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Degree of irrationality and Vector bundles on K3 and Enriques surfaces

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TÍTULO DE LA TESIS:

DEGREE OF IRRATIONALITY AND VECTOR BUNDLES ON K3 AND ENRIQUES SURFACES.

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ABSTRACT

Title: Degree of irrationality and Vector bundles on K3 and Enriques surfaces.

In recent years, the degree of irrationality has been studied by several authors as a measure of how much does a variety fails from being birational. In recent works, Stampleton in [Sta17] established an upper bound for K3 surface studying the maps induced by line bundles $L \in \mathrm{Pic}(X)$. In [Mor23] Moretti uses vector bundles technique to obtain a better upper bound for polarized K3 surfaces noticing that linear systems of a line bundle L are related to stable bundles E on X. In this work we provide a brief introduction to linear systems and the moduli space of K3 surfaces, describe in detail the techniques used by Moretti and apply them to get an upper bound to the degree of irrationality of polarized Enriques surfaces.

Keywords: *K3 surfaces, moduli space, stable bundles, Torelli Theorem, degree of irrationality, Enriques surfaces*

CONTENTS

INTRODUCTION

A K3 surface is a compact, connected smooth complex surface X with trivial canonical bundle $\mathcal{O}_X(K_X) = \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X) = 0$. In particular, the Kodaira dimension of X is $\kappa(X) = 0$. These surfaces can be found, for example, as solutions of the complex Einstein equation on the vacuum

$$\mathrm{Ric}(\omega) = \lambda\omega$$

when $\lambda = 0$, where Ric is the Ricci tensor and ω is a 2-form associated to the metric.

Their second integral cohomology group has the structure of a lattice, which gives these surfaces a rich structure and allows us to study its automorphisms and moduli spaces thanks to the Torelli type theorems.

In recent decades there has been a lot of study of K3 surfaces, from the foundational work of Saint Donat in [SD74], where he study linear systems on K3 surfaces and classify them as hyperelliptic and non-hyperelliptic, its algebraic and analytic moduli space, obtaining a bijection between the quasi-projective coarse moduli space $\mathcal{F}_{2d}$, constructed using geometric invariant theory, and the period domain Ω_{2d}/Γ_{2d} obtained as a 19 dimensional analytic manifold. Moreover, Mukai constructed in [Muk84a] the moduli space of stable vector bundles on such surfaces.

In more recent years, from Birational Geometry, new birational invariants have been studied. In the case of curves, the gonality, defined as

$$\mathrm{gon}(C) = \min\{\deg(\varphi) \text{ such that } \varphi : C \dashrightarrow \mathbb{P}^1 \text{ is not constant}\}$$

measures how far is C from being rational. In particular, C is rational if and only if $\mathrm{gon}(C) = 1$. There has been several generalizations of the gonality to higher dimensions, in the case of this work we will center in the degree of irrationality

$$\mathrm{irr}(X) = \min\{\deg(\varphi) \text{ such that } \varphi : X \dashrightarrow \mathbb{P}^{\dim(X)} \text{ is generically finite}\}.$$

In general is a very complicated problem to say if a variety is rational or not, and so is to compute $\mathrm{irr}(X)$. Several attempts have been made to get upper bounds to this invariant, for example in [BDPE$^+$17] the authors compute some bounds for hypersurfaces of large degree. In [Che19] the author showed that for very general abelian surfaces, the degree of irrationality is bounded above by 4.

In the case of K3 surfaces, since their Kodaira dimension is zero, we have that $\mathrm{irr}(X) > 1$. Contrary to the case of abelian surfaces, in this case we cannot have a general bound, and moreover, it was conjectured in [BDPE$^+$17] that $X \mapsto \mathrm{irr}(X)$ is unbounded and in [Sta17] the author showed that for a polarized K3 surface (X, L), we have that $\mathrm{irr}(X) \leq C\sqrt{L^2}$.

In a recent work [Mor23], Moretti used vector bundles techniques to find better upper bounds for the degree of irrationality of K3 surfaces and constructed a Brill-Noether theory to compute the degree for low genus.

This work has two main objectives.

- The first objective is to provide a brief and systematic introduction to the theory of K3 surfaces, describing its linear systems and their moduli space. This is done in Chapters 1 and 2.
- The second objective is to understand and expand the ideas originally presented in the article *The polarized degree of irrationality of K3 surfaces* [Mor23], by Federico Moretti, where the author gives an upper bound for the degree of irrationality of K3 surfaces. This is the content of Chapter 3.
  - A secondary objective is to provide all the necessary background to understand Moretti's method and Moretti's theorem.
  - Another secondary objective is to apply this method to Enriques surfaces and get an upper bound to the degree of irrationality of Enriques surfaces.

Chapter 1

PROJECTIVE MODELS OF K3 SURFACES

We will follow [SD74]. The paper presents the first results on projective models of K3 surfaces and demonstrates that if L is ample, then $L^{\otimes m}$ is very ample for $m \geq 3$. This is analogous to the case of elliptic curves.

As a consequence of the Riemann-Roch Theorem, we know that if C is a smooth, projective, irreducible curve of genus $g = g(C)$ and $L \simeq \mathcal{O}_C(D)$ is a line bundle, then L is ample if and only if $\deg(L) > 0$. Moreover, if $\deg(D) \geq 2g + 1$, then L is very ample. In particular, if C is an elliptic curve (i.e., $K_C = 0$ or equivalently $g = 1$), then

$$\deg(3D) = 3\deg(D) \geq 3 = 2g + 1,$$

hence $L^{\otimes 3}$ is very ample for every ample line bundle L.

The same result holds for abelian varieties (Lefschetz Theorem) and hyperelliptic varieties [CI14]. Specifically, if L is an ample divisor, then $L^{\otimes 3}$ is very ample. Furthermore, if L has no base divisor, then $L^{\otimes 2}$ is very ample.

1.1 Some results from classical projective geometry

Definition 1.1.1. *Let X be a variety and $r \in \mathbb{N}_{\geq 1}$. A **vector bundle of rank r** in X is a variety E with a surjective regular morphism $p : E \to X$ such that*

1. *For every $x \in X$, the fiber $p^{-1}(x) =: E_x$ is a k-vector space with $\dim_k(E_x) = r$. In*

particular, $E_x = \mathbb{A}^r$ *for every* $x \in X$.

2. *For every* $x \in X$*, there is an affine open neighborhood* $U \subset X$ *of* $x \in X$ *and a* ***trivialization*** *of* E *over* U*, i.e., an isomorphism*

$$\theta_U : p^{-1}(U) =: E|_U \longrightarrow U \times \mathbb{A}^r$$

such that the diagram commutes

$$\begin{tikzcd} p^{-1}(U) =: E|_U \arrow[rr, "\theta_U"] \arrow[dr, "p|_{p^{-1}(U)}"'] & & U \times \mathbb{A}^r \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

Definition 1.1.2. *Let* X *be a variety and* L *a line bundle (i.e., a vector bundle of rank 1) in* X *with surjective morphism* $p : L \to X$*. A* ***global section*** *of a line bundle* L *is a regular morphism* $s : X \to L$ *such that* $s(x) \in L_x := p^{-1}(x)$ *for all* $x \in X$*. The set of all global sections is denoted* $H^0(X, L)$.

Remark 1.1.3. *Let* $s, t \in H^0(X, L)$ *and* $\lambda \in \mathcal{O}_X(X)$*, then* $\lambda(x) \in k$ *for every* $x \in X$*, hence*

$$s(x) + \lambda(x)t(x) \in L_x \quad \forall x \in X$$

Then we define $s + \lambda t : X \to L$, $x \mapsto s(x) + \lambda(x)t(x)$*, giving* $H^0(X, L)$ *the structure of* $\mathcal{O}_X$*-module. In particular,* $H^0(X, L)$ *is a* k*-vector space of dimension* $h^0(X, L) := \dim_k H^0(X, L)$.

Definition 1.1.4. *Let* $L \in \mathrm{Pic}(X)$ *a line bundle on* X*. A* ***linear system*** M *on* X *is a finite dimensional linear subspace* $M \subset H^0(X, L)$*. In particular, if* $\dim H^0(X, L) < +\infty$ *we say that* $H^0(X, L)$ *is a* ***complete linear system.***

Recall 1.1.5. *Let* $\{s_0, \ldots, s_N\} \subset H^0(X, L)$ *be a basis of* $H^0(X, L)$*. Then we get a rational map* $\varphi_L : X \to \mathbb{P}^N$

$$\varphi_L(x) = [s_0(x) : \cdots : s_N(x)]$$

defined outside

$$Bs_L = \{x \in X : s(x) = 0 \text{ for all } s \in H^0(X, L)\}.$$

Moreover, if $V \subset H^0(X, L)$ *is a linear system and* $\{s_0, \ldots, s_r\}$ *is a basis of* V*, then there is a rational map*

$$\varphi_V : X \dashrightarrow \mathbb{P}(V), x \mapsto [s_0(x) : \cdots : s_r(x)].$$

1.1.1 Proj, $\mathbb{P}(E)$

Let us recall a construction from [Har77]. Let $S = \bigoplus_{i\in\mathbb{N}} S_i$ be a graded ring, i.e., S_i are abelian groups and there is an associative, commutative product such that $S_iS_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$. Let $S_+ := \bigoplus_{i>0} S_i$. Then we define

$$\mathbf{Proj}\ S = \{\mathfrak{p} \subseteq S \text{ homogeneous prime ideal such that } S_+ \not\subseteq \mathfrak{p}\}$$

We define the **Zariski topology** on **Proj** S defining the closed sets

$$V(I) := \{\mathfrak{p} \in \mathbf{Proj}\ S : I \subseteq \mathfrak{p}\}$$

Next, we define a sheaf $\mathcal{O}$ on **Proj** S which makes it into a scheme. For each $\mathfrak{p} \in$ **Proj** S, we consider the subring $S_{(\mathfrak{p})}$ of degree 0 of $T^{-1}S$, where T is the multiplicative set of homogeneous elements of S that are not in $\mathfrak{p}$. For any $U \subseteq$ **Proj** S open subset, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and such that s is locally a quotient of elements in S.

Example 1.1.6. *Let $S = k[X_0, \ldots, X_n]$ for an algebraically closed field k. Then, closed points of **Proj** S corresponds to $\mathbb{P}^n$. To see this, consider the ideals*

$$\mathfrak{p} = (\{x_iX_j - x_jX_i : 0 \le i, j \le n\})$$

Notice that

$$S/\mathfrak{p} \simeq k[X]$$

*Then $\mathfrak{p} \in \operatorname{spec} S$ and therefore $\mathfrak{p} \in$ **Proj** S. Let $\mathfrak{m} \in \operatorname{spec} S$ be a maximal ideal, as k is algebraically closed, there is $\xi \in k^{n+1}$ such that*

$$\mathfrak{m} = (X_0 - \xi_0, \ldots, X_n - \xi_n)$$

*Noticing $ht(\mathfrak{p}) = \dim S - 1$ and $\mathfrak{m} \notin$ **Proj** S, then $\mathfrak{p}$ are the closed points of **Proj** S. Finally, we get the homeomorphism $f : \mathbb{P}^n \to$ **Proj** S*

$$f([x_0 : \cdots : x_n]) = (\{x_iX_j - x_jX_i : 0 \le i, j \le n\})$$

Moreover, if V is a finite dimensionl k-vector space, $\{e_0, \ldots, e_n\}$ a basis of V and $\{e_0, \ldots, e_n\}$ its dual basis, then $\mathbb{P}(V)$ corresponds to the closed points of $\operatorname{Proj} SV^{\vee}$ since $SV^{\vee} \simeq k[X_1, \ldots, X_n]$ and hence we have a explicit morphism

$$\langle v\rangle \mapsto (\{e_i(v)e_j - e_j(v)e_i : 0 \le i, j \le n\})$$

Let X be a scheme and $\mathcal{F}$ a quasi coherent sheaf of $\mathcal{O}_X$-modules such that has a structure of graded $\mathcal{O}_X$-algebras,

$$\mathcal{F} \simeq \bigoplus_{d\in\mathbb{N}} \mathcal{F}_d, \quad \mathcal{F}_0 = \mathcal{O}_X$$

with $\mathcal{F}_1$ coherent and $\mathcal{F}$ locally generated by $\mathcal{F}_1$ as $\mathcal{O}_X$-module. Let $U = \operatorname{spec}(A)$ be an affine open subset of X. Then $\mathcal{F}(U)$ is a graded A-algebra, thus $A \hookrightarrow \mathcal{F}(U)$, that induces $\operatorname{spec} \mathcal{F}(U) \to \operatorname{spec} A$. Consider $\mathbf{Proj}\ \mathcal{F}(U)$ and the natural morphism $\pi : \mathbf{Proj}\ \mathcal{F}(U) \to U$. As $\mathcal{F}$ is quasi-coherent, if U, V are affine open sets, we have that $\pi_U(U \cap V) = \pi_V(U \cap V)$. Gluing $\mathbf{Proj}\ \mathcal{F}(U)$ we get a scheme $\mathbf{Proj}\ \mathcal{F}$ with a morphism $\pi : \mathbf{Proj}\ \mathcal{F} \to X$ such that $\pi^{-1}(U) \simeq \mathbf{Proj}\ \mathcal{F}(U)$.

Lemma 1.1.7. *Let X be a scheme and $\mathcal{F}$ a sheaf of graded algebra as before. Let $\mathcal{L}$ be an invertible sheaf on X. Then we can define a new sheaf of graded algebra $\mathcal{G} := \mathcal{F} \otimes \mathcal{L}$ given by $\mathcal{G}_d = \mathcal{F}_d \otimes \mathcal{L}^{\otimes d}$ for each $d \geq 0$. Then $\mathcal{G}$ satisfies the condition before and there is a natural morphism $\varphi : Q = \mathbf{Proj}\ \mathcal{G} \to \mathbf{Proj}\ \mathcal{F} = P$ commuting with the projections π_P and π_Q on X, i.e., the diagram*

$$\begin{tikzcd} Q = \mathbf{Proj}\ \mathcal{G} \arrow[rr, "\varphi"] \arrow[dr, "\pi_Q"'] & & P = \mathbf{Proj}\ \mathcal{F} \arrow[dl, "\pi_P"] \\ & X & \end{tikzcd}$$

commutes and having the property

$$\mathcal{O}_Q \simeq \varphi^* \mathcal{O}_P \otimes \pi_P^* \mathcal{L}$$

Definition 1.1.8. *Let X be a Noetherian scheme, and let $\mathcal{E}$ be a locally free coherent sheaf on X. We define the associated **projective space bundle** as follows. Let $\mathcal{F} = S\mathcal{E}$ be the symmetric algebra on $\mathcal{E}$, $\mathcal{F} = \bigoplus_{d\in\mathbb{N}} S^d\mathcal{E}$. Then X, $\mathcal{F}$ satisfies the conditions before, then we define $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}\ \mathcal{F}$. As such, it comes with a projection morphism $\pi : \mathbb{P}(\mathcal{E}) \to X$ and an invertible sheaf $\mathcal{O}(1)$. Moreover, if $\mathcal{E}$ is free of rank $n+1$, then $\pi^{-1}(U) \simeq \mathbb{P}^n_U$, so $\mathbb{P}(\mathcal{E})$ is a relative projective space over X.*

Proposition 1.1.9. *Let X, $\mathcal{E}$ and $\mathbb{P}(\mathcal{E})$ be as in the definition. Then*

1. *If $\operatorname{rank}(\mathcal{E}) \geq 2$, there is a canonical isomorphism of graded $\mathcal{O}_X$-algebras $\mathcal{F} \simeq \bigoplus_{d\in\mathbb{Z}} \pi_*(\mathcal{O}(d))$ with the grading on the right hand side given by d. In particular, for $d < 0$, $\pi_*(\mathcal{O}(d)) = 0$, for $d = 0$, $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}) = \mathcal{O}_X$ and for $d = 1$, $\pi_*(\mathcal{O}(1)) = \mathcal{E}$.*

2. *There is a natural surjective morphism $\pi^*\mathcal{E} \to \mathcal{O}(1)$.*

Remark 1.1.10. *The construction of $\mathbb{P}(E)$ is functorial, that is, if $f : E \to E'$ is a morphism of vector bundles, then there is an induced map $\operatorname{Proj}(f) : \mathbb{P}(E') \to \mathbb{P}(E)$.*

Another result that will be useful for the analysis is the Bertini's Theorem, from [Har77].

Theorem 1.1.11. *Let X be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where k is an algebraically closed field. Then there exists a hyperplane $H \subseteq \mathbb{P}^n_k$, not containing X, and such that the scheme $H \cap X$ is regular at every point. In fact, if $\dim X \geq 2$, then $H \cap X$ is connected, hence irreducible, and so $H \cap X$ is a nonsingular variety. Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.*

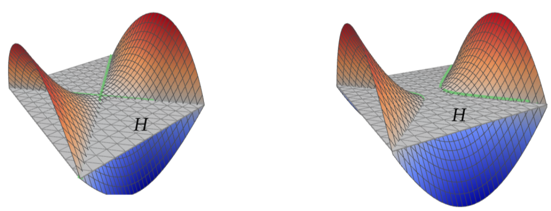


Figura 1.1. Hyperplane sections of a nonsingular variety.

Recall 1.1.12. *In Groethendieck notation, if $\pi : V \to W$, then*

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "\pi^*(f)"'] & W \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

from this it follows that there is a map $\mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$, $H_f \mapsto H_{\pi^*(f)}$, where $H_\ell := \ker(\ell)$.

Let $Y = \mathbb{P}^1$ and E a locally free sheaf on Y of rank r. Let $X = \mathbb{P}(E)$ and $\pi : X \to Y$ the canonical map, so $\pi_*(\mathcal{O}_X(1)) \simeq E$.

Consider the invertible sheaf $L = \mathcal{O}_X(1) \otimes \pi^*(\mathcal{O}_Y(1))$.

