.14: (Vojta. [HT01], Conjecture 2.2.)

Let (X', D') be a pair of log general type. Then, integral points on (X', D') are not potentially dense.

If $D = \emptyset$, then we recover the Bombieri-Lang conjecture (Conjecture 4.1.17).

The converse for Vojta's conjecture is false (cf. [HT01], 3 for a detailed discussion). In the remainder of this section, we will work toward stating a possible converse given in [HT01]. On the way, we will state a generalisation of the Chevalley-Weil theorem (Theorem 4.1.12).

Definition 4.2.15: ([HT01], Definition 3.1, 3.2.)

1. A dominant morphism of pairs $f : (X, D) \to (X', D')$ is **arithmetically continuous** if density of integral points on (X', D') implies potential density of integral points on (X, D).

2. A **pseudo-étale cover of pairs** is a dominant proper morphism of pairs $f : (X, D) \to (X', D')$ such that $f : X \to X'$ is generically finite and the map from the normalisation $\hat{X}'$ of X' to X' is étale on $X' \setminus D'$.

Proposition 4.2.16: ([HT01], Remark 3.3, 3.5.)

1. If X is a normal proper variety, $D = \emptyset$, and $f : \mathbb{P} \to X$ is a projective bundle, then f is arithmetically continuous.

2. If (X', D') is a pair, then there exists a pair (X, D) and a birational pseudo-étale morphism of pairs $f : (X, D) \to (X', D')$ such that X is smooth and D is normal crossings.

3. If $p : X \to X'$ is a generalised elliptic fibration and $D' \subset X'$ is a divisor such that the restriction $p|_{p^{-1}(X' \setminus D')} : p^{-1}(X' \setminus D') \to X' \setminus D'$ is isotrivial, then p is arithmetically continuous.

We may now state a generalisation of the Chevalley-Weil theorem (Theorem 4.1.12).

Theorem 4.2.17: ([HT01], Theorem 3.4.)

Let $f : (X, D) \to (X', D')$ be a pseudo-étale cover of pairs. Then, f is arithmetically continuous.

Finally, we state a possible converse to Vojta's conjecture.

Conjecture 4.2.18: (Possible converse to Vojta's conjecture. [HT01], Problem 3.7.)

Let (X', D') be a pair such that there does not exist any pseudo-étale cover $f : (X, D) \to (X', D')$ such that (X, D) dominates a pair of log general type. Then, integral points on (X', D') are potentially dense.

Chapter 5

Rational points on elliptic K3 surfaces

In this chapter we expand on the ideas originally in Bogomolov and Tschinkel's article *Density of Rational Points on Elliptic K3 Surfaces* [BT00], wherein they prove that if a K3 surface admits an elliptic fibration or has an infinite group of automorphisms, then it has potentially dense rational points, here Theorem 4.1.23.

In Section 5.1, we show that an immersed curve is *semi-regular*, i. e., that it admits a deformation of immersed curves.

In Section 5.2, we give the definition and properties of the *effective monoid* of an algebraic K3 surface. We also give a representation theorem for effective divisors by sums of rational curves.

In Section 5.3, we give the definition and properties of *multisections*. We introduce a generalisation of Jacobian elliptic surfaces. We also give the definition and properties of the *Tate-Shafarevich group*, which is one of the most important tools used in the article.

In Section 5.4, we give the definition and properties of the local and global *monodromy* of an elliptic surface.

In Section 5.5, we give a lower bound for the genus of a torsion multisection, and use the Tate-Shafarevich group to find infinitely many rational non-torsion multisections on an algebraic K3 surface.

Finally, in Section 5.6, we give properties of the automorphism group of an algebraic K3 surface, and give the idea of an argument that uses it to prove potential density. We also give an argument that uses multisections. We finish with the main theorem and its consequences.

5.1 A deformation argument

In this section we work with complex K3 surfaces. Recall our discussion about immersions of curves on surfaces (see Definition 1.2.16, Proposition 1.2.18, 1.2.17) and families and deformations of K3 surfaces (see Definition 3.2.3, Proposition 3.2.4). Figure 5.1 will serve as a visual aid throughout.

Notation 5.1.1

As before, we write

$$F(\mathcal{X}_\mathcal{T}, \mathcal{T}) : \mathcal{X}_\mathcal{T} \to \mathcal{T}$$

for a family with a distinguished point $0_\mathcal{T} \in \mathcal{T}$ and fibers $X_t := \mathcal{X}_{\mathcal{T},t}, t \in \mathcal{T}$. We abuse this notation slightly by using the same symbol $\mathcal{X}.$ for families over different base spaces

$$F(\mathcal{X}., \cdot) : \mathcal{X}. \to \cdot$$

as long as the distinguished fiber X_0 remains isomorphic. This will come in handy for dealing with subfamilies and (uni)versal deformations, and should not lead to confusion.

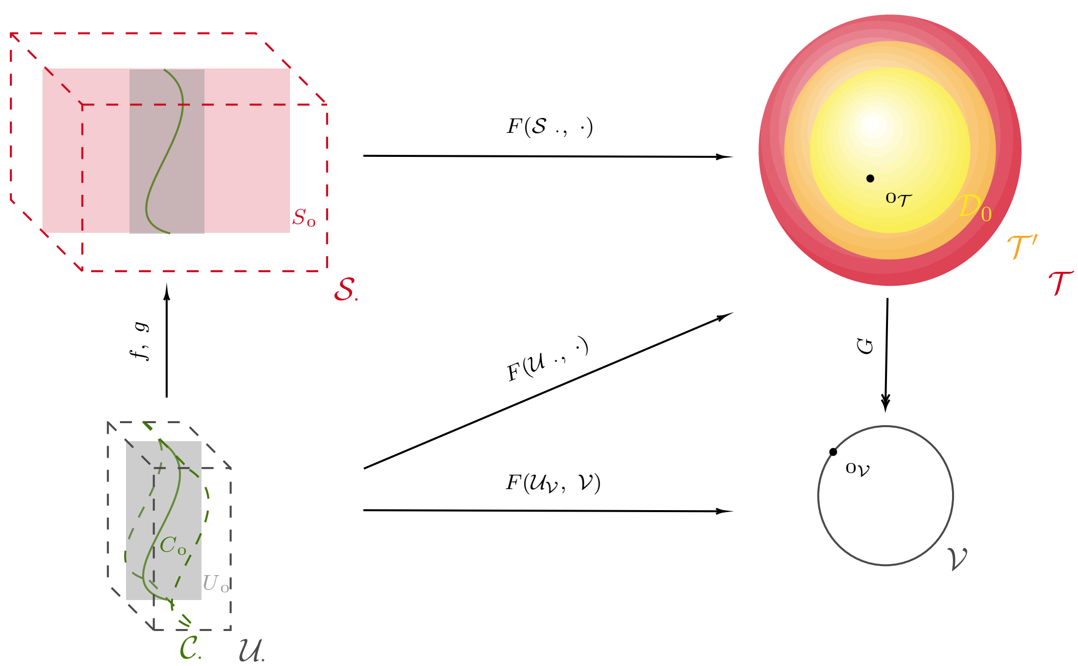

Figure 5.1: A deformation argument.

Let S_0 be a complex K3 surface, and let

$$F(\mathcal{S}_\mathcal{T}, \mathcal{T}) : \mathcal{S}_\mathcal{T} \to \mathcal{T}$$

be a (smooth) local deformation of S_0 as in Proposition 3.2.4 (i. e., $\mathcal{T}$ is a complex ball of dimension 20, and the fibers $S_t := \mathcal{S}_{\mathcal{T},t}, t \in \mathcal{T}$ are complex K3 surfaces).

Let C_0 be a smooth rational curve, and let $f_0 : C_0 \to S_0$ be an immersion. As a divisor of X, C_0 defines an algebraic class $\hat{C}_0 := (c_1 \circ f_0)(C_0) \in H^2(S_0, \mathbb{Z})$. If $t \in \mathcal{T}$, the deformed image $f_0(C_0)$ in S_t also defines a class $\hat{C}_t \in H^2(S_t, \mathbb{Z})$. Let

$$F(\mathcal{S}_{\mathcal{T}'}, \mathcal{T}') : \mathcal{S}_{\mathcal{T}'} \to \mathcal{T}'$$

be the smooth subfamily such that this class is algebraic. Clearly, $S_0 \subset \mathcal{S}_{\mathcal{T}'}$, so we may choose the same zero in the base spaces (i. e., $0_{\mathcal{T}} = 0_{\mathcal{T}'}$). It can be shown that $\mathcal{T}'$ is a complex ball of dimension 19.

Theorem 5.1.2: ([BT00], Proposition 2.2.)

There exist a smooth family of smooth curves

$$F(\mathcal{C}_{\mathcal{T}'}, \mathcal{T}') : \mathcal{C}_{\mathcal{T}'} \to \mathcal{T}'$$

and a holomorphic map $f : \mathcal{C}_{\mathcal{T}'} \to \mathcal{S}_{\mathcal{T}'}$ such that $C_0 = \mathcal{C}_{\mathcal{T}',0}$ and $f|_{C_0} = f_0$.

Proof. By Proposition 1.2.18, there exists a neighbourhood U_0 of C_0 of dimension 2 such that f_0 extends to a biholomorphic map $g_0 : U_0 \to g_0(U_0) \subset S_0$.

The smooth family $F(\mathcal{S}_{\mathcal{T}}, \mathcal{T})$ and g_0 induce a smooth family

$$F(\mathcal{U}_{\mathcal{T}}, \mathcal{T}) : \mathcal{U}_{\mathcal{T}} \to \mathcal{T}$$

and a holomorphic map $g : \mathcal{U}_{\mathcal{T}} \to \mathcal{S}_{\mathcal{T}}$ such that $U_0 = \mathcal{U}_{\mathcal{T},0}$ and $g|_{U_0} = g_0$.

It can be shown that there exists a versal deformation (see Proposition 3.2.4)

$$F(\mathcal{U}_{\mathcal{V}}, \mathcal{V}) : \mathcal{U}_{\mathcal{V}} \to \mathcal{V}$$

such that $U_0 = \mathcal{U}_{\mathcal{T},0} \cong \mathcal{U}_{\mathcal{V},0}$, $F(\mathcal{U}_{\mathcal{T}}, \mathcal{T})$ is the pullback of $F(\mathcal{U}_{\mathcal{V}}, \mathcal{V})$ by a surjective map $G : \mathcal{T} \to \mathcal{V}$, and $\mathcal{V}$ is a complex ball of dimension 1.