Lemma 1.1.13. *L is very ample if and only if E is generated by its sections.*

Proof. Let us assume E is generated by its sections (i.e., E is globally generated), then by definition

$$\mathcal{O}^n \to E \to 0.$$

By the recall, we get a commutative diagram

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[r, hook] & \mathbb{P}(\mathcal{O}_Y^{\oplus n}) = \mathbb{P}^{n-1} \times Y \arrow[dl, "pr_2"'] \arrow[dr, "pr_1"] & \\
\mathbb{P}^1 = Y & & \mathbb{P}^{n-1}
\end{tikzcd}$$

where j is the closed immersion defined induced by the recall. Since $pr_1^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \otimes pr_2^*(\mathcal{O}_Y(1))$ is very ample, we have that

$$L \simeq j^*(pr_1^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \otimes pr_2^*(\mathcal{O}_Y(1)))$$

is very ample.

Lets suppose now L is very ample. Then, by the Birkhoff-Grothendieck Theorem,

$$E = \bigoplus_{i=0}^{r} \mathcal{O}_Y(d_i).$$

If we fix $1 \leq j \leq r$, then we have the projection $E \to \mathcal{O}_Y(d_j) \to 0$ that induces a map

$$\begin{tikzcd}
\mathbb{P}(\mathcal{O}_Y(d_j)) \arrow[rr, hook, "s_j"] \arrow[dr] & & \mathbb{P}(E) = X \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

Then we have

$$s_j^*(L) = s_j^*(\mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_Y(1)) \simeq s_j^*(\mathcal{O}_X(1)) \otimes \mathcal{O}_Y(1) \simeq \mathcal{O}_Y(d_j+1)$$

is very ample, so $d_j \geq 0$, concluding that E is globally generated.

□

Lemma 1.1.14. *In the previous setting,* $H^i(X, L^{\otimes(1-i)}) = 0$ *for* $i \geq 1$.

Proof. First by Grothendieck vanishing we have that $H^p(X, L^{\otimes j}) = 0$ for $p > \dim(X) = r$. Now, for $j \in \mathbb{Z}$, we have $L^{\otimes j} = \mathcal{O}_X(j) \otimes \pi^*(\mathcal{O}_Y(j))$, and using the canonical isomorphisms

$$R^q\pi_*(\mathcal{O}_X(j) \otimes \pi^*\mathcal{O}_Y(j)) \simeq R^q\pi_*\mathcal{O}_X(j) \otimes \mathcal{O}_Y(j).$$

We get the Leray spectral sequence (see A.2)

$$E_2^{p,q}(j) := H^p(Y, R^q\pi_*\mathcal{O}_X(j) \otimes \mathcal{O}_Y(j)) \Rightarrow H^{p+q}(X, L^{\otimes j}).$$

If $\mathcal{E}$ is the sheaf of sections of E, then

$$R^q\pi_*\mathcal{O}_Y(j) \simeq R^q\mathcal{E}^{\otimes j} \simeq H^q(X, \mathcal{E}^{\otimes j})$$

since $\pi_*\mathcal{O}_Y(1) \simeq E$. Hence by Grothendieck vanishing $R^q\pi_*\mathcal{O}_Y(j) = 0$ for $q \neq 0$. It follows that

$$E_2^{p,q}(j) := H^p(Y, R^q\pi_*\mathcal{O}_X(j) \otimes \mathcal{O}_Y(j)) = H^p(Y, \mathcal{O}_Y(j)) = 0$$

for $1 \leq p \leq r$ and $j \geq 1-p$, therefore $E_2^{p,0}(j) \simeq H^p(X, L^{\otimes j})$. We conclude that for $i \geq 1$

$$H^i(X, L^{\otimes(1-i)}) \simeq E_2^{i,0}(1-i) \simeq H^i(Y, \pi_* L^{\otimes(1-i)}) = 0.$$

$\square$

Proposition 1.1.15. *Let us assume that E is generated by its sections and let $\varphi_L : X \to \mathbb{P}(H^0(X, L))$ the embedding of X defined by L.*

1. *The canonical map $u : S^\bullet H^0(X, L) \to \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})$ is surjective.*

2. *The kernel I of α is generated by its elements of degree 2.*

3. $\deg \varphi_L(X) = \operatorname{codim} \varphi_L(X) + 1$.

Proof. We'll just prove 3 by induction on the rank of r of E, the proof of 1 and 2 can be found in [SD74, Proposition 1.5]. If the rank is 1, i.e., E is a line bundle over $Y = \mathbb{P}^1$, then $\pi : \mathbb{P}(E) \to Y = \mathbb{P}^1$ satisfies

$$\pi^{-1}(x) = \mathbb{P}(E_x) \simeq \mathbb{P}(\mathbb{A}^1) \simeq \mathbb{P}^0 = \{p\}.$$

Hence we have that $\pi : \mathbb{P}(E) \to \mathbb{P}^1$ is an isomorphism, then $X \simeq \mathbb{P}^1$. If $L \simeq \mathcal{O}_{\mathbb{P}^1}(d)$ is globally generated, then is very ample by the Proposition 1.1.14 and $d > 0$. Since

$$H^0(X, L) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \simeq \mathbb{C}[X, Y]_d$$

we have $h^0(X, L) = d + 1$. It follows

$$\operatorname{codim} \varphi_L(X) + 1 = d - 1 + 1 = d = \deg(\varphi_L).$$

By induction, lets assume 3 is true for vector bundles of rank r and let E be a vector bundle of rank $r+1$ generated by its sections, and $X = \mathbb{P}(E)$. By Lemma 1.1.13 L is very ample. Let $H \subset \mathbb{P}(H^0(X, L))$ be a hyperplane in general position defined by a section s such that $\varphi_L(X) \cap H$ is irreducible and H doesn't contain any fibers of $\pi : X \to Y$. By the proof of the previous Lemma, we have that

$$s \in H^0(X, \mathcal{O}_X(1) \otimes \pi^* \mathcal{O}_Y(1)) \simeq H^0(Y, E \otimes \mathcal{O}_Y(1))$$

and so s induces an injection $\mathcal{O}(-1) \overset{s}{\hookrightarrow} E$. Let E' be the cokernel of s and so we have an exact sequence

$$0 \to \mathcal{O}_Y(-1) \xrightarrow{s} E \xrightarrow{f} E' \to 0.$$

We then have that E' is a vector bundle of rank $r-1$ generated by its sections. Let $X' = \mathbb{P}(E')$ be the projective bundle induced by E' with canonical morphism $\nu : X' \to Y$ and $L' = \mathcal{O}_{X'}(1) \otimes \nu^* \mathcal{O}_Y(1)$. From the exact sequence we have

$$0 \to \langle s \rangle_{H^0(X;L)} \to H^0(X,L) \to H^0(X',L') \to 0$$

and the commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}(H^0(X,L)) & \\
\mathbb{P}(H^0(X',L')) = H \arrow[ur, hook, "i"] & & X \arrow[ul, dashed, "\varphi_L"'] \\
 & X' \arrow[ul, dashed, "\varphi_{L'}"] \arrow[ur, "\mathrm{Proj}(f)"'] &
\end{tikzcd}$$

from which we identify $\varphi_L(X) \cap H \simeq \varphi_{L'}(X')$. Finally by induction hypothesis $\deg(\varphi_{L'}(X')) = \operatorname{codim}(\varphi_{L'}(X')) + 1$ and so

$$\deg(\varphi_L(X)) = \deg(\varphi_L(X) \cap H) = \operatorname{codim}(\varphi_L(X) \cap H) = h^0(X,L) - 1 - (\dim(\varphi_L(X)) - 1)$$

Concluding $\deg(\varphi_L(X)) = \operatorname{codim}(\varphi_L(X)) + 1$.

□

Proposition 1.1.16. *With the previous notation, let $s_1, \dots, s_N$ be a basis of $H^0(Y,E)$, $\{x, y\}$ a basis of $H^0(Y, \mathcal{O}_Y(1))$. We denote again by $s_i \otimes x$ (resp. $s_i \otimes y$) the image of $s_i \otimes x$ (resp. $s_i \otimes y$) by the canonical map*

$$H^0(Y,E) \otimes H^0(Y, \mathcal{O}_Y(1)) \to H^0(Y, E \otimes \mathcal{O}_Y(1)) \simeq H^0(X,L)$$

Then the minors of order 2 of the matrix

$$\begin{pmatrix} s_1 \otimes x & \dots & s_N \otimes x \\ s_1 \otimes y & \dots & s_N \otimes y \end{pmatrix}$$

de ne a basis of I_2*.*

Recall 1.1.17. *By Leray spectral sequence,*

$$H^0(X, \mathcal{O}_X(m) \quad \pi \quad M) \ ' \ H^0(Y, S^m E \quad M)$$

Proof. By Leray spectral sequence,

$$H^0(X, L \quad ^2) \ ' \ H^0(X, \mathcal{O}_X(2) \quad \pi \quad \mathcal{O}_Y(2)) \ ' \ H^0(Y, S^2 E \quad \mathcal{O}_Y(2))$$

Since

$$E \ ' \ \bigoplus_{i=1}^{r} \mathcal{O}_Y(d_i) \text{ then } E \quad \mathcal{O}_Y(1) \ ' \ \bigoplus_{i=1}^{r} \mathcal{O}_Y(d_i + 1)$$

and hence

$$S^2(E \quad \mathcal{O}_Y(1)) \ ' \ \bigoplus_{i \quad j} \mathcal{O}_Y(d_i + d_j + 2).$$

Therefore

$$\begin{aligned}
h^0(X, L \quad ^2) = h^0(Y, S^2 E \quad \mathcal{O}_Y(2)) &= \sum_{i=1}^{r} \sum_{j=1}^{i} h^0(Y, \mathcal{O}_Y(d_i + d_j + 2)) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{i} d_i + d_j + 2 + 1 \\
&= \sum_{i=1}^{r} \sum_{j=1}^{i} d_i + d_j + 3 \\
&= \sum_{i=1}^{r} i d_i + (r \quad i + 1) d_i + 3i \\
&= 3 \frac{r(r+1)}{2} + (r+1) \sum_{i=1}^{r} d_i
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\dim S^2 H^0(X, L) &= \frac{(\sum d_j + 2r)(\sum d_j + 2r + 1)}{2} \\
&= \frac{1}{2} \left(\sum d_j \right)^2 + r \sum d_j + \frac{1}{2} \sum d_j + r \sum d_j + 2r^2 + r \\
&= \frac{1}{2} \left(\sum d_j \right)^2 + 2r \sum d_j + \frac{1}{2} \sum d_j + 2r^2 + r.
\end{aligned}$$

Then

$$\begin{aligned}
\dim I_2 &= \dim S^2H^0(X,L) - \dim H^0(X,L^{\otimes 2}) \\
&= \frac{1}{2}\left(\sum d_j\right)^2 + 2r\sum d_j + \frac{1}{2}\sum d_j + 2r^2 + r - 3\frac{r(r+1)}{2} - (r+1)\sum_{i=1}^{r} d_i \\
&= \frac{1}{2}\left(\sum d_j\right)^2 + r\sum d_j - \frac{1}{2}\sum d_j + \frac{1}{2}r^2 - \frac{1}{2}r.
\end{aligned}$$

Noticing that the minors are linearly independent, the dimension of the space generated by the minors of order 2 is

$$\binom{N}{N-2} = \frac{N!}{(N-2)!2!} = \frac{N(N-1)}{2}$$

where $N = \dim H^0(Y,E) = \sum d_j + r$, hence the dimension is given by

$$\begin{aligned}
\frac{(\sum d_j + r)(\sum d_j + r - 1)}{2} &= \frac{1}{2}\left(\sum d_j\right)^2 + \frac{1}{2}r\sum d_j - \frac{1}{2}\sum d_j + \frac{1}{2}r\sum d_j + \frac{1}{2}r^2 - \frac{1}{2}r \\
&= \frac{1}{2}\left(\sum d_j\right)^2 + r\sum d_j - \frac{1}{2}\sum d_j + \frac{1}{2}r^2 - \frac{1}{2}r.
\end{aligned}$$

□

Definition 1.1.18. *A* ***rational scroll*** *is a variety $\varphi_L(X)$ given by the previous construction. In particular, all these varieties are rational.*

Following [SD74], we recall the following classical result from projective geometry, proven by del Pezzo.

Theorem 1.1.19. *Let X be a surface on $\mathbb{P}^n$ which does not lie in any hyperplane and such that $\deg X = n-1$, then X is one of the following:*

1. *The Veronese surface in $\mathbb{P}^5$.*
2. *A rational scroll.*
3. *A cone over a rational normal twisted curve.*

In particular, X is rational.

Definition 1.1.20. *Let C be an algebraic, projective, smooth, irreducible curve. We say that C is* ***hyperelliptic*** *if there exists a degree two regular morphism $f : C \to \mathbb{P}^1$.*

Definition 1.1.21. *Let D be a divisor on a curve X. We define the sheaf $\mathcal{O}_X(D)$ by*

$$\mathcal{O}_X(D)(U) := \{f \in \mathcal{O}_X(U) : D + (f) \geq 0\}.$$

Theorem 1.1.22. *(Riemann-Roch) Let X be an algebraic curve and D a divisor and K_X the canonical divisor of X. Then*

$$h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(K_X - D)) = 1 - g(X) + \deg(D)$$

Recall 1.1.23. *By historical reasons, some authors use the notation $\mathcal{L}(D) := \mathcal{O}_X(D)$ and $\ell(D) := h^0(X, \mathcal{L}(D))$.*

The following is a classical result about hyperelliptic curves (here we follow the proof of [Har77, Chapter IV, Proposition 5.2])

Theorem 1.1.24. *Let C be an algebraic projective smooth irreducible curve of genus $g(C) = g \geq 2$. Then, the canonical bundle $\mathcal{O}_C(K_C)$ is very ample if and only if C is not hyperelliptic.*

Proof. We recall that D is very ample if and only if $h^0(C, \mathcal{O}_C(D - p - q)) = h^0(C, \mathcal{O}_C(D)) - 2$ for every two points $p, q \in C$. Let $p, q \in C$, by Riemann-Roch for $D = p + q$

$$h^0(C, \mathcal{O}_C(p+q)) - h^0(C, \mathcal{O}_C(K_C - p - q)) = 1 - g + \deg(p+q) = 3 - g$$

hence K_C is very ample if and only if for every $p, q \in C$

$$2 - g = 2 - h^0(C, \mathcal{O}_C(K_C)) = 3 - g - h^0(C, \mathcal{O}_C(p+q))$$

or equivalently $h^0(C, \mathcal{O}_C(p+q)) = 1$ for every points $p, q \in C$. If there are points $p, q \in C$ such that $h^0(C, \mathcal{O}_C(p+q)) \geq 2$, then there is $M \subseteq H^0(C, \mathcal{O}_C(p+q))$ that induces a morphism $\varphi_M : C \to \mathbb{P}^1$ of degree two and therefore C is hyperelliptic. By the contrary, if C is hyperelliptic then there is a linear system $M \subseteq H^0(C, L)$ such that $\dim(M) = \deg(M) = 2$. Every non-zero section $s \in M \setminus \{0\}$ has $\operatorname{div}(s) = p + q$ and

$$h^0(C, \mathcal{O}_C(\operatorname{div}(s))) = h^0(C, \mathcal{O}_C(p+q)) = 2,$$

therefore K_C is not very ample.

□

1.2 Linear systems on K3 surfaces

Definition 1.2.1. *A **K3 surface** is a complex, compact manifold X with canonical bundle $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$.*

Remark 1.2.2. *If X is a K3 surface, then*

$$q(X) = h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X) = 0.$$

Recall that a **K3 surface** is a complex compact manifold X with canonical bundle $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and irregularity $q(X) = 0$. We will center in the study of projective surfaces over $\mathbb{C}$. In particular, we have that

$$q(X) = h^1(\mathcal{O}_X) = h^0(\Omega^1_X) = 0,$$

$$p_g(X) = h^0(X, \mathcal{O}_X(K_X)) = h^2(X, \mathcal{O}_X) = 1,$$

$$\chi(X) = 1 - q(X) + p_g(X) = 2.$$

And then, for $L \cong \mathcal{O}_X(D) \in \mathrm{Pic}(X)$, by Riemann-Roch we get

$$\chi(L) = \chi(X) + \frac{1}{2}(L^2 - L \cdot \mathcal{O}_X(K_X)) = 2 + \frac{1}{2}L^2.$$

Then, using Serre's duality and $K_X = 0$

$$h^0(L) - h^1(L) + h^2(L) = h^0(L) - h^1(L) + h^0(L^{-1}) = 2 + \frac{L^2}{2}.$$

If $D \geq 0$ (i.e., $h^0(\mathcal{O}_X(D)) \geq 1$) we have

$$h^2(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(-D)) = 0,$$

since if $h^1(L) \neq 0 \neq h^1(L^{-1})$, we would have $L \cong \mathcal{O}_X$.

Remark 1.2.3. *Since* $h^1(X, \mathcal{O}_X) = 0$ *we have that* $\mathrm{Pic}(X) = \mathrm{NS}(X) \cong \mathbb{Z}$.

Proposition 1.2.4. *Let C be an irreducible curve of (arithmetic) genus $p_a(C) = g$ on a K3 surface S. Then $C^2 = 2g - 2$ and $h^0(C) = g + 1$.*

Proof. By definition of arithmetic genus $C^2 = 2g - 2$. By Riemann-Roch

$$\chi(X, \mathcal{O}_S(C)) = 2 + \frac{1}{2}C^2 = 2 + g - 1 = g + 1$$

Since by Serre's duality $h^2(X, \mathcal{O}_S(C)) = h^0(X, \mathcal{O}_S(-C)) = 0$ we have that

$$h^0(X, \mathcal{O}_S(C)) - h^1(X, \mathcal{O}_S(C)) = g + 1.$$

hence it is left to show $h^1(X, \mathcal{O}_S(C)) = 0$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

then we have the exact sequence on cohomology

$$0 \cong H^0(S, \mathcal{O}_S(-C)) \to H^0(S, \mathcal{O}_S) \hookrightarrow H^0(C, \mathcal{O}_C) \to H^1(S, \mathcal{O}_S(-C)) \to H^1(S, \mathcal{O}_S) \cong 0$$

since $H^0(S, \mathcal{O}_S) \simeq H^0(C, \mathcal{O}_C) \simeq \mathbb{C}$ we have

$$1 = h^0(C, \mathcal{O}_C) = h^0(S, \mathcal{O}_S) + h^1(S, \mathcal{O}_S(-C)) = 1 + h^1(S, \mathcal{O}_S(-C)).$$

We conclude by Serre's duality that $h^1(S, \mathcal{O}_S(C)) = h^1(S, \mathcal{O}_S(-C)) = 0$.

□

Remark 1.2.5. *Notice that in the previous proof we only use the fact that $h^1(X, \mathcal{O}_X) = 0$.*

Proposition 1.2.6. *Let C be an smooth irreducible curve of genus $g > 1$. Then the system $|C|$ is base-point free, so defines a morphism $\varphi_C : S \to \mathbb{P}^g$ and the restriction of φ_C to C is the canonical map $\varphi_C|_C : C \to \mathbb{P}^{g-1}$ defined by K_C.*

Proof. By the adjunction formula, since $K_S = 0$, $\mathcal{O}_C(K_C) \simeq \mathcal{O}_S(C)|_C$ and then we have the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_S(C)|_C \simeq \mathcal{O}_C(K_C) \to 0$$

Since $g \geq 2$, $\mathcal{O}_C(K_C)$ is base-point free. Since S is smooth,

$$\mathrm{Div}(S)/\,\mathrm{PDiv}(S) \simeq \mathrm{WDiv}(S)/\,\mathrm{PWDiv(S)} = \mathrm{Cl}(S)$$

there is a section s such that $V(s) = C$. Hence, $|C|$ is base-point free.

□

The following result is classical about K3 surfaces and will not be proved. For a reference see [SD74, Proposition 2.6].

Proposition 1.2.7. *Assume $|L| \neq \emptyset$ and $\dim Bs(L) \leq 0$. Then either*

1. *$L^2 > 0$ and for $C \in |L|$ general irreducible curve*

$$p_a(C) = \frac{L^2}{2} + 1$$

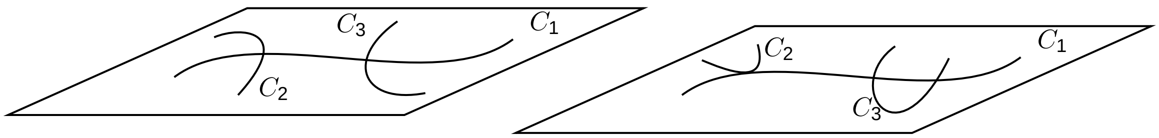
   and in this case $h^1(L) = 0$.

2. *$L^2 = 0$, then $L \simeq \mathcal{O}_X(kE)$, where $k \in \mathbb{Z}_{\geq 1}$ and E is an irreducible curve of arithmetic genus 1. In this case $h^1(L) = k - 1$ every member of $|L|$ can be written as a sum $E_1 + E_2 + \cdots + E_k$ where $E_i \in |E|$ for $i \in \{1, \dots, k\}$.*

1.2.1 Globally generated line bundles on K3 surfaces

Theorem 1.2.8. *Let X be a K3 surface and $C \subset X$ irreducible such that $C^2 > 0$. Then $|C|$ is base-point free.*

Definition 1.2.9. *Let X be a smooth complex projective surface and $D \geq 0$ a divisor. D is m-**connected** if for each $D_1, D_2 \geq 0$ such that $D = D_1 + D_2$, $D_1 \cdot D_2 \geq m$.*



$m = 1$ $m = 2$

Lemma 1.2.10. *If $q(X) = 0$ (e.g., X a K3 surface) and $D \geq 0$ is 1-connected, then $h^1(\mathcal{O}_X(-D)) = 0$. In particular, if X is a K3 surface, $h^1(\mathcal{O}_X(D)) = 0$.*

Proof. Since D is effective, by a previous Lemma

$$h^1(X, \mathcal{O}_X(-D)) = h^0(D, \mathcal{O}_D) - 1 = 0.$$

Moreover, if X is a K3 surface, then by Serre's duality

$$h^1(X, \mathcal{O}_X(D)) = h^1(X, \mathcal{O}_X(-D) + K_X) = h^1(X, \mathcal{O}_X(-D)).$$

□

Proposition 1.2.11. *Let X be a K3 surface and D a divisor on X. Then either D is effective or $-D$ is effective.*

Proof. By Riemann-Roch and the previous Proposition we have

$$h^0(X\mathcal{O}_X(D)) + h^0(X, \mathcal{O}_X(-D)) = h^0(X\mathcal{O}_X(D)) + h^2(X, \mathcal{O}_X(D)) = \frac{1}{2}D^2 + 1 \geq 1$$

hence either $h^0(X, \mathcal{O}_X(D)) > 0$ or $h^0(X, \mathcal{O}_X(-D)) > 0$.

□

1.3 Projective models of K3 surfaces

Throughout this Section we will follow [SD74] and [Bea96] to get a better understanding of the projective models of $K3$ surfaces. Let L be a line bundle on a K3 surface such that $L^2 > 0$ and $|L|$ has no fixed components. Then $L \simeq \mathcal{O}_X(C)$ for some irreducible curve C and L is base-point free.

We denote $\varphi_L : X \to \mathbb{P}^{p_a(L)}$ defined by L, where $g = p_a(L) = \frac{1}{2}L^2 + 1$. Note that $\dim \varphi_L(X) = 2$. The image is not contained in a hyperplane since $h^0(X, L) \geq h^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1)) = g \geq 3$, therefore $\deg \varphi_L(X) \geq p_a(L) - 1$.

Theorem 1.3.1. *Let $L \in \operatorname{Pic}(X)$ such that $L^2 \geq 4$. Then either:*

1. *φ_L is a birational morphism, its image has degree $2p_a(L) - 2$ and a generic curve of $|L|$ is non-hyperelliptic.*

2. *φ_L is a 2-to-1 morphism to a rational surface (possibly singular) degree $g - 1$ in $\mathbb{P}^g$. A generic curve of $|L|$ is then hyperelliptic.*

Proof. First of all, note that by adjunction if $C \in |L|$ is smooth, then $\mathcal{O}_C(K_C) = L|_C$ and since $\deg(L|_C) = L^2 \geq 4$ we have that $\mathcal{O}_C(K_C)$ is ample. Thus, it is natural to distinguish between the case when $\mathcal{O}_C(K_C)$ is very ample (i.e., C is not hyperelliptic) or not (i.e., C is hyperelliptic and $\mathcal{O}_C(K_C)$ defines a $2:1$ map $C \to \mathbb{P}^1$).

If C is non-hyperelliptic, by Theorem 1.1.24 the restriction of φ_L is an embedding, hence $\varphi_L^{-1}(\varphi_L(C)) = C$, it follows that $\deg(\varphi_L) = 1$ and therefore φ_L is birational. If C is hyperelliptic, then for a generic point $x \in X$, $\varphi_L^{-1}(\varphi_L(x))$ consists of 2 points and hence $\deg(\varphi_L) = 2$. Since $L^2 = 2g - 2$, the image $\varphi_L(X)$ is a surface of degree $g - 1$ in $\mathbb{P}^g$ (in particular $\deg(\varphi_L(X)) = g - 1 = (g-2) + 1 = \operatorname{codim}(\varphi_L(X)) + 1$). By Theorem 1.1.19 we have that $\varphi_L(X)$ is rational.

□

Example 1.3.2. *Let $S = S_{d_1,\ldots,d_r} = V(f_1, \ldots, f_r) \subset \mathbb{P}^{r+2}$ a complete intersection. Then $K_S = kH$, where H is a hyperplane and $k = \sum d_i - r - 3$. Moreover*

1. *$S_2, S_3, S_{2,2}$ are rational and so $\kappa(S)$[1] $= -\infty$.*

2. *$S_4, S_{2,3}, S_{2,2,2}$ have $K_S = 0$ and so $\kappa(S) = 0$.*

3. *All other surfaces have $\kappa(S) = 2$.*

In fact, let $E = \bigoplus_{i=1}^r \mathcal{O}(d_i)$ and $s = (f_1, \ldots, f_r) \in H^0(\mathbb{P}^{r+2}, E)$, then $S = V(s)$ is smooth of dimension 2. We compute

$$\det(E) = \det\left(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{r+2}}(d_i)\right) = \mathcal{O}_{\mathbb{P}^{r+2}}\left(\sum_{i=1}^{r} d_i\right).$$

Therefore by the adjunction formula

$$\begin{aligned}\mathcal{O}(K_S) &\simeq (\mathcal{O}(K_{\mathbb{P}^{r+2}}) \otimes \det(E))|_S \simeq (\mathcal{O}_{\mathbb{P}^{r+2}}(-r-3) \otimes \mathcal{O}_{\mathbb{P}^{r+2}}(d_1 + \cdots + d_r)) \\ &\simeq \mathcal{O}_{\mathbb{P}^{r+2}}\left(\sum_{i=1}^{r} d_i - r - 3\right)\Bigg|_S.\end{aligned}$$

We now analyze the sign of k.

[1] Here κ denotes the Kodaira dimension, see Definition 3.1.3

1. *If $k < 0$, then*
$$\sum_{i=1}^{r} d_i < r + 3.$$
If $d_i = 1$ for some i then S is contained in a hyperplane, so we study $d \geq 2$ and therefore
$$2r \leq \sum_{i=1}^{r} d_i < r + 3$$
it follows that the possible cases are $r < 3$. For $r = 1$ we have $d < 4$ so the only possible choices are $S_2, S_3 \subset \mathbb{P}^3$. If $r = 2$, then $d_1 + d_2 < 5$ and therefore the only choice is $S_{2,2} \subset \mathbb{P}^4$.

   Since $k < 0$, $H^0(S, \mathcal{O}_S(mK_S)) = 0$ for any $m \in \mathbb{Z}^{>0}$ and hence $\kappa(S) = -\infty$.

2. *For $k = 0$, we have that*
$$2r \leq \sum_{i=1}^{r} d_i = r + 3$$
and then the possible cases are $r \leq 3$. For $r = 1$ we have $d = 4$ and then $S_4 \subset \mathbb{P}^3$ is the only possibility. If $r = 2$, then $d_1 + d_2 = 5$ and therefore $S_{2,3} \subset \mathbb{P}^4$ is the only complete intersection. Finally, $r = 3$ implies $6 \leq d_1 + d_2 + d_3 = 6$ hence $S_{2,2,2} \subset \mathbb{P}^5$ is the only complete intersection.

   For this case we have trivial canonical bundle $\mathcal{O}_S(K_S) \simeq \mathcal{O}_S$ and therefore $\kappa(S) = 0$.

3. *If $k > 0$ then S has ample canonical bundle $\mathcal{O}_S(K_S)$ and hence for some $m > 0$ we have that $\varphi_{mK_S} : S \hookrightarrow \mathbb{P}^N$ is an embedding, therefore $\kappa(S) = 2$, so S is a surface of general type for all other cases.*

Moreover, in all cases, $q(S) = 0$, hence $S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$ and $S_{2,2,2} \subset \mathbb{P}^5$ are K3 surfaces.

Definition 1.3.3. *A polarized surface is a pair (S, L) where S is a surface and $L \in \mathrm{Pic}(S)$ an ample line bundle on S.*

Example 1.3.4. 1. *Let $S = S_4 \subset \mathbb{P}^3$ be a smooth quartic in $\mathbb{P}^3$ and $H \subset \mathbb{P}^3$, then $D = H \cap S_4$ defines an ample divisor in S_4 and $D \cdot D = 4$, then $L = \mathcal{O}_{S_4}(D)$ is an ample and (S_4, L) is a polarized K3 surface of degree $L^2 = 4$.*

2. *Let $S = S_{2,3} \subset \mathbb{P}^4$ be a complete intersection of a cubic and a quadric. Then $L = \mathcal{O}_{S_{2,3}}(1)$ is an ample line bundle and $L^2 = 6$, hence $(S_{2,3}, L)$ is a polarized K3 surface of degree $L^2 = 6$.*

3. *For $S = S_{2,2,2} \subset \mathbb{P}^5$ complete intersection of three quadrics, $L = \mathcal{O}_S(1)$ is an ample line bundle of degree $L^2 = 8$.*

Proposition 1.3.5. *Let (X,L) be a polarized K3 surface with $\mathrm{Pic}(X)=\mathbb{Z}\cdot[L]$ such that $L^2=4$. Then $X \simeq_{bir} S_4$ is birational to some quartic surface $S_4 \subset \mathbb{P}^3$.*

Proof. By Riemann-Roch

$$\chi(X,L)=2+\frac{1}{2}L^2=4.$$

Moreover, since $h^1(X,L)=h^2(X,L)=0$ we have $h^0(X,L)=4$. Since $\mathrm{Pic}(X)=\mathbb{Z}\cdot[L]$, every curve $C\in|L|$ is irreducible. By the previous results, we now that L is base-point free. Let us denote by $\varphi_L : X \to \mathbb{P}^3$ the associated morphism and by $S\subset\mathbb{P}^3$ its image, which is a surface in $\mathbb{P}^3$. By the projection formula, we know that $4=L^2=\deg(\varphi_L)\deg(S)$.

Assume $\varphi_L : X\to\mathbb{P}^3$ has degree 2, so φ_L is 2-1 to a rational surface $S=\varphi_L(X)\subset\mathbb{P}^3$ of degree 2, i.e., a quadric surface. If S is smooth, $S=V(Q)$ with Q non-degenerate quadratic form, which can be diagonalized and then $S\simeq\mathbb{P}^1\times\mathbb{P}^1$, therefore $\mathrm{Pic}(S)\simeq\mathbb{Z}\oplus\mathbb{Z}$, a contradiction. Indeed, in that case we would have that $L=\varphi_L^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$ with $a,b\geq 1$ as L is ample. In that case, we compute via the projection formula that $4=L^2=\deg(\varphi_L)(aF_1+bF_2)^2=2\cdot(a^2F_1^2+2abF_1\cdot F_2+b^2)=4ab$ and hence $L=\varphi_L^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. This would imply that any smooth curve $C\in|L|$ is the double cover of a line $\mathbb{P}^1=\ell\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ and that would mean that $g(C)=1\neq 3$ by Riemann-Hurwitz, which is absurd[2].

If S is singular, then $\mathcal{O}_{\mathbb{P}^3}(1)|_S$ has a square root as a Weil divisor (i.e., it is linearly equivalent to 2ℓ where $\ell\subset S$ is the line passing through the vertex of the singular quadric S), and its pull-back to X would yield a square root of L, which is absurd since in that case L would not be a primitive element in $\mathrm{Pic}(X)$.

□

Proposition 1.3.6. *Let X be a K3 surface and $L\in\mathrm{Pic}(X)$ ample non-hyperelliptic with $L^2=6$. Then $X\simeq_{bir} S_{2,3}$ is birational to some complete intersection of a cubic and a quadric $S_{2,3}\subset\mathbb{P}^4$.*

Proof. By Riemann-Roch,

$$h^0(X,L^{\otimes 2})=\chi(X,L^{\otimes 2})=2+\frac{1}{2}(4L^2)=14.$$

Since $g=L^2/2+1=4$, $\varphi_L : X\to\mathbb{P}^4$. Moreover,

$$h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(2))=\binom{4+2}{2}=\frac{6!}{2!4!}=15.$$

Since $L^{\otimes 2}=\varphi_L^*(\mathcal{O}_{\mathbb{P}^4}(2))$, S must be contained in a quadric Q. Now consider $h^0(X,L^{\otimes 3})=$

[2] Alternatively, if we consider the pullback of a ruling of $\mathbb{P}^1\times\mathbb{P}^1$ we would get a non-zero effective divisor $F\subseteq X$ such that $F^2=0$, which is absurd if $\mathrm{Pic}(X)=\mathbb{Z}\cdot[L]$.

$2+\frac{1}{2}(9L^2)=29$ and

$$h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(3))=\binom{4+3}{3}=\frac{7!}{4!3!}=35$$

hence there are six independent cubics going through $\varphi_L(X)$. Notice $h^0(X,L)=h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(1))=5$, there is at least a cubic P that is not the product of a quadric and a hyperplane. Finally, $\deg(\varphi_L(X))=L^2=6$ and therefore $S\simeq_{\mathrm{bir}} P\cap Q=S_{2,3}$. □

The following Theorem from Saint-Donat characterise hyperelliptic linear systems.

Theorem 1.3.7. *Let $|L|$ be a complete linear system of a K3 surface S without fixed components, and such that $L^2\geq 4$. Then L is hyperelliptic only in the following cases:*

1. *There exists an irreducible curve E such that $g(E)=1$ and $E\cdot L=2$.*
2. *There exists an irreducible curve C such that $g(C)=2$ and $L\simeq\mathcal{O}_S(2C)$.*

Proposition 1.3.8. *Let C be an irreducible curve on S such that $g(C)=2$ and let $L=\mathcal{O}_S(2C)$. Then $\varphi_L(S)$ is the Veronesse surface in $\mathbb{P}^5$; in fact $\varphi_L=\nu_2\circ\varphi_C$, where $\nu_2:\mathbb{P}^2\to\mathbb{P}^5$ is the Veronesse embedding.*

Remark 1.3.9. *Given any integer $g\geq 2$, there is a K3 surface S and an irreducible curve $C\subset S$ such that $g(C)=g$. See [Bea96, Proposition VIII.15] for details.*

Finally, we give a sketch of a proof for the Saint-Donat's Theorem, which was mentioned at the beginning of this Chapter.

Theorem 1.3.10. *Let X be a K3 surface and $L\in\mathrm{Pic}(X)$. If L is ample, then $L^{\otimes 3}$ is very ample.*

Proof. The idea of the proof is as follows. Notice that $L^{\otimes 3}\cdot E=3L\cdot E\in 3\mathbb{Z}$, then $L^{\otimes 3}\cdot E\neq 2$ for every curve E, and moreover

$$g(C)=h^0(X,\mathcal{O}_X(C))=\frac{9}{2}L^2+1\geq 5>4$$

for every $C\in|L^{\otimes 3}|$, since by Nakai's criterion, L ample implies $L^2>0$. By the Theorem 1.3.7 L is not hyperbolic. Then, by Theorem 1.3.1 we have that $\varphi_L:X\to\varphi_L(X)$ is birational. Finally Saint-Donat concludes by [SD74, Theorem 6.1] that L is very ample.