Since G is surjective, $\mathcal{D}_0 := G^{-1}(0_{\mathcal{V}})$ is a divisor on $\mathcal{T}$, and have $0_{\mathcal{T}} \in \mathcal{D}_0$. Furthermore, if $t \in \mathcal{D}_0$, then

$$C_0 \subset U_0 = \mathcal{U}_{\mathcal{T},0} \cong \mathcal{U}_{\mathcal{V},0} \cong \mathcal{U}_{\mathcal{T},t} = U_t. \tag{5.1}$$

Let $t \in \mathcal{T} \setminus \mathcal{T}'$. By definition, the deformed image $f_0(C_0)$ in S_t defines a class $\hat{C}_t \in H^2(S_t, \mathbb{Z})$ that is not algebraic. Thus, the pullback of this class by $g|_{U_t} : U_t \to g(U_t) \subset S_t$,

$$g^*(\hat{C}_t) := g|_{U_t}^*(\hat{C}_t|_{g(U_t)}) \in H^2(U_t, \mathbb{Z}) \tag{5.2}$$

is also not algebraic. Indeed, the non-degenerate nowhere vanishing holomorphic $(2,0)$ form ω_{S_t} (see Proposition 3.1.4) on S_t induces a non-degenerate $(2,0)$-form

$$g^*(\omega_{S_t}) := g|_{U_t}^*(\omega_{S_t}|_{g_t(U_t)})$$

on U_t. By the Lefschetz $(1,1)$ theorem (Theorem 1.3.21) and the Hodge decomposition of $H^2(S_t, \mathbb{Z})$ (Proposition 3.1.13), the integral of ω_{S_t} over $\hat{C}_t$ is non-zero. By definition, this implies that the integral of $g^*(\omega_{S_t})$ over $g^*(\hat{C}_t)$ is also non-zero. By a similar argument (g is a local isomorphism), the claim follows.

Now, let $t \in \mathcal{D}_0$. By construction, the image of C_0 in U_t (see Equation 5.1) defines the same class in $H^2(U_t, \mathbb{Z})$ as $g^*(\hat{C}_t)$ (see Equation 5.2). As this class is algebraic, we have $\mathcal{D}_0 \subset \mathcal{T}'$. As $\mathcal{T}'$ is irreducible, we have equality. The theorem follows by restricting the family $F(\mathcal{U}_{\mathcal{T}'}, \mathcal{T}')$ and g to these images. □

5.2 A representation theorem for effective divisors

Let S be an algebraic K3 surface over an arbitrary field.

Definition 5.2.1

The **effective monoid** $\Lambda_{\text{eff}}(S)$ is the monoid of all classes in $\mathrm{Pic}(S)$ represented by an effective divisor.

The aim of this section is to give a sketch of the proof of a theorem by Bogomolov and Mumford that allows us to represent a line bundle in the effective monoid by sums of rational curves. Another proof of the $x.x > 0$ case may be found in [MM82].

Lemma 5.2.2: (Cf. [BT00].)

Let x be a generator of $\Lambda_{\text{eff}}(S)$. Then, exactly one of the following holds:

1. $x.x = -2$, in which case x is represented by a smooth rational curve.
2. $x.x = 0$, in which case x is represented by an elliptic curve E and a singular rational curve C. Furthermore, there exists an elliptic fibration without multiple fibers $f : S \to \mathbb{P}^1$ such that its smooth fiber is E and its singular fibers contain C.
3. $x.x$ is positive and even.

Proof. By Proposition 3.1.11, the lattice $\mathrm{Pic}(S)$ is even. The claim if $x.x = 0$ follows by a proof of Proposition 3.3.3. For a detailed analysis of the effective cone of an algebraic K3 surface, see e. g. [Huy16]. □

Lemma 5.2.3: ([BT00], Corollary 2.7.)

Let $x \in \Lambda_{\text{eff}}(S)$ be primitive. Then, x is uniquely determined by its self-intersection $x.x$.

Proof. After a choice of marking on S, the lattice $\mathrm{Pic}(S)$ contains $\Lambda_{\text{eff}}(S)$ and is embedded in Λ_{K3}. A choice of marking on S is a choice of automorphism in $O(\Lambda_{\text{K3}})$. By Lemma 1.1.18, the orbit of x under the action of $O(\Lambda_{\text{K3}})$ is determined by its square. □

The proofs of Lemmas 5.2.4, 5.2.5 are a bit outside of the scope of this thesis, as they use technical results about the moduli spaces of hyperelliptic curves and K3 surfaces. We give a rough sketch, but more details may be found in [BT00]. The author made the conscious decision to give more complete arguments in the next sections instead, where we talk extensively about *multisections* of elliptic fibrations.

Lemma 5.2.4: ([BT00], Lemma 2.11)

Let $n \in \mathbb{Z}^+$ be even. Then, there exist a K3 surface S_n and a rational curve $C_n \subset S_n$ such that $C_n.C_n = n$.

Proof. (Idea.) First, one considers a curve C of genus 2, its Jacobian $J(C)$, the quotient $J(R)/(\mathbb{Z}/l\mathbb{Z})$, l odd (*a posteriori*, $l = n+3$), and the natural map $\pi : C \to J(R)/(\mathbb{Z}/l\mathbb{Z})$. Then, one proves that for a generic curve C of genus 2, $\pi(C)$ contains exactly 6 points in $J(R)/(\mathbb{Z}/l\mathbb{Z})[2]$, the 2-torsion of $J(R)/(\mathbb{Z}/l\mathbb{Z})$, and these points are smooth in $\pi(C)$ ([BT00], Lemma 2.9). To do this, one proves that, if x is a torsion point in $J(C)$ such that $x \in C$, then x is one of the 6 standard points of $J(C)[2]$, and then that this implies the claim. Here one uses the universal family

$$F(\mathcal{C}_{\mathcal{T}}, \mathcal{T}) : \mathcal{C}_{\mathcal{T}} \to \mathcal{T}$$

of smooth curves of genus 2 and level 2, which is embedded in the universal family

$$F(\mathcal{J}_{\mathcal{T}'}, \mathcal{T}') : \mathcal{J}_{\mathcal{T}'} \to \mathcal{T}'$$

of Jacobians of dimension 2 and level 2. The family $F(\mathcal{C}_{\mathcal{T}}, \mathcal{T})$ has 6 natural sections, which correspond to a point in $J(C_t)[2]$ for each $t \in \mathcal{T}$. The family $F(\mathcal{J}_{\mathcal{T}'}, \mathcal{T}')$ has 16 natural sections, and 6 of them are contained in $\mathcal{C}_{\mathcal{T}}$. One also uses the monodromy of the family $F(\mathcal{C}_{\mathcal{T}}, \mathcal{T})$ and torsion multisections of the family $F(\mathcal{J}_{\mathcal{T}'}, \mathcal{T}')$, in a similar way to Section 5.4, 5.5.

Then, one proves that the self-intersection $(\pi(C), \pi(C)) = 2l$ ([BT00], Lemma 2.10). One considers the blow-up S_{l-3} of the quotient $J(C)/D_{2l}$ at the images of the 16 points in $J(C)[2]$, where D_{2l} is the dihedral group (of order $2l$). Then, one proves that S_{2l} is a K3 surface, and that the rational curve is the image of $\pi(R)/(\mathbb{Z}/2\mathbb{Z})$ in S_{2l}, of self-intersection $l-3$. □

Lemma 5.2.5: ([BT00], Lemma 2.12.)

Let $n \in \mathbb{Z}^+$ be even, let S_n be as in Lemma 5.2.4, and let $x \in \Lambda_{\text{eff}}(S)$ be primitive such that $x.x = n$. Then, there exists a smooth family

$$F(\mathcal{S}_{\mathcal{T}''}, \mathcal{T}'') : \mathcal{S}_{\mathcal{T}''} \to \mathcal{T}''$$

of K3 surfaces such that:

1. $\dim(\mathcal{T}'') = 1$.
2. There exist $t_0, t_1 \in \mathcal{T}''$ such that $S_{t_0} := \mathcal{S}_{\mathcal{T},t_0} = S_n$ and $S_{t_1} := \mathcal{S}_{\mathcal{T},t_1} = S$.
3. For all $t \in \mathcal{T}''$, the deformation x_t of $x \in \text{Pic}(S)$ in $\text{Pic}(S_t)$ is in $\Lambda_{\text{eff}}(S_t)$.
4. The class $x_{t_1} \in \Lambda_{\text{eff}}(S_{t_1})$ is represented by a rational curve.

Proof. (Idea.) One considers the moduli space D_x of marked K3 surfaces with a fixed algebraic class x (see Remark 3.2.11). The quotient $D'_x := D_x/\operatorname{Stab}(x, O(D_x))$ is a coarse moduli space of K3 surfaces with a fixed algebraic class x. The quotient D''_x of D'_x by a subgroup of $\operatorname{Stab}(x)$ of finite index acting freely on D_x is a fine moduli space of K3 surfaces with a fixed algebraic class x, a polarisation of a generic point. One takes $x = C_n$ as in Lemma 5.2.4 and finds the base points t_n and t of S_n and S, respectively, in D''_x. One connects t_n and t by a curve, and argues about the images of x in the fibers on the curve. One uses Theorem 5.1.2. □

Theorem 5.2.6: ([BT00], Theorem 2.4.)

Let $x \in \Lambda_{\text{eff}}(S)$. Then, x is represented by a sum of rational curves. In particular, if x is primitive and a generator of $\Lambda_{\text{eff}}(S)$, then x is represented by an irreducible rational curve.

Proof. We may assume $x \in \Lambda_{\text{eff}}(S)$ is primitive. By Item 4 of Lemma 5.2.2, we may further assume $n = x.x$ is positive and even. Then, by Lemma 5.2.4, we may realise n as the self-intersection of a rational curve C_n on a K3 surface S_n. By Lemma 5.2.5, we may deform S_n and C_n to S and a rational curve representing x. □

5.3 Multisections and Tate-Shafarevich group

Let $p : S \to C$ be an elliptic fibration over $\mathbb{C}$ (i. e., S is a smooth projective surface, C is a smooth projective irreducible curve, and the generic fiber E is a smooth projective curve of genus 1, all over $\mathbb{C}$). Recall that we may also study the fibration *analytically* (see Proposition 2.1.7).