□

Chapter 2

MODULI SPACE OF POLARIZED K3 SURFACES

Throughout this Chapter do a quick review of [Deb20], where Debarre summarizes the construction of the moduli spaces of polarized K3 surfaces, and we also review [Kon20] to get a general perspective of lattice theory and its applications in studying K3 surfaces. In particular, the Torelli Theorem gives a connection between morphisms of $K3$ surfaces and homomorphisms of lattices.

2.1 Polarized K3 surfaces of low degree

A polarized variety is a pair (X, L) where X is an algebraic variety and $L \in \mathrm{Pic}(X)$. Following the models in §1.3 and [Deb20] we describe briefly the image of the embedding $\varphi_L : S \to \mathbb{P}^g$ for a polarized K3 surfaces (S, L) of low degree. We begin defining some constructions that appear as images of these embeddings.

Definition 2.1.1. *Let V be a vector space over a field k and $0 < r < \dim(V)$ a positive integer. We define the grassmannian*

$$\mathrm{Gr}(r, V) := \{W \subseteq V : \text{ subspace with } \dim(W) = r\}$$

as the set of subspaces of V of dimension r. If $V = k^n$ is the affine space, then we simply write $\mathrm{Gr}(r, n)$.

We define an algebraic atlas on $Gr(r,V)$ given by open sets

$$U_I := \{W \in Gr(r,V) : W \cap I = \{0\}\} \simeq \mathbb{A}^{n(n-r)}$$

parameterized by subspaces $I \subset V$ of dimension $n-r$, giving $Gr(r,V)$ a structure of algebraic variety. Moreover, the Plücking embedding $\varphi : Gr(r,V) \hookrightarrow \mathbb{P}^N$, $W \mapsto \bigwedge^m W \simeq k$ shows that $Gr(r,V)$ is a projective variety.

Remark 2.1.2. *Let V be a vector space over a field k and $0 < r < \dim(V)$ a positive integer.*

1. *The projective space is the grassmannian $\mathbb{P}(V) = Gr(1,V)$.*
2. *Similar to the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ we can construct a tautological vector bundle of rank r as follows: Consider the algebraic variety*

$$S := \{(x,W) \in V \times Gr(r,V) : x \in W\}$$

and the projection to the second coordinate $p : S \to Gr(r,V)$, $(x,W) \mapsto p(x,W) = W$, then $p^{-1}(W) = W \simeq \mathbb{A}^r$ is a subspace of dimension r, hence S is a vector bundle of rank r, called the ***tautological bundle*** *of $Gr(r,V)$.*

3. *The* ***tautological quotient bundle*** *Q is defined by*

$$Q := \{(x,W) \in V \times Gr(r,V) : x \notin W\}/\sim$$

where $(x,W) \sim (y,U)$ if and only if $x - y \in U = W$. It is clear that the map $q : Q \to Gr(r,V)$, $q([(x,W)]) = W$ is well defined and

$$q^{-1}(W) = \{(x,W) \in V \times Gr(r,V)\}/\sim \;\simeq \{(x,W) : x \in V/W\} \simeq \mathbb{A}^{\dim(V)-r}$$

then Q is a vector bundle of rank $\dim(V) - r$ and the quotient map $\pi : V \times Gr(r,V) \to Q$ is a morphism of vector bundles since $\pi_W : V = p^{-1}(W) \to q^{-1}(W) = V/W$ is the quotient map (and hence linear) and $q \circ \pi = p$. Moreover, $\ker(\varphi) = S$ (or equivalently $(V \times Gr(r,V))/S \simeq Q$).

Similarly to grassmannian varieties, we can consider now a k-vector space admitting a symmetric non-degenerate bilinear from b. Then, for a non negative integer $0 < r < n$, the set

$$\begin{aligned} OGr(r,V) &:= \{W \subset V : W \text{ subspace of } \dim W = r \text{ isotropic with respect to } b\} \\ &= \{W \in Gr(r,V) : q(W) = 0\} \end{aligned}$$

is called the **orthogonal grassmanian**, where $q(x) := b(x,x)$ is the quadratic form associated to b. Since $q(W) = 0$ is a closed condition, $OGr(r,V)$ is again an algebraic variety, and moreover, $OGr(r,V)$ is again projective.

Below we give a description of general polarized K3 surfaces of low degree, mainly given by Mukai and compiled by Debarre in [Deb20]. In the following (S,L) is a polarized K3 surface and $\varphi_L : S \to \mathbb{P}^g$ the morphism induced by L.

- $L^2 = 2$. The morphism $\varphi_L : S \to \mathbb{P}^2$ is a double cover branched over a smooth plane sextic curve. Conversely, any such double cover is a polarized K3 surface of degree 2.

- $L^2 = 4$. The morphism $\varphi_L : S \to \mathbb{P}^3$ induces an isomorphism between S and a smooth quartic surface (we proved in Proposition 1.3.5 the case when $\mathrm{Pic}(S) = \mathbb{Z}[L]$). Conversely, any smooth quartic surface in $\mathbb{P}^3$ is a polarized K3 surface of degree 4 as shown in Example 1.3.2.

- $L^2 = 6$. The morphism $\varphi_L : S \to \mathbb{P}^4$ induces an isomorphism between S and the intersection of a quadric and a cubic (see Proposition 1.3.6). Conversely, any smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$ is a polarized K3 surface of degree 6 (Example 1.3.2).

- $L^2 = 8$. The morphism $\varphi_L : S \to \mathbb{P}^5$ induces an isomorphism between S and the intersection of 3 quadrics. Conversely, any smooth complete intersection of 3 quadrics in $\mathbb{P}^5$ is a polarized K3 surface of degree 8.

- $L^2 = 10$. The morphism $\varphi_L : S \to \mathbb{P}^6$ is a closed embedding. Its image is obtained as the transverse intersection of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, a quadric $Q \subset \mathbb{P}^9$, and a $\mathbb{P}^6 \subset \mathbb{P}^9$. Conversely, any such smooth complete intersection is a polarized K3 surface of degree 10.

- $L^2 = 12$. The morphism $\varphi_L : S \to \mathbb{P}^7$ is a closed embedding. Its image is obtained as the transverse intersection of the orthogonal Grassmannian $\mathrm{OGr}(5,10) \subset \mathbb{P}^{15}$ and a $\mathbb{P}^8 \subset \mathbb{P}^{15}$. Conversely, any such smooth complete intersection is a polarized K3 surface of degree 12.

- $L^2 = 14$. The morphism $\varphi_L : S \to \mathbb{P}^8$ is a closed embedding. Its image is obtained as the transverse intersection of the Grassmannian $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$ and a $\mathbb{P}^8 \subset \mathbb{P}^{14}$. Conversely, any such smooth complete intersection is a polarized K3 surface of degree 14.

- $L^2 = 16$. General K3 surfaces of degree 16 are exactly the zero loci of general sections of the rank-7 vector bundle $\mathcal{O}(1)^{\oplus 4} \oplus \mathcal{S}(1)$ on $\mathrm{Gr}(3,6)$ (see [Muk89, Example 1]).

- $L^2 = 18$. General K3 surfaces of degree 18 are exactly the zero loci of general sections of the rank-8 vector bundle $\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{Q}^\vee(1)$ on $\mathrm{Gr}(2,7)$ (see [Muk89, Example 1]).

2.2 Lattice theory

It is fundamental for the study of moduli space of K3 surfaces and Torelli Theorem to notice that if X is a K3 surface, then $H^2(X,\mathbb{Z})$ has the structure of a Lattice. In fact, for all K3 surfaces $H^2(X,\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8^{\oplus 2} =: L_{K3}$ and this is one of the keys to construct the moduli space. In this Section we give a brief comment on lattice theory necessary for the following Sections, for this we will follow [Kon20, Chapter 1].

Definition 2.2.1. *Let V be a vector space over a field k of characteristic $\mathrm{char}(k) = 0$. A **lattice** is a free abelian group $\mathbb{Z}^{\dim(V)} \simeq L \subset V$ with a symmetric non-degenerate bilinear form $\langle\, , \rangle : L \times L \to \mathbb{Z}$.*

We denote $L^\vee$ the dual $\mathrm{Hom}(L,\mathbb{Z})$ and for $x \in L$, we define $f_x \in L^\vee$ by $f_x(y) = \langle x, y\rangle$. Since the product is non-degenerate, the natural map

$$L \to L^\vee, \quad x \mapsto f_x$$

is injective.

Two lattices L_1, L_2 are isomorphic if there is an isomorphism between free abelian groups L_1 and L_2 preserving the bilinear forms. An isomorphism of L to itself is called an **automorphism**. An isomorphism from L to itself is called an automorphism, and the group of automorphism is denoted $O(L)$.

Fixing a basis $\{e_1, \dots, e_r\}$ of L we denote $a_{ij} = \langle e_i, e_j\rangle \in \mathbb{Z}$. Writing $x = \sum_{i=1}^r x_i e_i$ we have that $f(x) = \langle x, x\rangle = \sum_{i,j=1}^r a_{ij} x_i x_j$ is a quadratic form. By Sylvester's Theorem there is a basis (of $L \otimes \mathbb{R}$) such that

$$f(x) = t_1^2 + \cdots t_p^2 - t_{p+1}^2 - \cdots - t_{p+q}^2$$

where $p + q = r$ and t_j are variables over $\mathbb{R}$. We say that the **signature** of L is (p,q) and define the $d(L) := |\det((a_{ij}))|$. A lattice is called **unimodular** if $d(L) = 1$.

Remark 2.2.2. *A lattice is unimodular if and only if $x \mapsto f_x$ is an isomorphism.*

A lattice is called even if $x^2 := \langle x, x\rangle$ is even for all $x \in L$. If it is not even is called an odd lattice.

Example 2.1. *1. We denote by $I_\pm$ the rank 1 lattice with the quadratic form $f(x) = \pm x^2$. Then $I_+^{\oplus p} \oplus I_-^{\oplus q}$ is an odd lattice with signature (p,q).*

2. *We denote by U the lattice of rank 2 defined by the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *This is an even unimodular lattice of signature (1, 1). And $U(m)$ is a lattice of rank 2 defined by the matrix* $\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$.

A negative definite lattice generated by elements of norm -2 (i.e., $x^2 = -2$) is called a root lattice.

Example 2.2. *Consider $\mathbb{Z}^{m+1}$ as $I^{(m+1)}$ and define the sublattice*

$$A_m = \left\{ (x_1, \dots, x_{m+1}) \in \mathbb{Z}^{m+1} : \sum_{i=1}^{m+1} x_i = 0 \right\}.$$

If $(e_1, \dots, e_{m+1})$ is the canonical basis, then $(r_1, \dots, r_m)$ is a basis of A_m, where $r_j := e_j - e_{j+1}$. Moreover

$$r_j^2 = e_j^2 - 2\langle e_j, e_{j+1}\rangle + e_{j+1}^2 = -2$$

To describe root lattices it is convenient to use Dynkin diagrams. We represent a vertex as ○ for each r_j, and the join of two vertexes by a $\langle r_i, r_j\rangle$-edge. In table 2.1 we find the main examples of Dynkin diagrams.

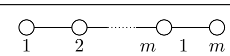
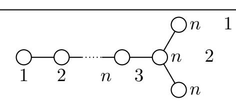
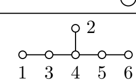
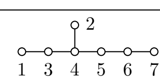
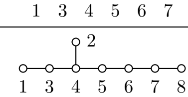
Lattice	Dynkin diagram
A_m $(m \geq 1)$	
D_n $(n \geq 4)$	
E_6	
E_7	
E_8	

Table 2.1. Lattice and Dynkin Diagrams

A root lattice is called irreducible if its Dynkin diagram is connected. Any root lattice is the orthogonal direct sum of irreducible lattices.

Proposition 2.2.3. *A connected Dynkin diagram is of type A_m, D_n, $n \geq 4$ or E_k for $k \in \{6, 7, 8\}$.*

Proposition 2.2.4. *Let L be an indefinite even unimodular lattice of signature (p,q). Then*

1. $L \simeq U^{\oplus p} \oplus E_8^{\oplus (q-p)/8}$ *if* $p \leq q$.
2. $L \simeq U^{\oplus q} \oplus E_8(-1)^{\oplus (p-q)/8}$.

In particular, the isomorphism class of L is determined by its signature.

Definition 2.2.5. *Let L, S be lattices. A linear map from S to L preserving the bilinear form is called an **embedding**. In this case, by identifying S with the image, S can be considered a sublattice of L. An embedding $S \subset L$ of lattices is called **primitive** is the quotient L/S if torsion free.*

A final construction that we will need is called the **overlattice**. Let L be an even lattice. A subgroup H of $A_L := L^\vee/L$ (considering L a sublattice of $L^\vee$ by the canonical injection) is called **isotropic** if $q_L|_H = 0$, where $q_L(x) = \langle x, x\rangle \mod 2\mathbb{Z} \in \mathbb{Q}/2\mathbb{Z}$. For an isotropic group we define

$$L_H := \{x \in L^\vee : x \mod L \in H\}$$

Then $(L_H, \langle\,,\rangle)$ is an even lattice because H is isotropic. It follows from the definition that

$$L \subset L_H \subset L_{\overline{H}} \subset L^\vee \quad \text{and} \quad d(L) = d(L_H) \cdot [L_H : L]^2.$$

An **overlattice** is an even lattice containing L as a sublattice of finite index. For example, L_H is an overlattice of L. Conversaly, for any overlattice L' of L, L'/L is an isotropic subgroup of A_L.

Theorem 2.2.6. *The set of overlattices of L bijectively corresponds to the set of isotropic subgroups of A_L.*

2.3 Period domain

Remark 2.3.1. *For a connected projective surface S, by Poincare's duality we have that $H^4(S,\mathbb{Z}) \simeq H_0(S,\mathbb{Z}) \simeq \mathbb{Z}$ and then the cup product*

$$\langle\,,\,\rangle : H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to \mathbb{Z}$$

is a non-degenerate symmetric bilinear form modulo torsion and hence $H^2(S,\mathbb{Z})/\{torsion\}$ is a lattice. From the exponencial sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

we have a long exact sequence in cohomology

$$\to H^1(S,\mathcal{O}_S) \to H^1(S,\mathcal{O}_S^*) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O}_S) \to$$

where c_1 *is the first Chern class. For a line bundle* $L, L' \in \mathrm{Pic}(S) \cong H^1(S,\mathcal{O}_S^*)$ *we denote* $L \cdot L' := c_1(L) \cdot c_1(L') = \langle c_1(L), c_1(L')\rangle$ *the cup product. For irreducible curves* $C, C' \subset S$ *their intersection number* $C \dot{C}'$ *is defined and coincides with the cup product* $\mathcal{O}_S(C) \cdot \mathcal{O}_S(C')$.

Proposition 2.3.2. *Let* X *be a K3 surface, then*

$$(H^2(X,\mathbb{Z}),\langle\,,\,\rangle) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$$

as lattices, where $\langle\,,\,\rangle$ *is the intersection product.*

Proof. By Noether's formula

$$c_1(X)^2 + c_2(X) = 12(p_g(X) - q(X) + 1) = 12(1 - 0 + 1) = 24$$

then $c_2(X) = 24$ since $c_1(X) = -c_1(\mathcal{O}_X(K_X)) = -c_1(\mathcal{O}_X) = 0$. By Gauss-Bonnet formula (see [GH14])

$$\chi(X) = c_2(X) = 24$$

and by definition

$$24 = \chi(X) = \sum_{i=0}^{4} h^i(X,\mathcal{O}_X) = 1 - 0 + h^2(X,\mathbb{Z}) - 0 + 1$$

it follows from the universal coefficient Theorem that $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}$ since $H_1(X,\mathbb{Z})$ is torsion free. By Hirzebruch's index Theorem,

$$b^+(X) - b^-(X) = \frac{1}{3}(c_1(X)^2 - 2c_2(X)) = -16$$

where $(b^+(X), b^-(X))$ is the signature of $H^2(X,\mathbb{R})$ and so $H^2(X,\mathbb{Z})$ has signature $(3,19)$. By Poincaré duality it follows that $H^2(X,\mathbb{Z})$ is unimodular. Finally, let $x \in H^2(X,\mathbb{Z}/2\mathbb{Z})$, then following the results and notation in Appendix A.1, we have

$$(\langle x, x\rangle, \mu) = (Sq^2(x), \mu)$$

where $\mu \in H_4(X,\mathbb{Z}/2\mathbb{Z})$ is the fundamental class, $(\ ,\mu) : H^4(X,\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is the kroneker index that can be understood as the evaluation the cycle in μ and $Sq^i : H^n(X,\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X,\mathbb{Z}/2\mathbb{Z})$ are operators such that $Sq^0 = 0$, $Sq^n(a) = a \cup a$ and $Sq^i = 0$ if $> n$. Moreover, if w_i are the Whitney classes of X, then

$$w_2 = Sq^0(v_2) + Sq^1(v_1) + Sq^2(v_0) = v_2 + Sq^1(v_1)$$

but since X is a complex manifold and therefore orientable, it follows from Appendix A.1 that

$$0 = w_1 = Sq^0(v_1) + Sq^1(v_0) = v_1$$

hence $w_2 = v_2$ and

$$(\langle x, x\rangle, \mu) = (Sq^2(x), \mu) = (\langle x, v_2\rangle, \mu) = (\langle x, w_2\rangle, \mu).$$

Recall that w_2 is the reduction modulo 2 of $c_1(X) = 0$ and then

$$(\langle x, x\rangle, \mu) = (\langle x, w_2\rangle, \mu) = 0$$

therefore $\langle x, x\rangle$ is even for every $x \in H^2(X, \mathbb{Z})$. We conclude by Proposition 2.2.4.

□

Remark 2.3.3. *We denote* $L_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}$.

A very important family of K3 surfaces are the **Kummer surfaces**. The construction is as follows: Let $A = \mathbb{C}^2/\Gamma$ be a complex torus and $-1_A : A \to A$ the isomorphism given by $a \mapsto -a$, which has exactly 16 fixed points. We then consider the closed subscheme $Z = \{p_0, \dots, p_{15}\}$ consisting in this fixed points and $\varepsilon : \mathrm{Bl}_Z(A) \to A$ the blow-up of A along Z. Since p_j are invariant under -1_A, there exists a unique automorphism $\sigma \in \mathrm{Aut}(\mathrm{Bl}_Z(A))$ such that $\epsilon \circ \sigma = -1_A \circ \varepsilon$, or equivalently, -1_A acts on $\mathrm{Bl}_Z(A)$, so we define $\mathrm{Km}(A) := \mathrm{Bl}_Z(A)/\langle -1_A\rangle$. We call this surface a Kummer surface.

Proposition 2.3.4. *Let* $X = \mathrm{Km}(A)$ *be a Kummer surface. Then* X *is a K3 surface.*

Proof. Lets start by showing that X is smooth. Let $(U, (x, y))$ be local coordinates around a point in A, then the blow-up around that point looks like the

$$\mathrm{Bl}_0(\mathbb{A}^2) = \{((x, y), [s : t]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt = sy\}.$$

It follows that the blow-up $\mathrm{Bl}_Z(A)$ has local coordinates $(U_1, (x, t))$ and $(U_2, (y, s))$ covering $\varepsilon^{-1}(U)$ with

$$U_1 = \{t \neq 0\} = \{xt = y\}, \quad U_2 = \{s \neq 0\} = \{x = sy\}.$$

On the intersection we have $xt = y$ and $x = sy$ and so $st = 1$ and $x = sy$. Then the blow-up map acts as $\varepsilon(x, t) = (x, xt)$ on U_1 and $\varepsilon(y, s) = (ys, y)$ on U_2. Since -1_A acts on U as $-1_A(x, y) = (-x, -y)$ we have that -1_A is lifted as $-1_A(x, t) = (-x, t)$ on U_1

and $-1_A(y,s)=(-y,s)$ on U_2. Then the quotient $\mathrm{Km}(A)$ by the action of -1_A has local coordinates (x^2,t) and (y^2,s), so $X=\mathrm{Km}(A)$ is non-singular.

Now, let $E_j=\varepsilon^{-1}(p_j)$ be the exceptional curves, then we have $g(E_j)=0$ and $E_j^2=-2$. By the formula for the canonical bundle for the blow-up we have that

$$\mathcal{O}_Y(K_Y)=\varepsilon^*\mathcal{O}_A(K_A)+\mathcal{O}_Y\left(\sum_{j=0}^{15}E_j\right)=\mathcal{O}_Y\left(\sum_{j=0}^{15}E_j\right).$$

Moreover, since $\sigma:Y\to X$ is a morphism of degree 2, we have that

$$\mathcal{O}_Y(K_Y)=\sigma^*\mathcal{O}_X(K_X)+\mathcal{O}_Y\left(\sum_{j=0}^{15}E_j\right)$$

it follows that $\sigma^*\mathcal{O}_X(K_X)\simeq\mathcal{O}_Y$ and hence $K_X=0$ (more details on [Huy16, Example 1.3 (iii)]). The complex torus A has $\chi(A)=0$ and then $\chi(Y)=16$, so by Noether's formula

$$16=\chi(Y)=2\chi(X)-16\chi(\mathbb{P}^1)=2\chi(X)-32$$

obtaining $\chi(X)=24$, hence

$$24=\chi(X)=12\sum(-1)^ih^i(X,\mathcal{O}_X)$$

from where we conclude that $q(X)=h^1(X,\mathcal{O}_X)=0$ and thus X is a K3 surface.

□

Let $X=\mathrm{Km}(A)$ be a Kummer surface associated to a complex tori $A=\mathbb{C}^2/\Gamma$. Then $\varepsilon:\mathrm{Bl}_Z(A)\to X$ is a double covering branched along $E_1,\dots,E_{16}$ and then $\frac{1}{2}\sum_{i=1}^{16}E_i\in S_X$. Moreover, from [Kon20, Corollary 6.20] we have the following characterization of Kummer surfaces.

Theorem 2.3.5. *Let X be a K3 surface and assume X contains 16 mutually disjoint non-singular rational curves $E_1,\dots,E_{16}$. Moreover, assume*

$$\frac{1}{2}\sum_{j=1}^{16}E_j\in S_X.$$

Then there exists a unique (up to isomorphism) complex tori A such that $X=\mathrm{Km}(A)$ and $E_1,\dots,E_{16}$ are the exeptional curves.

Remark 2.3.6. *The cohomology groups $H^i(X,\mathbb{C})$ admits a Hodge structure*

$$H^i(X,\mathbb{Z}) \simeq \bigoplus_{p+q=i} H^{p,q}$$

where $H^{p,q} := H^q(X,\Omega_X^p)$, in particular

$$H^2(X,\mathbb{Z}) \simeq H^0(X,\Omega_X^2) \oplus H^1(X,\Omega_X) \oplus H^2(X,\mathcal{O}_X(K_X)).$$

Since $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$, we have

$$H^0(X,\Omega_X^2) = H^0(X,\mathcal{O}_X(K_X)) \simeq H^0(X,\mathcal{O}_X) \simeq \mathbb{C}$$

hence there is a unique nowhere vanishing holomorphic 2-form ω_X on X up to constant. Then we have (extending the cup product to $H^2(X,\mathbb{C})$)

$$\langle \omega_X, \omega_X \rangle = \int_X \omega_X \wedge \omega_X \qquad \text{and} \qquad \langle \omega_X, \overline{\omega_X} \rangle = \int_X \omega_X \wedge \overline{\omega_X}$$

it follows that

$$\langle \omega_X, \omega_X \rangle = 0 \qquad \text{and} \qquad \langle \omega_X, \overline{\omega_X} \rangle > 0$$

or equivalently

$$\langle \mathrm{Re}(\omega_X), \mathrm{Re}(\omega_X) \rangle = \langle \mathrm{Im}(\omega_X), \mathrm{Im}(\omega_X) \rangle = 0 \quad \text{and} \quad \langle \mathrm{Im}(\omega_X), \mathrm{Re}(\omega_X) \rangle > 0.$$

Let $E(\omega_X) \subset H^2(X,\mathbb{R}) \simeq \mathbb{R}^{22}$ be the subspace generated by $\mathrm{Im}(\omega_X)$ and $\mathrm{Im}(\omega_X)$ and let $H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$ its orthogonal complement.

Lemma 2.3.7. *Let X be a K3 surface and ω_X the nowhere-vanishing holomorphic 2-form. Let $c \in H^2(X,\mathbb{Z})$, then the following are equivalent:*

1. *There exists $L \in \mathrm{Pic}(X)$ such that $c_1(L) = c$.*
2. *$c \in H^{1,1}(X,\mathbb{R})$.*
3. *$\langle c, \omega_X \rangle = 0$.*

Proof. The equivalence between 2 and 3 follows immediately since

$$\langle \omega_X, c \rangle = \langle \mathrm{Re}(\omega_X), c \rangle + i\langle \mathrm{Im}(\omega_X), c \rangle$$

where both products on the right are real valued and then $c \in H^{1,1}(X,\mathbb{R})$ if and only if $\langle c, \omega_X \rangle = 0$. Moreover, for $c \in H^2(X,\mathbb{Z})$, we have that this is also equivalent to $\langle c, \overline{\omega_X} \rangle = 0$.

From the exponential sequence we have

$$0 \to \mathrm{Pic}(X) \simeq H^1(X,\mathcal{O}_X) \to H^2(X,\mathbb{Z}) \xrightarrow{f} H^2(X,\mathcal{O}_X).$$

Recall that $H^2(X,\mathbb{Z})$ has a Hodge structure

$$H^2(X,\mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

where $H^{0,2}(X) \simeq H^2(X,\mathcal{O}_X)$. Moreover, f coincides with the projection $H^2(X,\mathbb{C}) \to H^{0,2}(X)$. It follows that if $c \in H^{1,1}(X,\mathbb{R})$, then $c \in H^{1,1}(X)$ and then $f(c) = 0$, hence by the exactness there exists an $L \in \mathrm{Pic}(X)$ such that $c_1(L) = c$.

□

We denote $S_X := c_1(\mathrm{Pic}(X))$ the Neron-Severi lattice.

Definition 2.3.8. *For a K3 surface X, we define*

$$\Delta(X) = \{\delta \in S_X : \langle\delta,\delta\rangle = -2\}.$$

Since $H^{1,1}(X,\mathbb{R}) = H^{1,1}(X) \cap H^2(X,\mathbb{R})$, we have that $H^{1,1}(X,\mathbb{R})$ has signature $(1,19)$ and then the cone

$$P(X) = \{x \in H^{1,1}(X,\mathbb{R}) : \langle x,x\rangle > 0\}$$

has two components. We denote by $P^+(X)$ the component containing the Kähler class[1] and is called the **positive cone**. We can decompose $\Delta(X)$ in the following:

$$\begin{aligned}\Delta(X)^+ &:= \{\delta \in \Delta(X) : \delta \text{ is effective}\},\\ \Delta(X)^- &:= \{-\delta \in \Delta(X) : \delta \in \Delta(X)^+\}.\end{aligned}$$

by Rieamnn-Roch (see 1.2) we have that for every $\delta \in S_X$, either δ is represented by an efective divisor or $-\delta$ is represented by an effective divisor, hence $\Delta(X) = \Delta(X)^+ \cup \Delta(X)^-$. This decomposition defines a fundamental domain (with respect to the subgroup $W \subset O(L_{K3})$ generated by reflections by $\Delta(X)$)

$$D(X) := \{x \in P^+(X) : \langle x,\delta\rangle > 0 \quad \forall \delta \in \Delta(X)^+\}.$$

Let $x \in P^+(X)$. By Riemann-Roch Theorem (see 1.2) an irreducible curve C has $g(C) \geq 1$ if and only if $C^2 \geq 0$ and $C^2 = -2$ if and only if C is a non singular rational curve ($g(C) = 0$). Since for any curve C with $C^2 \geq 0$ (C represents an element $y_C \in \overline{P^+(X)}$ we have that $\langle x, y_C\rangle > 0$ (see [Kon20, Lemma 2.3]) we have that $x \in D(X)$ if and only if x intersects

[1] Every K3 surfaces is Kälher [Siu83], that is, there is an hermitian metric on X such that the associated $(1,1)$ form is d closed

positive with every curve C. In particular, $D(X) \cap H^2(X, \mathbb{Z})$ is the set of ample classes. This leads to the following Proposition.

Proposition 2.3.9. *Let X, X' be K3 surfaces and $\phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ an isomorphism of lattices such that $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$. Then the following conditions are equivalent:*

1. *ϕ sends every effective divisor to an effective divisor.*
2. *ϕ sends every ample divisor to an ample divisor.*
3. *$\phi(D(X)) \subset D(X')$.*

Proof. Notice that $S_X \cap D(X)$ is the set of ample classes and then (2) and (3) are equivalent. Assume (2) and let C be an irreducible curve and $\delta \in S_X$ an ample class, then $\langle \delta, C \rangle > 0$. By Proposition 1.2.11 either $\phi(C)$ is effective or $-\phi(C)$ is effective. If $-\phi(C)$ is effective, since δ is ample, by hypothesis $\phi(\delta)$ is ample and then $\langle -\phi(C), \phi(\delta) \rangle > 0$, but ϕ is a lattice isomorphism, so

$$\langle \phi(\delta), \phi(C) \rangle = \langle \delta, C \rangle > 0$$

a contradiction, so we $\phi(C)$ is effective, concluding (1). Assume (2) and then $\phi(\Delta(X)^+) = \Delta(X')^+$, hence by definition of $D(X)$ we have (3). □

In the following, we define the **period domain**, that we will show that is isomorphic to the space of isomorphism classes of K3 surfaces.

Definition 2.3.10. *We define the period domain as*

$$\Omega := \{[\omega] \in \mathbb{P}(\mathbb{C} \otimes L_{K3}) : \langle \omega, \omega \rangle = 0, \quad \langle \omega, \overline{\omega} \rangle > 0\}$$

To see that Ω is well defined, we notice that

$$\langle \lambda\omega, \overline{\lambda\omega} \rangle = \lambda\overline{\lambda}\langle \omega, \overline{\omega} \rangle$$

where $\lambda\overline{\lambda} = |\lambda|^2 > 0$ for every complex number $\lambda \in \mathbb{C} \setminus \{0\}$ and hence $\langle \lambda\omega, \overline{\lambda\omega} \rangle$ and $\langle \omega, \overline{\omega} \rangle$ have the same sign. Moreover, Ω is an open subset of the projective quadratic hypersurface

$$Q = \{[\omega] \in \mathbb{P}(\mathbb{C} \otimes L_{K3}) : q(\omega) = \langle \omega, \omega \rangle = 0\}.$$

It follows that Ω is a 20-dimensional complex manifold.

For polarized K3 surfaces (X, L) of degree $2d$, we can consider a primitive element $h_{2d} \in L_{K3}$ such that $h_{2d}^2 = 2d$ and denote $L_{K3,2d}$ the orthogonal complement of h_{2d}. By [Kon20, Lemma 1.45] the isomorphism class of $L_{K3,2d}$ is independent of the choice of h_{2d}. In the rest of the

Chapter, we fix $h_{2d} \in L_{K3}$. We then define

$$\Omega_{2d} := \{[\omega] \in \mathbb{P}(\mathbb{C} \otimes L_{K3,2d}) : \langle \omega, \omega \rangle = 0, \quad \langle \omega, \overline{\omega} \rangle > 0\}$$

similar to Ω, since $\mathrm{rk}(L_{K3,2d}) = 21$, we have that Ω_{2d} is a 19-dimensional complex manifold. We now connect the period domain to the set of isomorphism classes of K3 surfaces.

Definition 2.3.11. *Let X be a K3 surface. A **marking** is an isomorphism of lattices $\alpha_X : H^2(X, \mathbb{Z}) \to L_{K3}$. A **marked K3 surface** is (X, α_X), where X is a K3 surface and α_X a mark on X. For polarized K3 surfaces (X, H) of degree $H^2 = 2d$ we define a marking as an isomorphism $\alpha_X : H^2(X, \mathbb{Z}) \to L_{K3}$ such that $\alpha_X(H) = h_{2d}$.*

Remark 2.3.12. *It is clear from the previous that $[\omega_X] \in \Omega$. Moreover, for a polarized K3 surface (X, H), from Lemma 2.3.7, we have that $\langle c_1(H), \omega_X \rangle = 0$ and then $[\omega_X] \in \Omega_{2d}$.*

Definition 2.3.13. *Let M be the set of isomorphism classes of marked K3 surfaces. We define the **period map** as $\lambda : M \to \Omega$ given by $\lambda((X, \alpha_X)) = \alpha(\omega_X)$.*

2.4 Density of the period of Kummer surfaces

This Section is dedicated to show that the set of periods of Kummer surfaces is dense in Ω. We follow [Kon20, Chapter 6] to first notice that $\langle x, x \rangle \equiv 0 \mod 4$ characterize the transcendental lattice of Kummer surfaces and then show that this property will give us the density.

Notice that for every $\omega \in \Omega$, the subspace $E(\omega) \subset L \otimes \mathbb{R}$ generated by $\mathrm{Re}(\omega), \mathrm{Im}(\omega)$ is 2-dimensional and positive define, and $(\mathrm{Re}(\omega), \mathrm{Im}(\omega))$ is an oriented basis. Conversely, for a 2-dimensional positive definite subspace E with oriented basis (x_E, y_E) satisfying $x_E^2 = y_E^2$, we have that $\omega = x_E + i y_E \in \Omega$. Let $G_2^+(L_{K3})$ be the set of such subspaces and notice that the map $\Omega \to G_2^+(L_{K3})$, $\omega \mapsto E(\omega)$ is bijective.

Let $X = \mathrm{Km}(A)$ be a Kummer surface and $E_1, \ldots, E_{16}$ sixteen non-singular rational curves on X. Consider Π the primitive sublattice of $H^2(X, \mathbb{Z})$ which is an overlattice of $\langle E_1, \ldots, E_{16} \rangle$, i.e.,

$$\Pi := \left\{ \sum_{i=1}^{16} a_i E_i \in H^2(X, \mathbb{Z}) : a_i \in \mathbb{Q} \right\} \subset S_X.$$

By [Kon20, Corollary 6.26] we have that $\Pi^\perp \simeq U(2)^{\oplus 3}$ and then $S_X^\perp =: T_X \subset \Pi^\perp \simeq U(2)^{\oplus 3}$, so

$$\langle x, x \rangle \equiv 0 \mod 4$$

for every $x \in T_X$. In fact, if (e_1, e_2) is a basis of $U(2)$ and $x = x_1e_1 + x_2e_2$, then

$$\langle x, x\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4x_1x_2 \equiv 0 \mod 4.$$

The next Theorem shows the converse.

Theorem 2.4.1. *Let T be an positive definite lattice of rank 2 satisfying for every $x \in T$*

$$\langle x, x\rangle \equiv 0 \mod 4.$$

Then there exists a Kummer surface X with $T_X \simeq T$.