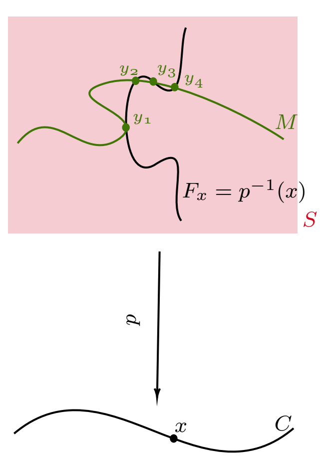

Figure 5.2: A multisection.

Definition 5.3.1

A **multisection** on S is an irreducible subvariety (resp., submanifold) M of S of dimension 1 such that the restriction $p|_M : M \to C$ is surjective (its degree $\deg(p|_M)$ is non-zero) and finite.

Definition 5.3.2

Let

$$\hat{M} = \sum_i n_i M_i$$

be a finite formal linear combination of multisections with integral coefficients. Then:

1. The **degree of** $\hat{M}$ is the degree of the restriction

$$d_S(\hat{M}) := \deg(p|_{\hat{M}}).$$

2. Let $i : S \hookrightarrow J(S)$ be the natural inclusion (which extends to algebraic 0-cycles), and let E be the generic fiber. The **class map of** $\hat{M}$ is the map

$$\tau_{\hat{M}} : S \to J(S), \quad x \mapsto d_S(\hat{M}) \cdot i(x) - i(\hat{M} \cap E).$$

Remark 5.3.3

Multisections generalise sections as they allow for more complex intersections with the fibers of p (e. g., tangential, multiple, etc.) Recall that the number of points (with multiplicities) on a fiber of $p|_M$ is finite, constant, and equal to $d_S(M)$ ([Har77], Chapter II, Proposition 6.9). In particular, as M is irreducible and of dimension 1, M is also a divisor on S, thus $d_S(M) = M.E$. Figure 5.2 may be useful.

The Picard number $\rho(E)$ of the generic fiber E is equal to 1. More specifically, the short exact sequence of abelian groups

$$0 \to \mathrm{Pic}^0(E) \hookrightarrow \mathrm{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \to 0$$

induces an isomorphism from the Néron-Severi group of E

$$\hat{\deg} : \mathrm{NS}(E) = \mathrm{Pic}(E)/\,\mathrm{Pic}^0(E) \xrightarrow{\sim} \mathbb{Z}.$$

Moreover, the restriction

$$r : \mathrm{Pic}(S) \to \mathrm{Pic}(E)/\,\mathrm{Pic}^0(E) \xrightarrow{\hat{\deg}} \mathbb{Z}, \quad \mathcal{L} \mapsto \hat{\deg}(\mathcal{L}|_E + \mathrm{Pic}^0(E))$$

is well-defined.

Definition 5.3.4

The **degree of** S is the index

$$d_S := [\mathbb{Z} : r(\mathrm{Pic}(S))].$$

Proposition 5.3.5

Let M be a multisection of degree $d_S(M)$ on S. Then, $d_S | d_S(M)$.

Proof. Simply note that, by Bezout's lemma,

$$r(\mathrm{Pic}(S)) = \langle r(\mathcal{L})\mathbb{Z} : \quad \mathcal{L} \in \mathrm{Pic}(S)\rangle = \gcd(\{r(\mathcal{L}) \in \mathbb{Z} : \quad \mathcal{L} \in \mathrm{Pic}(S)\})\mathbb{Z},$$

so

$$d_S = \gcd(\{r(\mathcal{L}) \in \mathbb{Z} : \quad \mathcal{L} \in \mathrm{Pic}(S)\}),$$

and that

$$d_S(M) = r(\mathcal{O}_S(M)).$$

□

Lemma 5.3.6

Let $D, D' \in \mathrm{Div}(S)$ such that D is not a multiple of D', D' is effective, and $D.D' < 0$. Then,

$$h^0(S, \mathcal{O}_S(D)) = 0.$$

Proof. If, for all D'' representing $\mathcal{O}_S(D)$, D'' is not effective, then claim follows. If there exists D'' effective, then $D''.D' = D.D' := \deg(\mathcal{O}_S(D)|_{D'})$, and there exists a point $P \in D \cap D'$ such that $v_P(D'') = v_P(D) < 0$, a contradiction. □

Proposition 5.3.7: ([BT00], Lemma 3.5.)

There exists a multisection M of degree $d_S(M) = d_S$ on S.

Proof. Let E be a general fiber. By the proof of Proposition 5.3.5, there exists $D \in \mathrm{Div}(S)$ such that

$$D.E := \deg(\mathcal{O}_S(D)|_E) = r(\mathcal{O}_S(D)) = d_S.$$

(Actually, this works if E is the generic fiber, but it is still true after specialising). Let $D_n =$

$D + n \cdot E, n \in \mathbb{Z}^+$. By the Riemann-Roch theorem for surfaces (Theorem 1.2.9),

$$\begin{aligned}
\chi(D_n) &= \chi(0_S) + \frac{D_n.(D_n - K_S)}{2} \\
&= \chi(0_S) + \frac{D.D + 2n(D.E) + n^2(E.E) - D.K_S - n(E.K_S)}{2} \\
\text{(Theorem 2.5.1)} \quad &= \chi(0_S) + \frac{D.D + 2n(D.E) + n^2(E.E) - A_{S,C}(D.E - n(E.E))}{2} \\
\text{(Lemma 5.2.2.2)} \quad &= \chi(0_S) + \frac{D.D + 2n(D.E) - A_{S,C}(D.E)}{2},
\end{aligned}$$

thus, there exists $n_0 \in \mathbb{Z}^+$ such that for all $n \geq n_0$, $\chi(D_n) > 0$. Fix $n \geq n_0$. Similarly, we also calculate

$$\begin{aligned}
(K_S - D_n).E &= K_S.E - D.E - n(E.E) \\
&= A_{S,C}(E.E) - D.E - n(E.E) \\
&= -D.E \\
&= -d_S < 0.
\end{aligned}$$

Then, by Serre duality and Lemma 5.3.6 (as $E.E = 0$, $K_S - D_n$ is not a multiple of E),

$$h^2(S, \mathcal{O}_S(D_n)) = h^0(S, \omega_S \otimes \mathcal{O}_S(D_n)^\vee) = 0.$$

This implies that

$$h^0(S, \mathcal{O}_S(D_n)) = \chi(\mathcal{O}_S(D_n)) + h^1(S, \mathcal{O}_S(D_n)) - h^2(S, \mathcal{O}_S(D_n)) > 0.$$

In other words, $\mathcal{O}_S(D_n) \in \mathrm{Pic}(S)$ is represented by an effective divisor, thus it is a multisection. Finally, note that

$$d_S(D_n) = D_n.E = D.E = d_S.$$

□

Corollary 5.3.8: ([BT00], Corollary 3.6.)

The order of $[S]$ in $H^1(C, J(S))$ is equal to d_S.

Proof. This is a consequence of the construction of the elliptic fibrations J^m which we will talk about later on. More details may be found in e. g. [Huy16] Chapter 11, Remark 4.4. □

Let $\hat{p} : J(S) \to C$ be the Jacobian elliptic fibration associated to S (see Section 2.2), let $\hat{F}_x = \hat{p}^{-1}(x)$ be its fiber over $x \in C$, and let $i : S \to J(S)$ be the natural inclusion.

Notation 5.3.9

If C is an algebraic curve, we will omit the inclusion $C \hookrightarrow J(C)$ from C to its Jacobian. In other words, if $x \in C$, we will also write $x \in J(C)$.

Definition 5.3.10

Let M be a multisection on S (see Figure 5.2). Then:

1. M is **torsion of order** m if $m \in \mathbb{Z}^+$ is minimal such that for all $x \in C$ and $y_1, y_2 \in M \cap F_x$, $i(y_1) - i(y_2) \in J(\hat{F}_x)[m]$ (i. e., $y_1 - y_2$ is torsion of order m in the Jacobian of $\hat{F}_x$).
2. M is **non-torsion** or a **nt-multisection** if for a general point $x \in C$ there exist $y_1, y_2 \in M \cap F_x$ such that $i(y_1) - i(y_2) \in J(\hat{F}_x)$ is non-torsion.

The following proposition is not absolutely trivial but serves as a reality check.

Proposition 5.3.11: ([BT00], Lemma 3.8.)

Let M be a multisection on S. If for all $m \in \mathbb{Z}^+$, M is not torsion of order m, then M is a nt-multisection.

Proof. For each $m \in \mathbb{Z}^+$ and fiber $\hat{F}_x$, the torsion $J(\hat{F}_x)[m]$ is a finite subgroup. Thus, the union T of all torsion multisections of S is an at most countable union of divisors. The intersection of M and T is of dimension 0 or 1. If it is of dimension 0, the claim holds. If it is of dimension 1, as M is irreducible, $M \subset T$. Again, as M is irreducible, M is contained in a torsion multisection, a contradiction. □

Notation 5.3.12

Let $p : S' \to C'$ be an elliptic surface. We will change our notation and write S'_x for the fiber above $x \in C'$.

Now, let $p : S \to \mathbb{P}^1$ be an elliptic K3 surface over $\mathbb{C}$. Recall the existence and properties of the Jacobian elliptic surface $\hat{p} : J(S) \to \mathbb{P}^1$ (see Section 2.2). The Jacobian elliptic surface of a K3 surface is a K3 surface. A general fiber $J(S)_x$ satisfies

$$J(S)_x \cong J(S_x) \cong \mathrm{Pic}^0(S_x) := \{\mathcal{L} \in \mathrm{Pic}(S_x) : \quad \deg(\mathcal{L}) = 0\}.$$

Let $m \in \mathbb{Z}^+$. There exists a similar construction which results in an elliptic K3 surface $p^m : J^m(S) \to \mathbb{P}^1$ such that a general fiber $J^m(S)_x$ satisfies

$$J^m(S)_x \cong \mathrm{Pic}^m(S_x) := \{\mathcal{L} \in \mathrm{Pic}(S_x) : \quad \deg(\mathcal{L}) = m\}.$$

Soon we will see that $J^0(S) \cong J(S)$ and that $J^1(S) \cong S$. The interested reader is referred to [Huy16], Chapter 11, Section 4, for more details, although we will soon state the properties we will use.

Fix $m \in \mathbb{Z}^+$. Let M be a multisection on S such that $d_S(M) = m$. M is also a divisor on S (see Remark 5.3.3), and the restriction $\alpha := \mathcal{O}_S(M)|_E \in \mathrm{Pic}(E)$ to a general fiber E of S is of degree

$$\deg(\alpha) := \deg(\mathcal{O}_S(M)|_E) = M.E = d_S(M) = m.$$

Therefore, a multisection M of degree m induces a section of $p^m : J^m(S) \to \mathbb{P}^1$.

Figure 5.3 summarises the situation.

$$
\begin{array}{ccc}
\left\{\begin{matrix}\text{Multisections } M \text{ on } S \\ \text{such that } d_S(M) = m\end{matrix}\right\} & & M \\
\big\downarrow & & \big\downarrow \\
\left\{\begin{matrix}\text{Classes of divisors } \alpha \in \operatorname{Pic}(E) \\ \text{such that } \deg(\alpha) = m\end{matrix}\right\} & & \alpha := \mathcal{O}_S(M)|_E \\
\big\updownarrow & & \\
\left\{\begin{matrix}\text{Sections of the elliptic K3 surface} \\ p^m : J^m(S) \to \mathbb{P}^1\end{matrix}\right\} & &
\end{array}
$$

Figure 5.3: A multisection induces a section of $p^m : J^m(S) \to \mathbb{P}^1$.

Proposition 5.3.13: (Cf. [BT00].)

Let $m, k \in \mathbb{Z}_0^+$. The following are rational maps of algebraic varieties which are regular in the open subvarieties obtained by removing all singular fibers:

1. $$J^m(S) \times_{\mathbb{P}^1} J^k(S) \to J^{m+k}(S).$$

2. $$\eta^m : J^k(S) \to J^{mk}(S).$$

Proof. 1. On smooth fibers $J^m(S)_x \cong \operatorname{Pic}^m(S_x)$ and $J^k(S)_x \cong \operatorname{Pic}^k(S_x)$ add the line bundles (the degree map is linear).

2. Either embed $J^k(S)$ diagonally in $J^k(S) \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} J^k(S)$ (m times) and apply the map in Item 1, or on a smooth fiber $J^k(S)_x \cong \operatorname{Pic}^k(S_x)$ multiply the line bundles by m.

□

Proposition 5.3.14

There exist isomorphisms $J(S) \xrightarrow{\sim} J^0(S) \xrightarrow{\sim} J^{d_S}(S)$, $S \xrightarrow{\sim} J^1(S)$. The isomorphism $J^0(S) \xrightarrow{\sim} J^{d_S}(S)$ is not canonical: it depends on a choice of multisection M on S of degree d_S.

Proof. Clearly, there exist biregular maps $J(S) \xrightarrow{\sim} J^0(S)$ and $S \xrightarrow{\sim} J^1(S)$. Choose a multisection M as in Proposition 5.3.7. Then, on a smooth fiber,

$$\operatorname{Pic}^0(S_x) \to \operatorname{Pic}^{d_S}(S_x), \quad \mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_{S_x}(M).$$

These biregular maps extend to isomorphisms. □

We will now work toward defining the (algebraic and complex) *Tate-Shafarevich group* of an elliptic K3 surface, which parametrises all elliptic K3 surfaces with a fixed Jacobian. We start with the case of an elliptic curve over a function field.

Definition 5.3.15

Let E be an elliptic curve over $\mathbb{C}(\mathbb{P}^1)$. Then:

1. An **E-torsor** is a smooth projective curve E' of genus $g(E') = 1$ over $\mathbb{C}(\mathbb{P}^1)$ with a simply transitive additive action of E on E'.

2. An **isomorphism of torsors** from E' to E'' is an isomorphism from E' to E'' that is compatible with the action.

3. The **Tate-Shafarevich group** $\mathrm{Sh}(E)$ is the group of E-torsors modulo isomorphism of torsors.

Proposition 5.3.16: (Cf. [Huy16], Chapter 11, Section 5.)

There is a natural bijection

$$\mathrm{Sh}(E) \leftrightarrow \{(E', f') : \quad f' : E \xrightarrow{\sim} J(E')\}/\sim,$$

where (E', f') and (E'', f'') are equivalent if there exists an isomorphism from E' to E'' compatible with the isomorphisms f' and f''.

Proposition 5.3.17: ([Huy16], Chapter 11, Remark 5.2.)

Let $E, \hat{E}$ be the generic fibers of $p : S \to \mathbb{P}^1, \hat{p} : J(S) \to \mathbb{P}^1$, respectively, and let $m \in \mathbb{Z}_0^+$. Then, E and the generic fiber E^m of $J^m(S)$ admit a canonical structure of $\hat{E}$-torsor. In particular, $\hat{E} \cong J(E) \cong J(E^m)$. The class $[E^m] \in \mathrm{Sh}(E)$ equals $m[E]$.

Definition 5.3.18

Let $J(S)^{\mathrm{smooth}}$ be the set of smooth points of the fibers of $\hat{p} : J(S) \to \mathbb{P}^1$, which is a group scheme on $\mathbb{P}^1$. Then:

1. A **$J(S)^{\mathrm{smooth}}$-torsor** is an elliptic surface $p' : S' \to \mathbb{P}^1$ with an action of $J(S)^{\mathrm{smooth}}$ on S'^{smooth} such that the fibers $S_t'^{\mathrm{smooth}}$ are $J(S)_t^{\mathrm{smooth}}$-torsors.

2. The **Tate-Shafarevich group** $\mathrm{Sh}(J(S))$ is the set of $J(S)^{\mathrm{smooth}}$-torsors modulo isomorphism of torsors.

Proposition 5.3.19: ([Huy16], Chapter 11, Corollary 5.5.)

Let $\hat{E}$ be the generic fiber of $p : J(S) \to \mathbb{P}^1$. Then:

1. There is a natural isomorphism from $\mathrm{Sh}(\hat{E})$ to $\mathrm{Sh}(J(S))$.
2. There is a natural bijection

$$\mathrm{Sh}(J(S)) \leftrightarrow \left\{ (S', f') : \begin{array}{c} S' \text{elliptic K3 surface,} \\ f' : J(S) \xrightarrow{\sim} J(S') \text{ compatible with} \\ \text{the group schemes } J(S)^{\mathrm{smooth}} \text{ and } J(S')^{\mathrm{smooth}} \end{array} \right\} / \sim,$$

where (S', f') and (S'', f'') are equivalent if there exists an isomorphism from S' to S'' compatible with the isomorphisms f' and f''.

Corollary 5.3.20

The class $[J^m(S)] \in \mathrm{Sh}(J(S))$ equals $m[S]$. In particular, in $\mathrm{Sh}(J(S))$ we have

$$[J(S)] = [J^0(S)] = [J^{d_S}(S)] = d_S[S] = 0, \quad [S] = [J^1(S)],$$

and $\{[J^m(S)]\}_{m \in \mathbb{Z}_0^+}$ is a cyclic subgroup of order d_S.

Proposition 5.3.21: (Cf. [Huy16], Chapter 11, proof of Proposition 5.6, Remark 5.13.)

There exist isomorphisms

$$\mathrm{Sh}(J(S)) \xrightarrow{\sim} H^1(\mathbb{P}^1, J(S)) \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})^{22-\rho(J(S))},$$

where in the middle term $J(S)$ is viewed as a sheaf of abelian groups on $\mathbb{P}^1$. In particular, $\mathrm{Sh}(J(S))$ is infinitely divisible and has torsion of all orders.

Remark 5.3.22: ([Huy16], Chapter 11, 5.3.)

If $p : S' \to \mathbb{P}^1$ is a complex elliptic K3 surface, it can be shown that S' is projective if and only if $d_{S'}$ is finite. In particular, if $p : S' \to \mathbb{P}^1$ is Jacobian, it is algebraic. The *analytic* Tate-Shafarevich group $\mathrm{Sh}^{\mathrm{an}}(J(S))$ is defined similarly to the algebraic Tate-Shafarevich group, but may parametrise non-algebraic elliptic K3 surfaces. Everything we have discussed so far still holds in the analytic case, except that $\mathrm{Sh}^{\mathrm{an}}(J(S))) \cong (\mathbb{C}/\mathbb{Z})^{22-\rho(J(S))}$. We also have that $\mathrm{Sh}(J(S))$ is isomorphic to the torsion subgroup of $\mathrm{Sh}^{\mathrm{an}}(J(S))$.

Notation 5.3.23

Let $m \in \mathbb{Z}_0^+$. We will write $J^m := J^m(S)$.

Proposition 5.3.24: ([BT00], Lemma 3.9.)

Let M be a multisection on J^k. Then:

1. If M is torsion of order t, $\eta^m(M)$ is a torsion multisection of order $t/\gcd(t,m)$ on J^{mk}.
2. If M is non-torsion or torsion of order coprime to m, the restriction

$$\eta^m|_M : M \to \eta^m(M)$$

is a birational map and $d_{J^k}(M) = d_{J^{km}}(\eta^m(M))$.

Proof. Let $x \in C$, and let $y_1 = \eta^m(x_1), y_2 = \eta^m(x_2) \in \eta^m(M) \cap J^{mk}_x$ such that $y_1 \neq y_2$. We want to find the minimal $A \in \mathbb{Z}^+$ such that

$$\begin{aligned} A(i(y_1) - i(y_2)) &= A(i(\eta^m(x_1)) - i(\eta^m(x_2))) \\ &= A(mi(x_1) - mi(x_2)) \\ &= Am(i(x_1) - i(x_2)) \\ &= 0 \in J(\hat{F}_x). \end{aligned}$$

If M is torsion of order t, $t \in \mathbb{Z}^+$ is minimal such that

$$t(i(x_1) - i(x_2)) = 0 \in J(\hat{F}_x),$$

so $t|Am$. Then, $Am = \operatorname{lcm}(m,t)$, and

$$A = \frac{\operatorname{lcm}(m,t)}{m} = \frac{t}{\gcd(t,m)}.$$

Now, if M is any multisection on J^k, the condition

$$\eta^m(x_1) = \eta^m(x) \tag{5.3}$$

is closed, the map η^m is rational, and M is irreducible, thus it is true either in a dense open subset of M (Case 1) or in a divisor D of M which contains x_1 (Case 2). Clearly, if $x = x_2$, Condition 5.3 implies

$$m(i(x_1) - i(x_2)) = 0,$$

so $t|m$. If M is non-torsion, Case 2 holds, and if M is torsion of order coprime to m, Case 2 holds and $D = x_1$. If Case 2 holds, the restriction

$$\eta^m|_M : M \to \eta^m(M)$$

is biregular in the open subvariety obtained by removing all intersections of M and $\eta^m(M)$ a singular fiber, and D and $\eta^m(D)$. Then, $d_{J^k}(M) = d_{J^{km}}(\eta^m(M))$. $\square$

Corollary 5.3.25: ([BT00], Corollary 3.10.)