Proof. The idea of the proof is to construct a complex torus A with transcendental lattice $T(1/2)$. First, by hypothesis $T(1/2)$ is an even lattice and then can be embedded primitively in $U^{\oplus 3}$. Let Γ be a free abelian group of rank 4 and consider an embedding of $T(1/2)$ into $\bigwedge^2 \Gamma^\vee \simeq U^{\oplus 3}$. Let (x, y) be an oriented basis of $T(1/2) \otimes \mathbb{R}$ with $x^2 = y^2$ and $\omega = x + yi$. Then $\mathbb{C}\omega \subset \bigwedge^2 \Gamma^\vee \otimes \mathbb{C}$ satisfy the Riemann condition, in fact,

$$\langle \omega, \omega\rangle = x^2 - y^2 = 0, \quad \langle \omega, \overline{\omega}\rangle = x^2 + y^2 > 0$$

and hence $\mathbb{C}\omega$ is an isotropic 1-dimensional subspace. Then there exists $\eta_1, \eta_2 \in \mathbb{C} \otimes \Gamma^\vee =: \Gamma^\vee_{\mathbb{C}}$ such that $\omega = \eta_1 \wedge \eta_2$. Let H be the 2-dimensional subspace of $\Gamma^\vee_{\mathbb{C}}$ generated by η_1 and η_2. Then it follows that $\bigwedge^2 H = \mathbb{C}\omega$ and $\eta_1 \wedge \eta_2 \wedge \overline{\eta_1} \wedge \overline{\eta_2} \neq 0$. We then have that $H \cap \tilde{H} = \{0\}$ and so $\Gamma^\vee_{\mathbb{C}} = H \oplus \overline{H}$. Moreover,

$$\Gamma \to \mathbb{C}, \gamma \mapsto (\eta_1(\gamma), \eta_2(\gamma))$$

is an embedding. Then for $A = \mathbb{C}^2/\Gamma$ we have that $H_1(X, \mathbb{Z}) = \Gamma$ and so by the universal coefficient Theorem $H^1(X, \mathbb{Z}) = \Gamma^\vee$ and $H^2(X, \mathbb{Z}) = \bigwedge^2 H^1(X, \mathbb{Z}) = \bigwedge^2 \Gamma^\vee$. By construction, $H^2(A, \mathbb{Z})$ has signature (3, 3), but the signature of $NS(A)$ is of the form $(1, 1 - \rho)$, so $T_A \simeq T(1/2)$.

□

In the following we give sufficient conditions for the density in Ω to finally get the density of the period of Kummer surfaces. The results and proofs are from [Kon20].

Lemma 2.4.2. *Let $m, n \in \mathbb{Z}_{\geq 1}$ and let M be a lattice. Suppose that the set*

$$R = \{\mathbb{R}e \in \mathbb{P}(M \otimes \mathbb{R}) : e \text{ is primitive in } M \text{ and } \langle e, e\rangle \equiv m \mod n\}$$

is not empty. Then R is a dense subset in $\mathbb{P}(M \otimes \mathbb{R})$.

Proof. Since R is not empty, there exists $e_0 \in R$. Let $V \subset \mathbb{P}(M \otimes \mathbb{R})$ be a non-empty open subset. Let $e \in M$ be a primitive element such that $\mathbb{R}e \in V$. If $e = \pm e_0$, then $\mathbb{R}e \in R \cap V$. Now we assume $e \neq \pm e_0$ and consider the primitive sublattice $M' = M \cap (\mathbb{Q}e + \mathbb{Q}e_0)$ in M of rank 2. Since e is primitive, there is an element $f \in M'$ such that $\{e, f\}$ is a basis of M' and then $e_0 = ae + bf$ for some $a, b \in \mathbb{Z}$. From our first assumption, e_0 is primitive and then $e_0 = \pm f$ or a and b are coprimes. Therefore, for any natural number N we have that $e_N := e_0 + Nbe = (a + Nb)e + bf$ is primitive too and

$$\langle e_N, e_N \rangle = e_0^2 + N(Nb^2e^2 + 2b\langle e, e_0 \rangle) \equiv m \mod n$$

for every N multiple of m, i.e., $e_N \in R$. Moreover, since V is open, then for sufficiently large N we have

$$\mathbb{R}e_N = \mathbb{R}\left(e + \frac{1}{bN}e_0\right) \in V$$

i.e., $e_N \in R \cap V$.

□

Lemma 2.4.3. *The set B of 2-dimensional subspaces of $L \otimes \mathbb{R}$ generated by lattices T of rank 2 satisfying $\langle x, x \rangle \equiv 0 \mod 4$ for every $x \in T$ is dense in $G_2^+(L_{K3})$.*

The proof is rather technical and uses similar tools from lattice theory, and apply the previous Lemma so it will be omitted.

Finally, we are in condition to show the density of the periods of marked Kummer surfaces. For $\omega \in \Omega$, define

$$S_\omega = \{x \in L_{K3} : \langle x, \omega \rangle = 0\}, \quad T_\omega = S_\omega^\perp.$$

By the last Lemma it follows:

Theorem 2.4.4. *Let S be the subset of Ω consisting of ω satisfying the following conditions:*

1. $\mathrm{rank}(T_\omega) = 2$.
2. $x^2 \equiv 0 \mod 4$ *for every* $x \in T_\omega$.

Then S is dense in Ω.

Proof. It follows directly from the Lemma and the fact that there is a bijection between Ω and $G_2^+(L_{K3})$.

□

Corollary 2.4.5. *The set of periods of marked Kummer surfaces is dense in Ω.*

Proof. If $\omega = \alpha_X(\omega_X)$ is the period of a Kummer surface, then $T_\omega = T_X$ so the two conditions on Theorem 2.4.4 are satisfied. Conversely, if $\omega \in \Omega$ is such that T_ω satisfy the two conditions, then by Theorem 2.4.1 there is a Kummer surface X such that $T_X \cong T_\omega$ and then $\alpha_X(\omega_X) = \omega$. We conclude by Theorem 2.4.4.

□

2.5 Torelli Theorem

This Section is dedicated to state and give a sketch of the proof some important results for K3 surfaces, the Torelli type Theorems, that will be fundamental to show the injectivity and surjectivity of λ.

The Torelli Theorem was originally stated for Riemann surfaces: If S is a Riemann surface, we can associate to S an abelian variety called its jacobian $J(S)$, given by

$$J(S) = H^0(S, \Omega^1_S)^\vee / H_1(S, \mathbb{Z})$$

the Torelli Theorem says that if S, S' are Riemann surfaces such that $J(S) \cong J(S')$, then $S \cong S'$, i.e., we recover S from its jacobian. In this Section we give Torelli type Theorems for K3 surfaces that relates isomorphisms of K3 surfaces with isomorphisms of lattices and a sketch of the proof, starting with the Torelli Theorem for complex tori, then for Kummer surfaces, and finally using the density of the period of the Kummer surfaces in Ω to conclude the Theorem for every K3 surface.

We start by giving a local version of the Torelli Theorem: Let X be a K3 surface. Then, by [Kon20, Corollary 5.18] there is a complex analytic family $\pi : \mathcal{X} \to B$ of $X = X_{t_0} := \pi^{-1}(t_0)$ ($t_0 \in B$), i.e., π is holomorphic and proper and $\mathrm{rank}(J(\pi)) = \dim(B)$. Assume that the base space B is contractible. Then, the locally constant sheaf $R^2\pi_*(\mathbb{Z})$ is trivial. Let $\alpha : R^2\pi_*(\mathbb{Z}) \to L_B$ be an isomorphism of sheaves, where L_B is a constant sheaf over B. Thus, by fixing an isomorphism $L_B \to L_{K3}$, we can consider that each fiber of the complex analytic family π is a marked K3 surface (X_t, α_{X_t}). Associating $\lambda(t) = \alpha(\omega_{X_t})$ with a non-zero holomorphic 2-form ω_{X_t} on X_t we obtain an holomorphic map $\lambda : B \to \Omega$. We call λ the period map of the complex analytic family π. From [Kon20, Theorem 6.16] we get the following Theorem, called the Local Torelli Theorem.

Theorem 2.5.1. *The period map λ is isomorphic around a neighborhood of $t_0 \in B$.*

Corollary 2.5.2. *Any K3 surfaces are deformation equivalent.*

Recall that a complex tori is a quotient $A = V/\Gamma$, where V is a complex vector space and Γ is a free abelian group of rank $\dim(V)$. The Torelli Theorem for Complex Tori goes as follows (for the proof see for example [Kon20, Theorem 4.35]):

Theorem 2.5.3. *Let $A_1 = \mathbb{C}^2/\Gamma_1$ and $A_2 = \mathbb{C}^2/\Gamma_2$ be complex tori and $\phi : H^2(A_1, \mathbb{Z}) \to H^2(A_2, \mathbb{Z})$ be an ismomorphism of lattices such that*

1. *$(\phi \otimes \mathbb{C})(H^0(A_1, \Omega^2_{A_1})) = H^0(A_2, \Omega^2_{A_2})$,*
2. *there exists $\psi_2 : \Gamma_1 \otimes \mathbb{F}_2 \to \Gamma_1 \otimes \mathbb{F}_2$ such that $\phi = \psi_2 \wedge \psi_2 \mod 2$.*

Then there exists a unique isomorphism of complex manifolds $\varphi : A_2 \to A_1$ such that $\phi = \varphi^*$.

The Theorem says that we can recover ismorphisms of complex manifolds of the ones of lattices if they preserve 2-forms. Since Kummer surfaces are constructed from complex tori, the Torelli Theorem for Kummer surfaces follows:

Theorem 2.5.4. *Let X, Y be K3 surfaces and ω_X, ω_Y its non-vanishing 2-forms. Assume X is a Kummer surface and let $\phi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be an isomorphism of lattices such that:*

1. *$\phi(\omega_X) \in \mathbb{C}\omega_Y$.*
2. *$\phi(D(X)) = D(Y)$.*

Then, there is a unique isomorphism of complex manifolds such that $\varphi : Y \to X$ such that $\varphi^* = \phi$.

Proof. Let $E_1, \ldots, E_{16}$ be non-singular rational curves on X. Condition (2) and Proposition 2.3.9 shows that ϕ preserves effective divisors, hence $\phi(E_i)$ is effective for every i. Lets assume $\phi(E_i)$ is reducible, and let

$$\phi(E_i) = \sum_j m_j C_j$$

be the irreducible decomposition. As shown in the proof of Proposition 2.3.9 $\phi^{-1}(C_j)$ must be effective and

$$E_i = \sum_j m_j \phi^{-1}(C_j)$$

it follows by Riemann-Roch that

$$h^0(X, \mathcal{O}_X(E_i)) = 2 + \frac{1}{2}\left(\sum_j m_j C_j\right)^2 \geq 2$$

since $C_j^2 \geq 0$, which is a contradiction to the fact that E_i is rational (and then $h^0(X, \mathcal{O}_X(E_i)) = 1$), so $\phi(E_i)$ is irreducible. We have found 16 disjoint non-singular rational curve on Y, so by Theorem 2.3.5 Y is a Kummer surface. Let A, A' be complex tori such that $X = \mathrm{Km}(A)$ and $Y = \mathrm{Km}(A')$. Notice that ϕ induces an isomorphism $\psi : H^2(A, \mathbb{Z}) \to H^2(A', \mathbb{Z})$ which by (1) preserves holomorphic 2-forms. Moreover, ψ satisfies the condition of the Torelli Theorem for complex tori, and then there exists a unique isomorphism $\tilde{\psi} : A' \to A$ with $\tilde{\psi}^* = \psi$. Since $\tilde{\psi}$ preserves double points, it induces an isomorphism $\varphi : X \to Y$ and $\varphi^* = \phi$. □

We are now left to use Corollary 2.4.5 to show that the density of the period of Kummer surfaces implies the Torelli Theorem for K3 surfaces.

Theorem 2.5.5. *Let X, X' be K3 surfaces and let $\omega_X, \omega_{X'}$ be non-zero holomorphic 2-forms on X and X' respectively. Suppose that an isomorphism of lattices $\phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfies the following two conditions.*

1. $\phi(\omega_X) \in \mathbb{C}\omega_X$.
2. $\phi(D(X)) = D(X')$.

Then there exists a unique isomorphism $\varphi : X' \to X$ of complex manifolds with $\varphi^* = \phi$.

Proof. We only give a sketch of the proof. For more details refer to [Kon20, Chapter 6]. Let $\pi : \mathcal{X} \to B$, $\nu : \mathcal{X}' \to B$ be a complex analytic families (i.e., π, ν proper and $\operatorname{rank}(J(\pi)) = \operatorname{rank}(J(\nu)) = \dim(B)$) with $X_0 = X$, $X'_0 = X'$ $(0 \in B)$ and B contractible. ϕ induces isomorphisms $\phi_t : H^2(X_t, \mathbb{Z}) \to H^2(X'_t, \mathbb{Z})$. Let $\lambda : B \to \Omega$ be the local period map and $\omega = \lambda(0) \in \Omega$. Then, since the period of Kummer surfaces is dense in Ω (see Corollary 2.4.5) there is a sequence $\{\omega_n\} \subset \Omega$ converging to ω such that ω_n are periods of Kummer surfaces. By the local Torelli Theorem, λ is bijective and so $t_n := \lambda^{-1}(\omega_n) \to 0$ and $X_{t_n} = \pi^{-1}(t_n)$ are Kummer surfaces. By the Torelli Theorem for Kummer surfaces, there exists isomorphism of complex manifolds

$$\varphi_{t_n} : X'_{t_n} \to X_{t_n}$$

such that $\varphi_{t_n}^* = \phi_{t_n}$. As a final step, by [Kon20, Theorem 6.50] we have that there is a unique isomorphism $\varphi : X' \to X$ such that $\varphi^* = \phi$. To show this last result, Kondo follows esencially the following steps using that K3 surfaces are Kähler:

- Consider a sequence $t_i \subset K$ that converges to 0, with K a dense subset of B and let $\Gamma_i \subset X_{t_i} \times X'_{t_i}$ be the graphs of φ_{t_i}. Show that the graphs are bounded and then there is a subsequence of $\{\Gamma_i\}$ that converges to a 2-dimensional complex analytic subset Γ_0 in $X \times X'$.
- Show that the limit Γ_0 can be decompose as a graph of an isomorphism between X' and X plus the sum of some products of curves.
- Show that the coefficients by the products of curves are zero and therefore Γ_0 is the graph of an isomorphism, concluding the proof.

$\square$

Corollary 2.5.6. *(Weak Torelli) Let X, X' be K3 surfaces. Suppose that an isomorphism of lattices $\phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfies the condition $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$. Then X and X' are isomorphic.*

Proof. Notice that either $\phi(P(X)^+) = P(X')^+$ or $-\phi(P(X)^+) = P(X')^+$, so we may assume the first case. Then by lattice theory, there is a reflection induced by a root w such that $w \circ \phi(D(X)) = D(X')$. We conclude by the Torelli Theorem applied to $w \circ \phi$.

$\square$

Theorem 2.5.7. *Let (X, H), (X', H') be polarized K3 surfaces. Suppose that an isomorphism of lattices $\phi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfies the two conditions*

1. $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$,
2. $\phi(H) = H'$.

Then there exists a unique isomorphism $\varphi : X' \to X$ of complex analytic manifolds with $\varphi^* = \phi$.

Proof. Condition (2) implies by Proposition 2.3.9 that $\phi(D(X)) \subset D(X')$ and then we conclude by Torelli Theorem.

$\square$

2.6 Moduli Space of K3 surfaces

This Section is dedicated to the study of the period map λ and define a moduli space for marked K3 surfaces and polarized K3 surfaces.

Recall that the period map for marked K3 surfaces is defined by $\lambda : \mathcal{M} \to \Omega$, $\lambda((X, \alpha_X)) = \alpha_X(\omega_X)$, where $\mathcal{M}$ is the set of isomorphism classes of marked K3 surfaces. By all the previous discussion on this Chapter, we finally conclude.

Theorem 2.6.1. *The period map $\lambda : \mathcal{M} \to \Omega$ is bijective.*

Proof. Let (X, α_X), $(X'\alpha'_X)$ be two marked K3 surfaces such that $\lambda(X, \omega_X) = \lambda(X', \omega_{X'})$, then

$$\alpha_X(\omega_X) = \alpha_{X'}(\omega_{X'}) = \omega$$

We get an isomorphism $\phi = \alpha_{X'}^{-1} \circ \alpha_X : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ such that

$$\phi(\omega_X) = \alpha_{X'}^{-1}(\alpha_X(\omega_X)) = \alpha_{X'}^{-1}(\omega) = \omega_{X'}$$

In particular, $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$. We conclude by the weak Torelli Theorem that $X \cong X'$. Finally, the local Torelli Theorem gives us the surjectivity.

□

We can then say that Ω is the moduli space of K3 surfaces, although it is important to notice that this construction is completely analytical, that is, Ω is a complex manifold of dimension 20 as mention in previous Sections.

We know study the case of polarized K3 surfaces. Recall that for polarized K3 surfaces (X, H) of degree $2d$ we fix an element $h \in L_{K3}$ with $h^2 = 2d$ and define markings as isomorphism $\alpha_X : H^2(X, \mathbb{Z}) \to L_{K3}$ such that $\alpha_X(H) = h$. Let now

$$\Gamma_{2d} = \{\gamma \in O(L_{K3}) = \mathrm{Aut}(L_{K3}) : \gamma(h) = h\}$$

the set of automorphisms of L_{K3} fixing h. Then Γ_{2d} is a subgroup of Ω_{2d} that acts on Ω_{2d} properly discontinuously, so Ω_{2d}/Γ_{2d} has the structure of a complex analytic space. Let $\mathcal{M}_{2d}$ be the set of isomorphism classes of polarized K3 surfaces of degree $2d$. To each marked polarized K3 surface (X, H, α_X) of degree $2d$ we associate $\alpha_X(\omega_X) \mod \Gamma_{2d}$ in Ω_{2d}/Γ_{2d} which is independent of the choice of the marking. We have constructed a period map

$$\lambda_{2d} : \mathcal{M}_{2d} \to \Omega_{2d}/\Gamma_{2d}.$$

Similar to before, we have the following result as a direct consequence of the Torelli Theorem for polarized K3 surfaces.

Proposition 2.6.2. *The period map $\lambda_{2d} : \mathcal{M}_{2d} \to \Omega_{2d}/\Gamma_{2d}$ is injective.*

The surjectivity of the period map λ_{2d} is also obtained, but we will not prove it. For a sketch of the proof using theory of degenerations see [Kon20, Chapter 7].