Let $q \in \mathbb{Z}^+$ be prime, let $p : S \to C$ be an elliptic fibration such that $d_S = q$, and let M be a torsion multisection of order t on $J^1 = S$. Then, there exists a surjective map from M to a torsion multisection of order q on S or to a non-zero torsion multisection of order q on $J(S)$.

Proof. W. l. o. g., assume $q|t$, and write $t = q^k r, \gcd(q, r) = 1, k \geq 1$. If $k = 1$, choose $a \in \mathbb{Z}^+$ such that $ar = 1 \mod q$. By Proposition 5.3.24, $\eta^{ar}(M)$ is a torsion multisection of order

$$\frac{t}{\gcd(t, ar)} = \frac{qr}{\gcd(qr, ar)} = q$$

on $J^{ar} \cong J^1 \cong S$. If $k > 1$, $\eta^{q^{k-1}r}(M)$ is a torsion multisection of order

$$\frac{t}{\gcd(t, q^{k-1}r)} = \frac{q^k r}{\gcd(q^k r, q^{k-1}r)} = q$$

on $J^{q^{k-1}r} \cong J^0 \cong J(S)$. As $\eta^{\cdot}$ acts as integer multiplication on $J(S)$, this multisection is non-zero. $\square$

5.4 Monodromy

Let $p : S \to C$ be an elliptic fibration over $\mathbb{C}$.

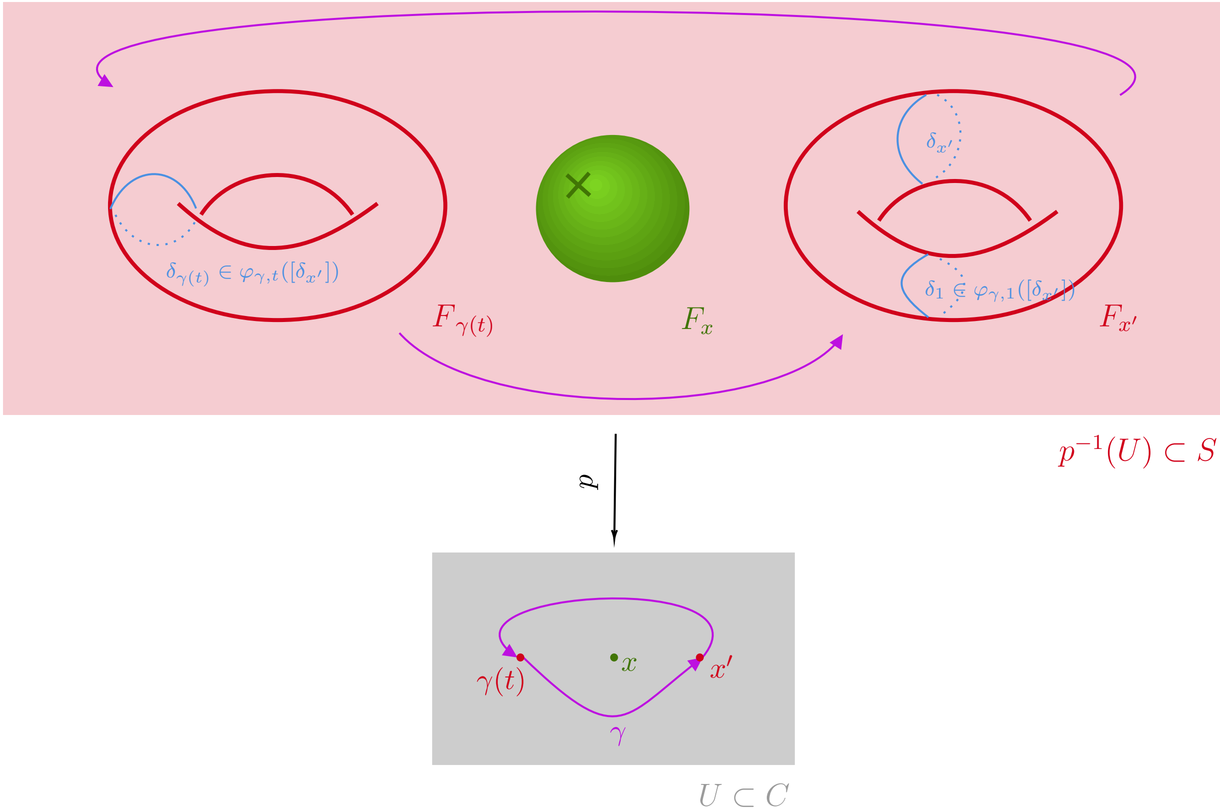

Figure 5.4: The local monodromy action of $\pi_1(U \setminus \{x\})$ on $H_1(F_{x'}, \mathbb{Z})$.

Restrict the fibration to a sufficiently small neighbourhood $U \subset C$ of a singular fiber above $x \in U$ that does not contain any other singular fiber. Let $x' \in U \setminus \{x\}$. Choose a simple loop $\gamma : [0,1] \to U$ that starts and ends at x' (i. e., $\gamma(0) = \gamma(1) = x'$) and winds around x in the counterclockwise direction.

As $F_t, t \in U \setminus x$ is an elliptic curve (passing to the Jacobian $J(S)$ if necessary), we may view it as a quotient $F_t \cong \mathbb{C}/L_t$ for some lattice $L_t \cong \mathbb{Z}^2$ of rank 2.

Proposition 5.4.1: (Cf. [Mir89], VI.2.)

Let $t \in U \setminus \{x\}$. The first integral homology group $H_1(F_t, \mathbb{Z})$ is isomorphic (as a lattice) to a lattice $L_t \cong \mathbb{Z}^2$ of rank 2 such that $F_t \cong \mathbb{C}/L_t$, i. e.,

$$H_1(F_t, \mathbb{Z}) \cong L_t.$$

Proof. (Idea.) There is an isomorphism d from $\mathbb{C}^*$ to $H^0(F_t, \Omega^1_{F_t})$. The isomorphism from L_t to $H_1(F_t, \mathbb{Z})$ is given by

$$L_t \to H_1(F_t, \mathbb{Z}), \quad x \mapsto \gamma_x := \{tx : \quad t \in [0,1]\} \mod L_t,$$

and the inner products are given by the integrals of dx^* along γ_x. $\square$

Proposition 5.4.2: (Cf. [Mir89], VI.2.)

Let $t \in [0,1]$. The loop γ induces an isomorphism

$$\varphi_{\gamma,t} : H_1(F_{x'}, \mathbb{Z}) \xrightarrow{\sim} H_1(F_{\gamma(t)}, \mathbb{Z}).$$

In particular, γ induces an automorphism $\varphi_{\gamma,1} \in \mathrm{Aut}(H_1(F_{x'}, \mathbb{Z}))$ which depends only on the homotopy class $[\gamma] \in \pi_1(U \setminus \{x\})$ (i. e., it does not depend on the representative γ or on the point x'). In other words, the group $\pi_1(U \setminus \{x\})$ acts on $H_1(F_{x'}, \mathbb{Z})$ (see Figure 5.4).

Furthermore, if we fix a basis $\{e_1, e_2\}$ of the lattice $L_{x'}$, γ induces a unique automorphism of $L_{x'}$ $\hat{\varphi}_{\gamma,1} \in O(L_{x'}) \cong \mathrm{SL}^{\pm}(2, \mathbb{Z})$ such that $\hat{\varphi}_{\gamma,1} \in \mathrm{SL}(2, \mathbb{Z})$. If not, γ induces a conjugacy class in $\mathrm{SL}(2, \mathbb{Z})$.

Proof. (Idea.) Let $[\delta_{x'}] \in H_1(F'_x, \mathbb{Z})$. We have an inclusion $j : H_1(F'_x, \mathbb{Z}) \hookrightarrow H_1(F'_x, \mathbb{C}) \cong \mathbb{C}$. $j([\delta_{x'}])$ varies continuously as t varies along $[0,1]$ (and $\gamma(t)$ varies along γ), giving the isomorphism $\varphi_{\gamma,t}$ and the automorphism $\varphi_{\gamma,1}$ (i. e., the action of $\pi_1(U \setminus \{x\})$ on $H_1(F_{x'}, \mathbb{Z})$).

Let $\{e_1, e_2\}$ be a basis of the lattice $L_{x'}$. By Proposition 5.4.1, if $t \in [0,1]$, the isomorphism of abelian groups $\varphi_{\gamma,t}$ induces an isomorphism of lattices

$$\hat{\varphi}_{\gamma,t} : L_{x'} \xrightarrow{\sim} L_t$$

which preserves the orientation of the basis $\{e_1, e_2\}$. □

Definition 5.4.3

The **local monodromy around** x is the conjugacy class in $\mathrm{SL}(2, \mathbb{Z})$ induced by γ in Proposition 5.4.2. The **local monodromy group around** x is the cyclic subgroup T_x of $\mathrm{SL}(2, \mathbb{Z})$ generated by a representative of the local monodromy. In other words, if we fix a basis $\{e_1, e_2\}$ of the lattice $L_{x'}$, T_x is the image of $\pi_1(U \setminus \{x\})$ in $\mathrm{SL}(2, \mathbb{Z})$.

We state the following fact without proof.

Proposition 5.4.4: ([Mir89], VI.2.1.)

The local monodromy around x depends only on the type of the singular fiber above x (i. e., it does not depend on any other information of the elliptic fibration $p : S \to C$!) The classification is given by Table 5.1.

Let $X := \{x_i\}_{1 \le i \le n} \in C$ be the bases of the singular fibers, and let $x' \in C \setminus X$. The construction "globalises" naturally by giving an action of $\pi_1(C \setminus X)$ on $H_1(x', \mathbb{Z})$.

Definition 5.4.5

Fix a basis $\{e_1, e_2\}$ of the lattice $L_{x'}$. The **global monodromy group** $\Gamma = \Gamma(S)$ is the image of $\pi_1(C \setminus X)$ in $\mathrm{SL}(2, \mathbb{Z})$.

Singular fiber type	Representative of local monodromy
I_n	$A := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
II	$B := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
III	$C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
IV	$D := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
I_n^*	$-A$
II^*	$-D$
III^*	$-C$
IV^*	$-B$

Table 5.1: Classification of local monodromy around singular fibers ([Mir89], VI.2.1.)

We also state the following facts without proof. For Item 3, the idea is that a generic elliptic fibration $p : S \to \mathbb{P}^1$ has only nodal singular fibers. If

$$a := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are the standard generators of $\mathrm{SL}(2, \mathbb{Z})$, this implies that there are two subsets X_a, X_b of X of equal cardinality such that $X = X_a \sqcup X_b$ and

$$T_{x_i} = \begin{cases} \langle a \rangle, & x_i \in X_a, \\ \langle b \rangle, & x_i \in X_b. \end{cases}$$

In particular, choosing $x_a \in X_a$ and $x_b \in X_b$,

$$\mathrm{SL}(2, \mathbb{Z}) = \langle T_{x_a} \cup T_{x_b} \rangle \subset \Gamma \subset \mathrm{SL}(2, \mathbb{Z}).$$

For Item 4, the idea is that the global monodromy group is completely determined by an isolated set of points (the bases of the singular fibers).

Proposition 5.4.6: ([BT00], Remark 3.13, 3.14, 3.18)

1. Let $\hat{p} : J(S) \to C$ be the Jacobian elliptic fibration associated to S. Then, the global monodromy group $\Gamma(S)$ of S is isomorphic to the local monodromy global monodromy group $\Gamma(J(S))$ of $J(S)$.

2. If the elliptic fibration $p : S \to C$ is locally isotrivial (e. g., if it is isotrivial, see Definition 2.3.9), the global monodromy group Γ is finite. If it is not isotrivial, the index $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$ is finite.

3. The global monodromy group $\Gamma(S)$ of a generic elliptic fibration $p : S \to \mathbb{P}^1$ is equal to $\mathrm{SL}(2,\mathbb{Z})$.

4. Let
$$F(\mathcal{S},\mathcal{T}) : \mathcal{S} \to \mathcal{T}$$
be a family of elliptic fibrations such that $\mathcal{T}$ is an algebraic variety. Then, there exists an algebraic subvariety (generally of high codimension) $\mathcal{T}'$ of $\mathcal{T}$ such that, for all $t_1, t_2 \in \mathcal{T} \setminus \mathcal{T}'$, $\Gamma(S_{t_1}) = \Gamma(S_{t_2})$.

A Jacobian elliptic fibration $p : S \to \mathbb{P}^1$ is given in Weierstrass form

$$y^2 = x^3 + A(t)x + B(t), \quad A, B \in \mathcal{O}(\mathbb{P}^1).$$

In particular, the degrees of A and B satisfy

$$\deg(A) = 4r, \deg(B) = 6r, \quad r \in \mathbb{Z}^+,$$

and the j-map is

$$j : \mathbb{P}^1 \to \mathbb{P}^1, \quad t \mapsto \frac{4A(t)^3}{4A(t)^3 + 27B(t)^2}.$$

This discussion implies the following.

Remark 5.4.7: ([BT00], Remark 3.15.)

Let $\mathcal{F}$ be the family of all Jacobian elliptic fibrations $p : S \to \mathbb{P}^1$. Then, there exist families $\mathcal{F}_r, r \in \mathbb{Z}^+$ such that
$$\mathcal{F} = \bigsqcup_{r=1}^{\infty} \mathcal{F}_r.$$
If $p : S \to \mathbb{P}^1$ belongs to the family $\mathcal{F}_r$, the degree of the j-map is bounded

$$0 \leq \deg(j) \leq 12r.$$

Proposition 5.4.8: ([BT00], Proposition 3.16.)

If the elliptic fibration $p : S \to C$ is not isotrivial, the index $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$ is bounded

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma] \leq 2\deg(j).$$

Proof. (Idea.) The j-map is defined identical for S and $J(S)$, so we may assume $p : S \to C$ is Jacobian. The j-map $j : C \to \mathbb{P}^1$ restricts to the smooth fibers $j^* : C^* \to (\mathbb{P}^1)^*$, and it can be shown that it induces a map to a quotient of the upper half-plane $j_H^* : C^* \to \Gamma\backslash H$ (a smooth fiber corresponds to a point in $\Gamma\backslash H$).

Then, if $r_\Gamma : \Gamma\backslash H \to \mathrm{SL}(2,\mathbb{Z})\backslash H$ is the restriction and $\hat{j} : \mathrm{SL}(2,\mathbb{Z})\backslash H \to \mathbb{P}^1$ is the j-invariant, we have

$$j^* = \hat{j} \circ r_\Gamma \circ j_H^*.$$

Therefore, the degrees satisfy

$$\deg(j) \geq \deg(j^*) \geq \deg(r_\Gamma),$$

and it can be shown that

$$\deg(r_\Gamma) = \begin{cases} [\mathrm{SL}(2,\mathbb{Z}) : \Gamma], & \text{if } \pm I_2 \subset \Gamma, \\ \frac{[\mathrm{SL}(2,\mathbb{Z}):\Gamma]}{2}, & \text{if not.} \end{cases}$$

□

Corollary 5.4.9: ([BT00], Corollary 3.17.)

Let $\mathcal{F}$ be a family of non-isotrivial elliptic fibrations $p : S \to C$, and suppose there exists $A > 0$ such that for all $p : S \to C$ in $\mathcal{F}$

$$0 \leq \deg(j) \leq A.$$

Then, the set

$$\{\Gamma(S) : \quad p : S \to C \in \mathcal{F}\}/ \text{ isomorphism}$$

is finite.

Proof. (Idea.) By Proposition 5.4.8, if there exists such a constant $A > 0$, for all $p : S \to C$ in $\mathcal{F}$,

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma(S)] \leq 2A.$$

In the case of $\mathrm{SL}(2,\mathbb{Z})$, this implies an upper bound for the number of possible subgroups $\Gamma(S)$. □

Corollary 5.4.10: ([BT00], Corollary 3.17.)

If $p : S \to \mathbb{P}^1$ belongs to the family $\mathcal{F}_r$ in Remark 5.4.7, the index $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$ is bounded

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma] \leq 24r.$$

Proof. This follows directly from Remark 5.4.7 and Proposition 5.4.8. □

Finally, we give the following fact without proof.

Proposition 5.4.11: ([BT00], Example 3.19.)

There exists a subvariety $\mathcal{F}'_r$ of $\mathcal{F}_r$ in Remark 5.4.7 such that

$$\dim(\mathcal{F}'_r) \leq \frac{\dim(\mathcal{F}_r)}{2} + 1$$

such that, if $p : S \to \mathbb{P}^1$ belongs to the family $\mathcal{F}_r$,

$$\Gamma(S) = \begin{cases} \mathrm{SL}(2,\mathbb{Z}), & \text{if } S \text{ belongs to } \mathcal{F}_r \setminus \mathcal{F}'_r, \\ \Gamma(S) \subsetneq \mathrm{SL}(2,\mathbb{Z}), & \text{if } S \text{ belongs to } \mathcal{F}'_r. \end{cases}$$

5.5 Torsion multisections and genus estimates

Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$, and let $p : S \to \mathbb{P}^1$ be a non-isotrivial elliptic fibration over $\mathbb{C}$ with global monodromy group isomorphic to Γ.

We state the following correspondence without proof.

Lemma 5.5.1: (Cf. [BT00].)

If $p : S \to \mathbb{P}^1$ is Jacobian and $m \in \mathbb{Z}^+$, there is a bijective correspondence

$$\big\{\text{Multisections } M \text{ of order } m \text{ on } S\big\} \leftrightarrow \left\{\begin{matrix} \text{For each } x \in \mathbb{P}^1, \\ \text{an orbit of the action of } \Gamma \text{ on } y \in J(F_x)[m] \\ \text{such that } y \notin J(F_x)[n] \text{ for all } n \leq m. \end{matrix}\right\},$$

$$M \mapsto \{M \cap F_x\}_{x \in \mathbb{P}^1}.$$

We also state the following group-theoretic fact without proof (many thanks are due to Prof. Sebastián Herrero, who took the time to convince the author of this fact).

Lemma 5.5.2

Let $L \cong \mathbb{Z}^2$ be a lattice of rank 2, and let $T = \mathbb{C}/L$ be a complex torus (an elliptic curve). $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{C}$, thus also on T. Let $m \in \mathbb{Z}^+$, and let $x \in T[m]$ be an m-torsion point such that $x \notin T[n]$ for all $n \leq m$. Then, the orbit of the action of $\mathrm{SL}(2,\mathbb{Z})$ on x has cardinality equal to

$$m^2 \prod_{p \text{ prime},\ p|m} (1 - 1/p^2).$$

In particular, if Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$, the orbit of the action of Γ on x has cardinality equal to

$$\frac{m^2}{[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]} \prod_{p \text{ prime},\ p|m} (1 - 1/p^2).$$

Proposition 5.5.3: ([BT00], Proposition 3.20.)

If $p : S \to \mathbb{P}^1$ is Jacobian and M is a torsion multisection of order m on S, the degree $d_S(M)$ is bounded:

$$d_S(M) > \frac{6m^2}{\pi^2[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]}.$$

Proof. Note that, if $x \in \mathbb{P}^1$,

$$d_S(M) \geq \# M \cap F_x.$$

By Lemma 5.5.1, it is sufficient to give a lower bound on the cardinality of an orbit of the action of Γ on $J(F_x)[m]$ satisfying an additional condition. As $p : S \to \mathbb{P}^1$ is Jacobian, the fiber F_x is an elliptic curve (isomorphic to its Jacobian), so by Lemma 5.5.2, if $y \in J(F_x)[m]$ is such that $y \notin J(F_x)[n]$ for all $n \leq m$, the orbit of the action of Γ on y has cardinality

$$\frac{m^2}{[\mathrm{SL}(2,\mathbb{Z}) : \Gamma]} \prod_{p \text{ prime},\ p|m} (1 - 1/p^2).$$

Recall that

$$1/\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})$$

and $\zeta(2) = \pi^2/6$, where $\zeta(\cdot)$ is the Riemann zeta function. As adding primes to the product makes it smaller, the bound holds. □

Proposition 5.5.4: ([BT00], Proposition 3.21.)