2.7 Moduli spaces for polarized K3 surfaces

There is another way to get the moduli space of K3 surface. We can follow the construction of the moduli space of algebraic curves. Let C be a smooth projective irreducible curve of genus $g \geq 3$ such that C is not hyperelliptic (or equivalently K_C is very ample, see Theorem 1.1.24). Then the canonical embedding $\varphi_C := \varphi_{K_C} : \hookrightarrow |K_C| \cong \mathbb{P}^{g-1}$ is unique up to change of coordinates in $\mathrm{Aut}(\mathbb{P}^{g-1}) \cong \mathrm{PGL}_g(k)$. The image has degree $\deg(\varphi_C(C)) = 2g - 2$. Following this results, and recalling that is C is algebraic, projective, smooth and irreducible, then $3K_C = 6g - 6 \geq 2g + 1$ and then $L := \mathcal{O}_C(K_C)$ is very ample, moreover by Riemann-Roch:

$$h^0(C, L) = 1 - g + 3(2g - 2) = 5g - 5$$

then we have that L determines a closed embedding $\varphi_L : C \hookrightarrow \mathbb{P}^{5g-6}$ which image $\varphi_L(C)$ is a curve of degree $6g-6$ in $\mathbb{P}^{5g-6}$. Moreover, this curves have a fixed Hilbert polynomial

$$P(m) = (6m-1)(g-1).$$

By [Gro60] there is a Hilbert scheme $\mathcal{H} := \mathrm{Hilb}_{P,5g-6}$ parameterizing subvarieties Z of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P. A open subset $\mathcal{U} \subset \mathcal{H}$ parameterize smooth irreducible curves and then Mumford constructed the quotient $\mathsf{M}_g = \mathcal{U}/\operatorname{PGL}(5g-5)$ which is a quasi-projective variety that parameterize smooth irreducible curves of genus g.

For K3 surfaces we can make a similar construction, which is based in a result from Saint-Donat in [SD74, Theorem 8.3]. Recall Theorem 1.3.10 from Chapter 1:

Theorem 2.7.1. *Let X be a K3 surface and $L \in \operatorname{Pic}(X)$. If L is ample, then $L^{\otimes 3}$ is very ample.*

So, following the construction of M_g, we have that every polarized K3 surface (X, L) of a fixed degree $2e = L^2$ admits an embedding $\varphi_{L^{\otimes 3}} : X \hookrightarrow \mathbb{P}^N$, where by Riemann-Roch

$$N + 1 = h^0(X, L^{\otimes 3}) = \frac{9}{2}L^2 + 2 = 9e + 2.$$

Moreover, we notice that

$$P_X(m) = h^0(\mathbb{P}^{9e+1}, \mathcal{O}_{\mathbb{P}^{9e+1}}(m)) = h^0(X, L^{\otimes 3m}) = 2 + \frac{1}{2}L^{\otimes 3} \cdot L^{\otimes 3} = 9em^2 + 2.$$

Now, with e and $P = P_e$ fixed, again by [Gro60] there is a Hilbert scheme $\mathcal{H} := \mathrm{Hilb}_{P,9e+1}$ parameterizing subvarieties Z of dimension 2 of $\mathbb{P}^{9e+1}$ with Hilbert polynomial $P_Z = P$. We consider the open subset $\mathcal{U} \subset \mathcal{H}$ of smooth surfaces. Finally, using Geometric Invariant Theory (not without difficulties) it is possible to take the quotient by the canonical action of $\operatorname{PGL}(9e+2)$, constructing a quasi-projective coarse moduli space $\mathcal{F}_{2d}$ over $\mathbb{C}$. As in the previous Section, we will have that there is a bijection between $\mathcal{F}_{2d}$ and Ω_{2d}/Γ_{2d}, having an analytic and an algebraic construction.

Chapter 3

DEGREE OF IRRATIONALITY OF K3 SURFACES

This chapter is dedicated to the study of the degree of irrationality of K3 surfaces mainly on the method described by Moretti in [Mor23] and extend the result to the case of Enriques surfaces. First §3.1 offers an introduction and preliminaries on the degree of irrationality as a measure of how irrational a variety is, in the sense of how far is this variety from being birational to $\mathbb{P}^{\dim X}$. Then §3.2 consists of an introduction to stability of vector bundles and the study of the moduli space of stable bundles, since Moretti's method consists on finding stable vector bundles with low top chern class. In §3.3 there are some Lemmas that allows us to understand Moretti Theorem and we give a proof of the latest. Finally in §3.4 we applied Moretti's construction to Enriques surfaces, which are quotients of K3 surfaces by an involution.

The main results of this Chapter are Corollary 3.3.10 (see also Remark 3.3.11) in the case of K3 surfaces and Theorem 3.4.10 in the case of Enriques surfaces.

3.1 Preliminaries on the degree of irrationality

Definition 3.1.1. *Let X and Y be algebraic varieties. A* ***rational map*** *from X to Y is a regular morphism $\varphi : U \subseteq X \to Y$, where U is an open subset of X, we denote this by $\varphi : X \dashrightarrow Y$. Moreover, X and Y are said to be* ***birational*** *if there are rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ such that $f \circ g = id_{\mathrm{dom}(g)}$ and $g \circ f = id_{\mathrm{dom}(f)}$.*

*An algebraic variety X is **rational** if X is birational to $\mathbb{P}^{\dim(X)}$ and **unirational** if there is a dominant rational map $\varphi : X \dashrightarrow \mathbb{P}^{\dim(X)}$.*

Being birational can be understood as admitting a set of parameters in the projective space $\varphi : \mathbb{P}^N \dashrightarrow X$. For example, the regular complex algebraic curve

$$C = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$$

can be parametrized by

$$t \mapsto \left(\frac{1 - t^2}{1 + t^2}, \frac{t^2}{1 + t^2} \right)$$

i.e., there is a rational map $\varphi : \mathbb{P}^1 \dashrightarrow C$ defined on the dense open set $\mathbb{P}^1 \setminus \{[1 : i], [1, -i]\}$ given by

$$\varphi([z : w]) = \left(\frac{w^2 - z^2}{w^2 + z^2}, \frac{2zw}{w^2 + z^2} \right).$$

Moreover, φ has an inverse $\psi(x, y) = [y : x + 1]$ defined on the dense open set $C \setminus \{(-1, 0)\}$, in fact

$$\varphi \circ \psi(x, y) = \varphi([y : x + 1]) = \left(\frac{(x + 1)^2 - y^2}{(x + 1)^2 + y^2}, \frac{2(x + 1)y}{(x + 1)^2 + y^2} \right)$$

since $(x, y) \in C$ we have $(x + 1)^2 + y^2 = x^2 + 2x + 1 + y^2 = 2(x + 1)$ and $(x + 1)^2 - y^2 = x^2 + 2x + 1 - (1 - x^2) = 2x(x + 1)$, hence

$$\varphi \circ \psi(x, y) = \left(\frac{2x(x + 1)}{2(x + 1)}, \frac{2y(x + 1)}{2(x + 1)} \right) = (x, y).$$

And

$$\psi \circ \varphi([z : w]) = \left[\frac{2zw}{w^2 + z^2} : \frac{w^2 - z^2}{w^2 + z^2} + 1 \right] = [2zw : w^2 - z^2 + (w^2 + z^2)] = [z : w].$$

Then we conclude C is rational. On the contrary, the curve

$$E = \{(x, y) \in \mathbb{C}^2 : y^2 - x^3 + x = 0\}$$

is not rational. In fact, for every homogeneous polynomials $P, Q \in \mathbb{C}[T]$ such that $\varphi = (P, Q) : \mathbb{P}^1 \dashrightarrow E$ we have $\mathrm{Im}(\varphi) \not\subset X$, i.e., $Q^2 - P^3 + P \not\equiv 0$.

A central problem in algebraic geometry is how to determine if two varieties are birational, which is an open and very hard question. One important birational invariant is kodaira dimension. A very well known result is the following.

Theorem 3.1.2. *Let X and Y be projective, irreducible, smooth algebraic varieties, and* *assume X and Y are birational. Then*

$$H^0(X, K_X^{\otimes m}) \simeq H^0(Y, K_Y^{\otimes m}) \qquad \forall m \in \mathbb{N}_{\geq 1}.$$

In particular, the plurigenera $P_m(Z) = h^0(X, K_Z^{\otimes m})$ is a birational invariant.

Definition 3.1.3. *Let X be a projective, irreducible, smooth algebraic variety. The* ***Kodaira dimension*** *of X, denoted by $\kappa(X)$, is the Iitaka dimension of the canonical bundle K_X, i.e.,*

$$\kappa(X) \overset{\text{def}}{=} \begin{cases} \max\limits_{m \in \mathbb{N}_{\geq 1}} \dim(\overline{\varphi_{K_X{}^{m}}(X)}) & \text{if } \exists m_0 \in \mathbb{N}_{\geq 1} \text{ such that } P_{m_0}(X) \neq 0 \\ -\infty & \text{if } P_m(X) = 0 \quad \forall m \in \mathbb{N}_{\geq 1} \end{cases}.$$

In particular, $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim(X)\}$.

From the Theorem, we have that Kodaira dimension is a birational invariant. In particular, if we consider $E \subseteq \mathbb{P}^2$ as before, i.e.,

$$E = \{[x : y : z] \in \mathbb{P}^2 : y^2 z - x^3 + xz^2 = 0\}$$

then $E \simeq \mathbb{C}/\Lambda$ is an elliptic curve and therefore is compact. Moreover, is an abelian variety and hence $K_E \simeq \mathcal{O}_E$. Then

$$\kappa(E) = \dim(\overline{\varphi_{\mathcal{O}_E}(X)}) = 0.$$

Since $h^0(E, \mathcal{O}_E) = 1$.

In the case of the projective space, we have from Euler's exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

that $K_{\mathbb{P}^n} \simeq \Lambda^n \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$, then

$$P_m(\mathbb{P}^n) = h^0(\mathbb{P}^n, K_{\mathbb{P}^n}{}^{\otimes m}) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m(d+1))) = 0 \quad \forall m \in \mathbb{N}_{\geq 1}$$

hence $\kappa(\mathbb{P}^n) = -\infty$. In particular, E is not rational since $\kappa(E) \neq \kappa(\mathbb{P}^1)$.

Going back to the study of K3 surfaces, we get a similar analysis as in the case of the elliptic curve. Let X be a K3 surface, then since X is compact $h^0(X, \mathcal{O}_X) = 1$. Then

$$\kappa(X) = \max_{m \in \mathbb{N}_{\geq 1}} \overline{\varphi_{K_X{}^{m}}(X)} = \dim \overline{\varphi_{\mathcal{O}_X}(X)} = 0$$

hence X is not rational.

Having established criteria to determine whether an algebraic variety is not rational, we could estimate how much does the variety fails from being rational. In the case of curves, we have the gonality, defined by

$$\mathrm{gon}(C) = \min\{\deg(\varphi) : \varphi : C \dashrightarrow \mathbb{P}^1 \text{ is not constant}\}.$$

Then we have that C is rational if and only if $\mathrm{gon}(C) = 1$. Extending this definition, one measure for the irrationality is the **degree of irrationality** defined by

$$\mathrm{irr}(X) = \min\{\deg(\varphi) : \varphi : X \dashrightarrow \mathbb{P}^n \text{ dominant rational map}\}.$$

This invariant has been the object of interest of some authors, such as Bastianeli et al., who studied in [BDPE⁺17] measures of irrationality of hypersurfaces of large degree, in particular computing $\mathrm{irr}(X)$ and getting lower bound for other invariants, Chen, who got in [Che19] an upper bound for general abelian surfaces, in particular showed that $\mathrm{irr}(X) \leq 4$ for abelian surfaces with Picard number $\rho = 1$. For K3 surfaces, Stapleton showed in [Sta17] that if (X, L) is a polarized K3 surfaces of degree d, then $\mathrm{irr}(X) \geq 3(\sqrt{2})\sqrt{d}$ and conjectured that $\mathrm{irr}(X)$ grows asymptotically as $\sqrt{g}$. In the following Sections we study the method that Moretti used to get a better lower bound and support Stapleton's conjecture.

3.2 Moduli space of stable bundles

Following [HL10] we show some results about stability of vector bundles that will be useful get Moretti's construction.

Definition 3.2.1. *Let (X, H) be a polarized surface and let $\pi : E \to X$ be a vector bundle. We define the **slope of E with respect to H** as*

$$\mu_H(E) = \frac{c_1(E) \cdot H}{\mathrm{rank}(E)}.$$

*We say that E is H**-stable** (respectively μ_H-semi-stable) if for every subsheaf F, with rank $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$,*

$$\mu_H(F) < \mu_H(E) \quad \textit{(respectively } \mu_H(F) \leq \mu_H(E)\textit{)}.$$

Proposition 3.2.2. *Let $E \to X$ be an μ_H-semi-stable vector bundle on a polarized variety*

(X,H) and $N \subsetneq E$ a proper subsheaf of E. If $F = E/N$, then

$$\mu_H(N) \leq \mu_H(E) \leq \mu_H(F)$$

Proof. We have the short exact sequence

$$0 \to N \to E \to F \to 0$$

and then $c_1(E) = c_1(N) + c_1(F)$. Since N is proper and E is μ_H-semi-stable

$$\mu_H(N) = \frac{c_1(E) - c_1(F) \cdot H}{\mathrm{rk}(N)} = \frac{c_1(E) \cdot H}{\mathrm{rk}(N)} - \frac{c_1(F) \cdot H}{\mathrm{rk}(N)} \leq \mu_H(E)$$

then

$$\frac{c_1(E) \cdot H}{\mathrm{rk}(N)} - \mu_H(E) \leq \frac{c_1(F) \cdot H}{\mathrm{rk}(N)}.$$

It follows

$$\frac{\mathrm{rk}(E) - \mathrm{rk}(N)}{\mathrm{rk}(N)}\mu_H(E) \leq \frac{\mathrm{rk}(F)}{\mathrm{rk}(N)}\mu_H(F)$$

and hence

$$\mu_H(N) \leq \mu_H(E) \leq \mu_H(F).$$

□

Remark 3.2.3. 1. *The previous Proposition is actually an equivalence, i.e., if $\mu_H(E) \leq \mu_H(G)$ for every quotient $E \twoheadrightarrow G$, then E is H-semi-stable.*

2. *Using this equivalence and the exact sequences*

$$0 \to N \to E \to E/N \to 0 \quad \textit{if and only if} \quad 0 \to (E/N)^\vee \to E^\vee \to N^\vee \to 0$$

we have that E is μ_H-semi-stable (respectively μ_H-stable) if and only if $E^\vee$ is μ_H-semi-stable (respectively μ_H-stable).

3. *Every line bundle L is μ_H-stable.*

4. *If L is a line bundle and E a vector bundle, then $E \otimes L$ is μ_H-stable if and only if E is μ_H-stable (see [Tak72, Proposition 1.4]).*

Proposition 3.2.4. *Let $E, F \to X$ be two μ-semi-stable vector bundles over a polarized variety (X,H). If $\mu_H(E) > \mu_H(F)$, then $\mathrm{Hom}(E,F) = \{0\}$.*

Proof. Let $f : E \to F$ be a non zero morphism and $G = \mathrm{Im}(f)$. Then by the previous Proposition and the fact that E is μ_H-semi-stable

$$\mu_H(E) \leq \mu_H(E/\ker(f)) = \mu_H(G).$$

Then

$$\mu_H(E) \quad \mu_H(G).$$

Moreover, since F is H-semi-stable, then $\mu_H(G) \quad \mu_H(F)$ getting

$$\mu_H(E) \quad \mu_H(G) \quad \mu_H(F)$$

and hence

$$\mu_H(E) \quad \mu_H(F)$$

which is a contradiction, hence there is no such f.

□

Definition 3.2.5. *Let H be an ample divisor on a projective surface X. The Hilbert polynomial is defined by*

$$P(E,m) := \int_X \left(1, mH, \frac{m^2H^2}{2}\right) \quad v(E)$$

*for E a coherent sheaf. Writting $P(E,m) = \sum_{i=0}^{\dim(E)} a_i(E)\frac{m^i}{i!}$ we define the **reduced Hilbert polynomial** by*

$$p(E,m) := \frac{P(E,m)}{a_{\dim(E)}(E)}.$$

We say a pure dimensional sheaf is Gieseker semi-stable (respectively stable) if for every proper subobject $0 \neq F \subsetneq E$, $p(F,m) \quad p(E,m)$ (respectively $p(F,m) < p(E,m)$) for all $m \quad 0$.

Both types of stability share some of the same properties, thought this chapter we'll mainly refer to μ-stability because it is easier to calculate. There's a relation between booth definitions of stability

E is μ-stable $\Longrightarrow$ E is Gieseker-stable $\Longrightarrow$ E is Gieseker-semi-stable $\Longrightarrow$ E is μ-semi-stable

Definition 3.2.6. *A vector bundle on E is called **simple** if $\mathrm{End}(E,E) \cong \mathbb{C}$.*

It is always true that if E is μ-stable but the reciprocal is in general not true (see for example [HL10]). In some cases, we do get the reciprocal, as shown in [Mar75].

Theorem 3.2.7. *Let X be a non-singular projective variety over k with $\dim(X) \in \{2,3\}$ and $\mathrm{Pic}(X) \cong \mathbb{Z}$. Then a vector bundle E of rank 2 on X is stable if and only if E is simple.*

For the proof refer to the Maruyama's paper.