There exists a constant $m_0(\Gamma)$ (depending only on Γ and not $p : S \to \mathbb{P}^1$) such that, if $p : S \to C$ is Jacobian and has at least 4 singular fibers, $m \in \mathbb{Z}^+$ is a positive integer such that $m > m_0(\Gamma)$, and M is a torsion multisection of order m on S, the genus of M is bounded:

$$g(M) \geq 2.$$

Proof. (Idea.) By the Riemann-Hurwitz formula,

$$2g(M) - 2 = \deg(f)(2g(\mathbb{P}^1) - 2) + \sum_{x \in M} e_x,$$

where e_x is the ramification index at $x \in M$. This may be rewritten as (see e. g. [Has03], 9.1)

$$2g(M) - 2 = \deg(f)(2g(\mathbb{P}^1) - 2) + \sum_{x \in \mathbb{P}^1} (\deg(f) - \mathrm{orbits}(x)),$$

where orbits(x) is the number of orbits of the local monodromy group $T_x \subset \mathrm{SL}(2, \mathbb{Z})$ around x on m-torsion points of nearby fibers. Note that $g(\mathbb{P}^1) = 0$, so

$$2g(M) - 2 = \deg(f) \left(-2 + \sum_{x \in \mathbb{P}^1} \left(1 - \frac{\mathrm{orbits}(x)}{\deg(f)} \right) \right). \tag{5.4}$$

As $\deg(f) = d_S(M) = \#\{M \cap F_x\}, x \in \mathbb{P}^1$, by Lemma 5.5.1 and Proposition 5.5.3, it is sufficient that

$$\sum_{x \in \mathbb{P}^1} \left(1 - \frac{\mathrm{orbits}(x)}{\deg(f)} \right) > 2,$$

as m tends to infinity. It can be shown that (cf. [Has03], Table 1):

1. If F_x is a smooth fiber, then the term is zero.

2. If F_x is a multiplicative fiber, then orbits$(x) \approx 2m$.

3. If F_x is an additive fiber, then orbits$(x) \approx g(x)$, where $g(x) \leq 1/2(m^2 + c), c > 0$.

Now, by Proposition 5.5.3 again, $\deg(f) = d_S(M) > c_\Gamma m^2, 0 < c_\Gamma < 1$, so $1/\deg(f) < c'_\Gamma / m^2, c'_\Gamma > 1$. *This is not enough!* However, by doing more local calculations, it can be shown that:

1. If F_x is a multiplicative fiber, then $\mathrm{orbits}(x)/\deg(f) \xrightarrow{m \to \infty} 0$, so the term is 1.

2. If F_x is an additive fiber, then $\mathrm{orbits}(x)/\deg(f) \xrightarrow{m \to \infty} c \leq 1/2$, so the term is $\geq 1/2$.

As, by hypothesis, $p : S \to \mathbb{P}^1$ has at least 4 singular fibers, this is enough. □

Now, to obtain a similar result for the non-Jacobian case, recall our discussion about the Tate-Shafarevich group of S (see Section 5.3), and consider the following construction: Let M be a torsion multisection on S, and define the elliptic surface $p_\mathcal{M} : \mathcal{M} \to (\mathbb{P}^1)^*$ on smooth fibers by

$$\mathcal{M}_t := \{x \in F_t : \quad i(x) - i(M|_{F_t}) \subset J(\hat{F}_t)[d_S]\} = i(M|_{F_t}) + J(\hat{F}_t)[d_S] \cong i(M|_{F_t}) + (\mathbb{Z}/d_S\mathbb{Z})^2.$$

This elliptic surface is a $J(S)^*$-torsor, where $\hat{p}^* : J(S)^* \to (\mathbb{P}^1)^*$ is the restriction to the smooth fibers. Therefore, it defines a class $[\mathcal{M}] \in \mathrm{Sh}(J(S)^*)$. There is a natural inclusion $\mathrm{Sh}(J(S)^*) \hookrightarrow \mathrm{Sh}(J(S))$, and it can be shown that the degree $d_S | d_\mathcal{M}$.

The global monodromy action of $\pi_1((\mathbb{P}^1)^*)$ on a smooth fiber $\mathcal{M}_t$ is a morphism

$$A : \pi_1((\mathbb{P}^1)^*) \to \mathrm{Aut}(\mathcal{M}_t) \cong \mathrm{ASL}(2, \mathbb{Z}/d_S\mathbb{Z}).$$

The linear part of A factors as

$$A_{\text{linear}} : \pi_1((\mathbb{P}^1)^*) \to \Gamma \subset \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/d_S\mathbb{Z}).$$

It can be shown that the connected components of $\mathcal{M}$ are in bijective correspondence with orbits of the action of $\pi_1((\mathbb{P}^1)^*)$ on $\mathcal{M}$. By (a similar lemma to) Lemma 5.5.1, these are in bijective correspondence with torsion multisections M' on $\mathcal{M}$, and thus M defines many torsion multisections on S (one of which is M).

Lemma 5.5.5: ([BT00], similar to Lemma 3.23.)

If the natural projection $\Gamma \to \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ is surjective, $p : S \to \mathbb{P}^1$ is not Jacobian, $q \in \mathbb{Z}^+$ is a prime such that $d_S = q$, and M is a torsion multisection of order q, for a general $x \in \mathbb{P}^1$, the cardinality

$$\#M \cap F_x > q^2.$$

Proof. (Idea.) If $A = A_{\text{linear}}$, then, for a general fiber, the set $i(M|_{F_t})$ is a singleton, and M is a section, a contradiction. Then, $A_{\text{linear}}(\pi_1((\mathbb{P}^1)^*)) \ncong \mathrm{SL}(2, \mathbb{Z}/d_S\mathbb{Z})$, and it can be shown that $A(\pi_1((\mathbb{P}^1)^*))$ contains the group of translations by $(\mathbb{Z}/q\mathbb{Z})^2$, thus the orbit of the action of $\pi_1((\mathbb{P}^1)^*)$ on $\mathcal{M}_t$ corresponding to $M|_{F_t}$ contains at least q^2 points. □

Proposition 5.5.6: ([BT00], similar to Proposition 3.22.)

There exists a constant $q_0(\Gamma)$ (depending only on Γ and not $p : S \to \mathbb{P}^1$) such that, if $p : S \to \mathbb{P}^1$ is not Jacobian and has at least 4 singular fibers, $q \in \mathbb{Z}^+$ is a prime such that $q > q_0(\Gamma)$ and $d_S = q$, and M is a torsion multisection of order q on S, the genus of the normalisation $\hat{M}$ of M is bounded:

$$g(\hat{M}) \geq 2.$$

Proof. (Idea.) It can be shown that the minimal index of a proper subgroup of $\mathrm{SL}(2, \mathbb{Z}/q\mathbb{Z})$ strictly increases with q. Then, the natural projection $\Gamma \to \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ is surjective, and, by Lemma 5.5.5 and Equation 5.4, the conclusion follows. □

Proposition 5.5.7: ([BT00], Proposition 3.24.)

There exists a constant $q_0(\Gamma)$ (depending only on Γ and not $p : S \to \mathbb{P}^1$) such that, if $p : S \to \mathbb{P}^1$ is not Jacobian and has at least 4 singular fibers, $q \in \mathbb{Z}^+$ is a prime such that $q > q_0(\Gamma)$ and $d_S = q$, and M is an arbitrary torsion multisection on S, the genus of M is bounded:

$$g(M) \geq 2.$$

Proof. By Proposition 5.3.25, there exists a surjective map from M to a torsion multisection of order q on S or to a non-zero torsion multisection of order q on $J(S)$. The claim follows by Proposition 5.5.6. □

Proposition 5.5.8: ([BT00], Proposition 3.25.)

If $p : S \to \mathbb{P}^1$ is not Jacobian and has at least 4 singular fibers, $q \in \mathbb{Z}^+$ is a prime such that $q > q_0(\Gamma)$ and $q \not| d_S$, and $p' : S' \to \mathbb{P}^1$ is an elliptic surface representing the class $[S]/q \in H^1(\mathbb{P}^1, J(S))$, there aren't any rational or elliptic torsion multisections on S'.

Proof. Note that there exists an elliptic surface representing the class $[S]/q \in H^1(\mathbb{P}^1, J(S))$ by 5.3.21.

By Corollary 5.3.8, as the order of $[S]$ in $H^1(\mathbb{P}^1, J(S))$ is d_S, the order of $[S]/q$ in $H^1(\mathbb{P}^1, J(S))$ is $qd_S = d_{S'}$. Define $J'^m, m \in \mathbb{Z}_0^+$ such that $J'^0 = J(S'), J'^1 = S', \eta'^m : J'^1 \to J'^m$ as in Section 5.3. Let $S'' := J'^{d_S}$. As S'' represents the class $d_S[S]/q$, its degree $d_{S''}$ is equal to q. S'' is also not Jacobian and has the same singular fibers as S. Let M be a torsion multisection on S'. By Proposition 5.3.24, $\eta'^{d_S}(M)$ is a torsion multisection on S''. By Proposition 5.5.7,

$$g(\eta'^{d_S}(M)) \geq 2.$$

By (the proof of) Proposition 5.3.24, the restriction

$$\eta'^m|_M : M \to \eta'^m(M)$$

is a birational map (otherwise Condition 5.3 implies that η'^m maps M to an algebraic 0-cycle on J^m, a contradiction). As the genus is a birational invariant, the conclusion follows. □

Remark 5.5.9: (Cf. [Has03].)

In [Has03], Hassett notes that the original version of Proposition 5.5.8 in [BT00] is not true if the elliptic surface is isotrivial, and that "the Kummer surface associated to a product of general elliptic curves is a counterexample". Hassett then proves the results in [BT00] by three different approaches, the first two of which being a modification of the approach in [BT00].

Theorem 5.5.10: ([BT00], Corollary 3.28.)

If S is an algebraic K3 surface such that $\rho(S) \leq 19$, there are infinitely many rational nt-multisections on S.

Proof. By 3.3.9, $p : S \to \mathbb{P}^1$ has at least 4 singular fibers. Let $p' : S' \to \mathbb{P}^1$ be an elliptic fibration such that S' is an algebraic K3 surface, S' is not Jacobian (if S is not Jacobian, choose $S' = S$), and $J(S') = J(S)$. S' has the same singular fibers as S. Choose a sequence $\{q_n\}_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $q_n > q_0(\Gamma)$, $q_n \not| d_{S'}$, and $p'_n : S'_n \to \mathbb{P}^1$ is an elliptic fibration representing the class $[S']/q_n$. By Proposition 5.5.8, there aren't any rational or elliptic torsion multisections on S'_n. Let M_n be an irreducible rational curve representing a horizontal primitive generator of $\Lambda_{\text{eff}}(S'_n)$ (see Theorem 5.2.6). Clearly, M_n is a multisection, and it must be non-torsion. Define $J_n^m, m \in \mathbb{Z}_0^+$ such that $J_n^0 = J(S'_n), J_n^1 = S'_n, \eta_n^m : J_n^1 \to J_n^m$ as in Section 5.3. By Proposition 5.3.24, $\eta_n^{q_n}(M_n)$ is a rational nt-multisection on S' of degree $d_{S'_n}(M_n)$. By Proposition 5.3.5, $d_{S'_n} = q_n d_{S'} | d_{S'_n}(M_n)$, thus infinitely many different multisections M_n map to different multisections $\eta_n^{q_n}(M_n)$. If S is Jacobian, the same argument allows us to map multisections on S' to multisections on S. Throughout the argument, we were allowed to say that the rational multisections actually are immersed rational curves by Section 5.1 and the fact that all the elliptic surfaces parametrised by the Tate-Shafarevich group are deformation equivalent (cf. [Huy16]). □

5.6 Density

5.6.1 Automorphisms

Let S be an algebraic K3 surface over a number field K.

The following lemma (which we state without proof) is a consequence of the strong Torelli theorem for K3 surfaces.

Lemma 5.6.1: ([Has03], Lemma 6.8.)

There exists a finite field extension K' of K such that all automorphisms of the complex manifold $S(\mathbb{C})$ are realised as algebraic morphisms of $S_{K'}$.

Lemma 5.6.2: ([BT00], Lemma 4.9.)

If $\text{Aut}(S)$ is infinite, then $\Lambda_{\text{eff}}(S)$ is infinitely generated.

Proof. (Idea.) Let Λ_{K3} be the K3 lattice (see Important examples 1.1.15). It can be shown that a subgroup G of $\text{Aut}(S)$ of finite index $[\text{Aut}(S) : G]$ is isomorphic to the subgroup of $O(\Lambda_{\text{K3}})$ that fixes $\Lambda_{\text{eff}}(S)$. In particular, if a full set of generators of $\Lambda_{\text{eff}}(S)$ is finite, then G is finite and $\text{Aut}(S)$ is finite, a contradiction. □

Lemma 5.6.3: (Cf. [BT00], proof of Theorem 4.10.)

If $\text{Aut}(S)$ is infinite, then there exists a generator x of $\Lambda_{\text{eff}}(S)$ such that the orbit of the action of $\text{Aut}(S)$ on x is infinite.

Proof. (Idea.) $O(\Lambda_{\text{K3}})$ embeds into $\text{SL}(22, \mathbb{Z}_3)$. The subgroup of $\text{SL}(22, \mathbb{Z}_3)$ consisting of matrices congruent to the identity matrix $\mod 3$ is normal and all of its elements are of infinite order. Thus, G (see proof of Lemma 5.6.2) has a subgroup of finite index such that all of its elements are of infinite order. Let $g \in G$. There exists a generator x of $\Lambda_{\text{eff}}(S)$ such that the orbit of the action of g on x is infinite. □

Theorem 5.6.4: ([BT00], Theorem 4.10.)

If $\mathrm{Aut}(S)$ is infinite, then K-rational points on S are potentially dense.

Proof. Let x be a generator of $\Lambda_{\text{eff}}(S)$ as in Lemma 5.6.3. By Theorem 5.2.6, the class x is represented by an irreducible rational curve x_{rational}. As the orbit of x_{rational} is infinite, it is not contained in any divisor of S. The result follows by Corollary 4.1.10 and Lemma 5.6.1. □

Before moving on to the subsection which explains the main consequence of the constructions in this chapter, we state a few more useful results about automorphisms on K3 surfaces.

Definition 5.6.5

S is **singular** if $\rho(S) = 20$, i. e., if the Picard number of S is maximal (cf. $\mathrm{char}(K) = 0$).

The following is a result by Shioda-Inose.

Proposition 5.6.6: ([SS10], Lemma 13.2.)

If S is singular, then $\mathrm{Aut}(S)$ is infinite.

Proposition 5.6.7: ([BT00], Corollary 4.12.)

If $\rho(S) \geq 2$ and $\mathrm{Pic}(S)$ does not have any classes of square zero or square -2, then $\mathrm{Aut}(S)$ is infinite.

Clearly, in either of these cases, we may apply Theorem 5.6.4 to conclude that K-rational points are potentially dense.

5.6.2 Multisections

Let $p : S \to \mathbb{P}^1$ be an elliptic fibration over a number field K, let $\hat{p} : J(S) \to C$ be the Jacobian elliptic fibration associated to S, and let $i : S \to J(S)$ be the natural inclusion.

Theorem 5.6.8: ([BT00], Proposition 4.1.)

If there exists a nt-multisection M on S such that K-rational points on M are dense, the set

$$\{x \in \hat{p}(i(M)(K)) : \quad \#F_x(K) < \infty\}$$

is finite. In other words, K-rational points on S are dense.

Proof. (Cf. [Has03], proof of Proposition 4.13). The base change $J(S)_M = J(S) \times_C M$ admits an elliptic fibration $p_M : J(S)_M \to M$ with a section

$$s_M : M \to J(S)_M, \quad x \mapsto (x, x).$$

As M is non-torsion and K-rational points on M are dense, this is true for $s_M(M)$ and $n \cdot s_M(M), n \in \mathbb{Z}$ (here, the multiplication may be done in the generic fiber of $J(S)_M$, see Remark 2.4.1). This implies that $\mathbb{Z} \cdot s_M(M)$ is dense in $J(S)_M$, as it is not contained in any divisor. Therefore, K-rational points on $J(S)_M$ are dense. As $J(S)_M$ dominates $J(S)$ and S, the result follows by (the proof of) Proposition 4.1.9. An alternative proof uses Merel's theorem (cf. [BT00], proof of Proposition 4.1). □

Theorem 5.6.9: ([BT00], Corollary 4.2.)

If S is K3, then K-rational points on S are potentially dense.

Proof. If $\rho(S) \leq 19$, by Theorem 5.5.10, there exist infinitely many rational nt-multisections on S. Therefore, the theorem follows as a corollary of Theorem 5.6.8 and Corollary 4.1.10.

If $\rho(S) = 20$, by Proposition 5.6.6, the group of automorphisms of S Aut(S) is infinite, therefore the theorem follows by Theorem 5.6.4. □

Remark 5.6.10: (Cf. [BT00], Proposition 4.4.)

There are similar approaches to potential density of K-rational points on elliptic surfaces also exhibited by Bogomolov and Tschinkel in *Density of rational points on Enriques surfaces* [BT98]. They consider *saliently ramified multisections*, which are multisections that intersect each smooth fiber F_x with multiplicity greater than or equal to two. They prove that saliently ramified multisections are non-torsion, which allows them to apply Theorem 5.6.8 if the multisection is rational or elliptic. They conclude that, if the surface admits two non-isomorphic elliptic fibrations over $\mathbb{P}^1$ (e. g., if the surface is K3 and admits two non-isomorphic elliptic fibrations), then a general fiber of one fibration is a saliently ramified multisection of the other, thus K-rational points are potentially dense.

5.6.3 Consequences

We start by giving specific examples.

Examples 5.6.11

1. ([BT00], Example 4.8.) There exists a K3 surface S with Néron-Severi lattice given by the Gram matrix
$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}.$$
As there are not any elements of square zero, by Proposition 3.3.3, S does not admit an elliptic fibration. Furthermore, as there are elements of square -2 and $\rho(S) \geq 2$, by Proposition 5.6.7, $\mathrm{Aut}(S)$ is not necessarily infinite. Therefore, we cannot apply any of our theorems about potential density of K-rational points.

2. The surfaces in Item 1 and 2 in Examples 3.3.2 admit elliptic fibrations by Corollary 3.3.5. The surfaces in Item 3 and 4 admit elliptic fibrations by construction. Therefore, K-rational points are potentially dense by Theorem 5.6.9.

3. ([Has03], Example 6.11.) Let p_{11} and p_{22} be bihomogeneous forms of type $(1,1)$ and $(2,2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, respectively. The complete intersection
$$S = \{(x,y) \in \mathbb{P}^2 \times \mathbb{P}^2 : \quad p_{11}(x,y) = p_{22}(x,y) = 0\}$$
is a K3 surface. Its Néron-Severi lattice is generated by the polarisations induced by the $\mathbb{P}^2$ factors, and is given by the Gram matrix
$$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}.$$
The projections $S \to \mathbb{P}^2$ are double covers, and the free subgroup of $\mathrm{Aut}(S)$ generated by the two induced involutions is non-abelian and infinite. Alternatively, as there are not any elements of square zero or -2 and $\rho(S) \geq 2$, by Proposition 5.6.7, $\mathrm{Aut}(S)$ is infinite. Therefore, K-rational points are potentially dense by Theorem 5.6.4.

4. ([Has03], Example 6.11.) Let $n \geq 4$. There exists a K3 surface with Néron-Severi lattice given by the Gram matrix
$$\begin{pmatrix} 2 & n \\ n & 2 \end{pmatrix}.$$
Similarly, K-rational points are potentially dense by Theorem 5.6.4.

Finally, we re-state and sketch the proof of Corollary 4.1.24, which gives the general classification of potential density of K-rational points on K3 surfaces as per the results in this thesis.

Corollary 5.6.12: (Cf. [BT00].)

If X is a K3 surface, the following conditions imply that $X(K)$ is potentially dense:

1. $\rho(X_{\mathbb{C}}) = 1$, and X is a double cover of $\mathbb{P}^2$ ramified in a singular curve of degree six.
2. $\rho(X_{\mathbb{C}}) = 2$ and X does not contain a -2-curve (i. e., a curve of self-intersection equal to 2).
3. $\rho(X_{\mathbb{C}}) = 3$, except possibly for 6 isomorphism classes of lattices $\mathrm{Pic}(X_{\mathbb{C}})$.
4. $\rho(X_{\mathbb{C}}) = 4$, except possibly for 2 isomorphism classes of lattices $\mathrm{Pic}(X_{\mathbb{C}})$.
5. $\rho(X_{\mathbb{C}}) \geq 5$.

Proof. (Idea.)

1. Cf. [BT00].
2. If X does not contain a -2-curve, then, by Proposition 3.3.3 and 5.6.7, either X admits an elliptic fibration or $\mathrm{Aut}(X)$ is infinite.
3. This follows by Nikulin's classification of Néron-Severi lattices of algebraic K3 surfaces [Nik87].
4. Idem. In particular, 17 lattices contain classes of square zero or -2, and 15 of these contain classes of square zero, thus admit elliptic fibrations.
5. This follows by Corollary 3.3.5, as then X admits an elliptic fibration.

□

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