Definition 3.2.8. *Let X be a K3 surface. We define the lattice*

$$\tilde{H}(X,\mathbb{Z}) := H^0(X,\mathbb{Z}) \quad H^2(X,\mathbb{Z}) \quad H^4(X,\mathbb{Z}) \cong \mathbb{Z} \quad H^2(X,\mathbb{Z}) \quad \mathbb{Z}$$

with the product

$$\langle u_1, u_2 \rangle := \alpha_1 \alpha_2 - r_1 s_2 - s_1 r_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$$

where $u_i = (r_i, \alpha_i, s_i) \in \tilde{H}(X, \mathbb{Z})$. *Given a vector bundle bundle* $E \to X$*, we have a map* $E \mapsto v(E)$ *defined by*

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$$

called the ***Mukai vector*** *of* E*, which we identify with* $(\mathrm{rk}(E), c_1(E), \chi(E) - \mathrm{rk}(E))$*, where* $c_1(E) := c_1(\det(E))$ *is its first Chern class.*

Let X be a K3 surface, $L \in \mathrm{Pic}(X)$ ample, then Mukai in [Muk84a] showed that there exists $M_X(r, L, s)$ the moduli space of semi-stable vector bundles E of rank r, with $c_1(E) = L$ and $\chi(E) = r + s$. Moreover, Mukai also showed in [Muk84b] that the Moduli space of simple bundles exists and denoted by Spl_X, is smooth and has a symplectic structure. Moreover, for any $E \in \mathrm{Spl}_X$, mukai showed that

$$\dim_E \mathrm{Spl}_X = c_1(E)^2 - 2\,\mathrm{rk}(E)(\chi(E) - \mathrm{rk}(E)) + 2.$$

In particular, by Theorem 3.2.7, we have that the component of simple bundle with mukai vector $(2, L, s)$ is $M_X(2, L, s)$. Mukai also showed that in the limit case $v^2 = -2$, $M_X(v)$ is empty or a reduced point and when $v^2 = 0$ we have $M_{L;X}(v) \neq \emptyset$ and therefore is a surface (in particular is a K3 surface), where $M_{L;X}(v)$ is the moduli space of μ_L-semi-stable bundles with $v(E) = v$. Later on, continuing Mukai's work, in [MYY14] the authors defined Bridgeland's stability conditions and showed the existence of the moduli space of vector bundles with that stability conditions and proved that if $\mathrm{Pic}(X) = \mathbb{Z}[L]$, then for a general stability conditions σ, the moduli space of σ-semi-stable with mukai vector $v = (r, L, s)$ is isomorphic to the space of Gieseker semi-stable bundles $M_{L;X}(v)$ (c.f., §3) and in [BM14] the authors proved that the for every $v \in \tilde{H}(X, \mathbb{Z})$ such that $v^2 \geq -2$, the moduli space of σ-semi-stable bundles is not empty (c.f., §6). By all the work before, in particular we have the following Proposition.

Theorem 3.2.9. *Let* (X, L) *be a polarized K3 surface with* $\mathrm{Pic}(X) \cong \mathbb{Z}[L]$ *and* $v = (r, L, s) \in \tilde{H}(X, \mathbb{Z})$ *primitive with* $v^2 \geq -2$*. Then* $M_{L;X}(v) \neq \emptyset$*. In particular, if* $v^2 = 0$ *then* $M_{L;X}(v)$ *is a K3 surface and if* $v^2 = -2$ *then* $M_{L;X}(v)$ *is a reduced point.*

3.3 The polarized degree of irrationality of K3 surfaces

From general theory, we know that if L is a line bundle and $V \subseteq H^0(X, L)$ a subspace, then there is an induced rational map

$$\varphi_V : X \dashrightarrow \mathbb{P}^{\dim V - 1}$$

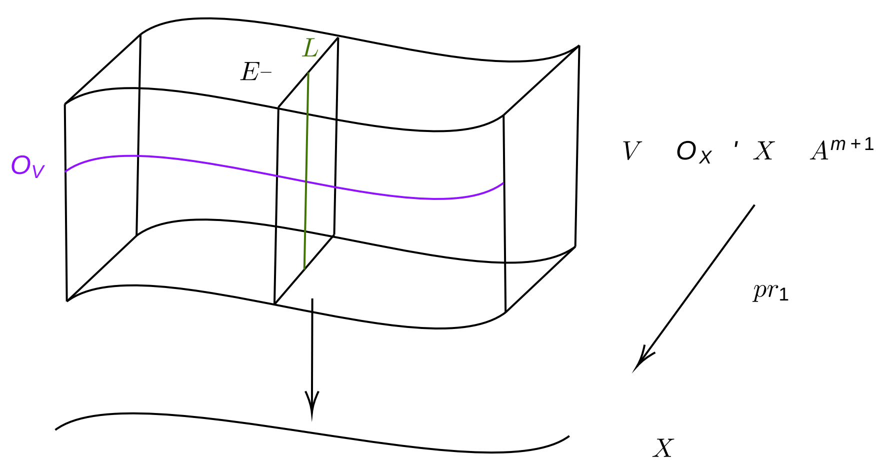
through the next two Chapters, we shall fix the notation φ_V to describe this rational map induced by a subspace V of the 0th cohomology group of a line bundle.
For a polarized n-dimensional variety (X, L), where X is an n-th dimensional variety and L an ample divisor, we define the **polarized degree of irrationality** as

$$\deg_L(X) = \min\{\deg(\varphi_V : X \dashrightarrow \mathbb{P}^n) \text{ where } V \in \mathrm{Gr}(n+1, H^0(X, L)) \text{ defined in codim } n\}$$

In the following Sections, we will center in the study of complex surfaces, in particular §3.3 is dedicated to follow the work of Moretti in [Mor23] and Moretti and Rojas in [MR24] who studied the degree of irrationality of K3 surfaces and §3.4 consists on the study of Enriques surfaces and applying Moretti's methods to study its degree of irrationality.
Let X be a variety of $\dim(X) = m$ and L an ample divisor. Let I be a sheaf of ideals such that $V = |L \otimes I| \in \mathrm{Gr}(m+1, H^0(X, L))$. We have an exact short sequence

$$0 \longrightarrow E^* \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L \otimes I \longrightarrow 0$$

where E is a reflexive sheaf, i.e., $E^{**} \simeq E$. We have that $V^* \subseteq H^0(X, E)$.



Remark 3.3.1. *Let E_1, E_2, E_3 vector bundle and the exact sequence*

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0.$$

Then

$$\det(E_2) \cong \det(E_1) \otimes \det(E_3) \Longrightarrow c_1(E_2) = c_1(E_1) + c_1(E_3).$$

In our case $c_1(E_3) = L$.

Lemma 3.3.2. *Let X be a smooth projective variety of dimension n. Let E be a rank n vector bundle with first Chern class L together with $V_- \in \mathrm{Gr}(n+1, H^0(X,E))$ such the determinant map $\bigwedge^n V_- \to V \subseteq H^0(X,L)$ is an isomorphism. Then*

$$\deg(\varphi_V) = c_n(E) - \deg(Z(V_-))$$

where $Z(V_-) = \bigcap_{s \in V_-} Z_{\mathrm{cycle}}(s)$ is the intersection of the zero loci of the sections of V_- (i.e., we are taking the degree of a general element minus the degree of the fixed part as zero cycles).

Proof. Observe that $E = \varphi_V^*(T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1))$. In fact, from

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow T\mathbb{P}^n \longrightarrow 0$$

twisting by $\mathcal{O}_X(-1)$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow T\mathbb{P}^n(-1) \longrightarrow 0.$$

Then

$$0 \longrightarrow \varphi_V^*(\mathcal{O}_{\mathbb{P}^n}(-1)) \cong L_- \otimes I \longrightarrow \mathcal{O}_X^{n+1} \cong V \otimes \mathcal{O}_X \longrightarrow \varphi_V^*(T\mathbb{P}^n(-1)) = E \longrightarrow 0.$$

From this $E = \varphi_V^*(T\mathbb{P}^n \otimes \mathcal{O}(-1))$ and we have a well defined pullback $\varphi_V^* : H^0(X, T\mathbb{P}^n(-1)) \to H^0(X,E)$. Now notice that if $\sigma \in H^0(\mathbb{P}^n, T\mathbb{P}^n)$ is a general section, then $Z(\sigma)$ consist in $c_n(T\mathbb{P}^n)$ points and by Gauss-Bonnet third formula (see [GH14])

$$c_n(T\mathbb{P}^n) = \chi_{\mathrm{top}}(\mathbb{P}^n) = n + 1.$$

Now, by [EH16] chapter 5 we have that

$$\begin{aligned} c_n(T\mathbb{P}_n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) &= \sum_{\ell=0}^{n} \binom{n-\ell}{n-\ell} c_1(\mathcal{O}_{\mathbb{P}^n}(-1))^{n-\ell} c_\ell(T\mathbb{P}^n) \\ &= \sum_{\ell=0}^{n} \binom{n+1}{\ell} H^\ell(-H)^{n-\ell} \\ &= \sum_{\ell=0}^{n} \binom{n+1}{\ell} (-1)^{n-\ell} \\ &= 1. \end{aligned}$$

Hence a general section of $T\mathbb{P}^n(-1)$ vanishes exactly in one point (in fact every section vanishes in exactly one point). Now, the degree of φ_V may be computed by $|Z(\varphi_V) \cap (X - Bl)|$, where Bl denotes the base locus of φ_V. Let $s \in H^0(\mathbb{P}^n, T\mathbb{P}^n(-1))$, then $Z(s) = \{p\}$ for some $p \in \mathbb{P}^n$, hence the fiber of $\varphi_V : X - Bl \to \mathbb{P}^n$ over p is $Z(\varphi^* s) \cap (X - Bl)$.

□

We have that

$$Z(V^-) := \bigcap_{s \in V^-} Z_{\mathrm{cycle}}(s) \text{ “=” } \mathrm{Bs}(\varphi_V) \text{ where } V^- \subset H^0(X, E).$$

The idea to get the main result, a bound to the degree of irrationality, is to find a suitable vector bundle E with fixed parameters and calculate the degree using the Lemma. The Theorem from Mukai will grant us the existence of such vector bundles. The following Lemmas have the purpose of show that the vector bundles in $M_S(r, L, s)$ are in fact the ones we need to get the minimal bound.

Lemma 3.3.3. *Let X be a smooth n-dimensional variety and $E \to X$ a rank r vector bundle. Let $x \in X$, then*

$$h^0(X, E \otimes \mathfrak{m}_x^d) \geq h^0(X, E) - r\binom{n+d-1}{n}.$$

Proof. Let us consider the ideal sheaf $\mathfrak{m}_x$ and the corresponding exact sequence

$$0 \to \mathfrak{m}_x^d \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{m}_x^d \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x^d \to 0$$

with $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ a skyscraper sheaf on x. Then

$$0 \to E \otimes \mathfrak{m}_x^d \to E \to E \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x^d \to 0.$$

Since $E \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x^d$ is a skyscraper sheaf of length $r\binom{n+d-1}{n}$ since

$$\mathcal{O}_{X,x}/\mathfrak{m}_x^d \simeq \mathbb{C}[X_1,\dots,X_n]_{\le d-1} \simeq \mathbb{C}[X_0,\dots,X_n]_d$$

from

$$0 \to H^0(X, E \otimes \mathfrak{m}_x^d) \to H^0(X,E) \xrightarrow{f} H^0(X, E \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x^d)$$

we get

$$h^0(X,E) = h^0(X, E \otimes \mathfrak{m}_x^d) + \mathrm{Im}(f) \le h^0(X, E \otimes \mathfrak{m}_x^d) + r\binom{n+d-1}{d}$$

and then

$$h^0(X, E \otimes \mathfrak{m}_x^d) \ge h^0(X,E) - r\binom{n+d-1}{n}.$$

□

Remark 3.3.4. *Furthermore, if* $I = \mathfrak{m}_{x_1}^{n_1} \otimes \cdots \otimes \mathfrak{m}_{x_d}^{n_d}$ *the Chinese Reminder Theorem gives*

$$h^0(X, E \otimes I) \ge h(X,E) - r\sum_{i=1}^{d}\binom{n+n_i-1}{n_i-1}$$

In particular, since we are studying surfaces,

$$h^0(X, E \otimes I) \ge h(X,E) - r\sum_{i=1}^{d}\binom{n_i+1}{n_i-1} = h(X,E) - r\sum_{i=1}^{d}\binom{n_i+1}{2}.$$

Definition 3.3.5. *A line bundle $L \in \mathrm{Pic}(S)$ is said to be indecomposable if L cannot be written as $L = \mathcal{O}_S(M+N)$, with $M, N > 0$ effective divisors.*

Lemma 3.3.6. *Let $L \in \mathrm{Pic}(S)$ be indecomposable, then $\varphi_L : S \to \mathbb{P}H^0(X,L)$ is dominant.*

We need one last Lemma about stable vector bundles to give the proof of the Moretti's result. This Lemma was discuss with Andrés Rojas, who helped me undestand the result and the proof.

Lemma 3.3.7. *Let (X,L) be a polarized K3 surface and $E \to X$ an L-semi-stable vector bundle. Then $h^2(X,E) = 0$ and $\chi(X,E) \le h^0(X,E)$.*

Proof. By Serre's duality

$$H^2(X,E) \simeq H^0(X,E^\vee)^\vee \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, E^\vee)^\vee.$$

Notice both $\mathcal{O}_X$ and E^{-} are L-stable and since L is ample

$$\mu_L(E^{-}) = \frac{c_1(E^{-})\cdot L}{\mathrm{rk}(E^{-})} = \frac{-c_1(E)\cdot L}{\mathrm{rk}(E^{-})} = \frac{-L^2}{r} < 0 = \mu_L(\mathcal{O}_X).$$

Then by Proposition 3.2.4, $h^2(X, E) = \dim \mathrm{Hom}(\mathcal{O}_X, E^{-}) = 0$.

$\square$

Theorem 3.3.8. *Let (S, L) be a polarized K3 surface of genus $g = 2 + 2n(n+1)$ such that $\mathrm{Pic}(S) = \mathbb{Z}[L]$. Then $\mathrm{irr}(S) \leq 2 + n$.*

Proof. If an L-stable vector bundle E is such that its rank is 2 and

$$c_1(E) = L, \quad c_2(E) = 2 + n(n+1) = \frac{g}{2} + 1.$$

Then, by Riemann-Roch for surfaces and Proposition 3.3.7

$$\begin{aligned} h^0(X,E) \geq \chi(X,E) &= r\chi(\mathcal{O}_X) + \frac{1}{2}(c_1(E)^2 - c_1(E)c_1(K_X)) - c_2(E) \\ &= 2 \cdot 2 + \frac{1}{2}L^2 - c_2(E) \\ &= 3 + g - \frac{g}{2} - 1 \\ &= 2 + \frac{g}{2}. \end{aligned}$$

Notice that having $r = 2$ and $s = g/2$ then

$$\dim M_S(2, L, g/2) = L^2 - 2 \cdot 2 \cdot \left(\frac{g}{2}\right) + 2 = 2g - 2 - 2g + 2 = 0.$$

Therefore such a vector bundle exists, since by Theorem 3.2.9 $M_S(2, L, g/2) \neq \emptyset$. Then

$$h^0(X, E) \geq 2 + \frac{g}{2} = 3 + n(n+1).$$

Let $P \in S$ be any point, then by the Lemma 3.3.3

$$h^0(X, E \otimes \mathfrak{m}_P^n) \geq h^0(X,E) - 2\binom{n+1}{2} \geq 3 + n(n+1) - n(n+1) = 3$$

there exists a vector space $V_P \subset H^0(E \otimes \mathfrak{m}_P^n)$ of dimension 3. Any section of V_P vanishes at P with order n^2. Finally by the Lemma 3.3.2,

$$\deg \varphi_{V_P} \leq c_2(E) - n^2 = 2 + n$$

where φ_{V_P} is the map $S \dashrightarrow \mathbb{P}^2$ induced by $V_P \subset H^0(L)$.

$\square$

Following the same analysis, Moretti gets to the general result.

Corollary 3.3.9. *Let (S, L) be a polarized K3 surface of genus $g = 2 + 2(n+1) + k < 2 + 2(n+1)(n+2)$ (i.e., $k < 4n+4$), where $\operatorname{Pic}(S) = \mathbb{Z}[L]$ and L is indecomposable. Then*

$$\operatorname{irr}(S) \leq 2 + n + \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

The proof is exactly the same as in the Theorem, but twisting by the ideal

$$I = \mathfrak{m}_P^n \cap \mathfrak{m}_{Q_1} \cap \cdots \cap \mathfrak{m}_{Q_{\lfloor k/4 \rfloor}}$$

The upper bound in Moretti's corollary is not optimal, since in the general case $g = 2 + 2n(n+1) + k$ in the proof Moretti considers a point P with multiplicity n and the rest with multiplicity 1, but following Remark 3.3.4 with the ideal

$$I = \mathfrak{m}_{P_1}^{n_1} \cap \cdots \cap \mathfrak{m}_{P_r}^{n_r}$$

we get an optimization problem

$$\operatorname{irr}(X) \leq \min \left\{ \left\lceil \frac{g}{2} \right\rceil + 1 - \sum_{i=1}^{r} n_i^2 : \sum_{i=1}^{r} n_i(n_i+1) \leq \left\lfloor \frac{g}{2} \right\rfloor - 1, n_i \in \mathbb{Z}^{\geq 1} \right\}.$$

Corollary 3.3.10. *Let (X, L) be a polarized K3 with $\operatorname{Pic}(X) \simeq \mathbb{Z}[L]$. Then for any choice of integers $n_1, \ldots, n_r \in \mathbb{Z}^{>0}$ such that*

$$\sum_{i=1}^{r} n_i(n_i+1) \leq \left\lfloor \frac{g}{2} \right\rfloor - 1$$

we get an upper bound

$$\operatorname{irr}_L(X) \leq \left\lceil \frac{g}{2} \right\rceil + 1 - \sum_{i=1}^{r} n_i^2.$$

Proof. Indeed, since L is indecomposable, any Mukai vector $(2, L, s)$ is primitive. Hence by 3.2.9, if $v = (2, L, s)$ is isotropic, then there exists an stable vector bundle E such that $v(E) = v$, i.e.,

$$\begin{aligned} 0 = \dim M_X(2, L, s) = L^2 - 4s + 2 &= L^2 - 4(\chi(X, E) - 2) + 2 \\ &= L^2 - 4\chi(X, E) + 10. \end{aligned}$$

Then

$$
\begin{aligned}
0 = L^2 \quad 4\chi(X,E) + 10 = L^2 + 10 \quad 4\left(4 + \frac{L^2}{2} \quad c_2(E)\right) \\
= 4c_2(E) \quad L^2 \quad 6.
\end{aligned}
$$

In term of genus

$$
0 = 4c_2(E) \quad (2g \quad 2) \quad 6 = 4c_2(E) \quad 2g \quad 4
$$

hence $c_2(E) = \left\lceil \frac{g}{2} \right\rceil + 1$. It follows that

$$
c_2(E) = \left\lceil \frac{g}{2} \right\rceil + 1 \quad \text{and} \quad h^0(X,E) \quad \chi(X,E) = 2 + \left\lfloor \frac{g}{2} \right\rfloor.
$$

Hence, by the Lemmas we get the constrain

$$
h^0(X, E \quad I) \quad h^0(X,E) \quad \sum_{i=1}^{r} n_i(n_i+1) \quad 3,
$$

therefore

$$
\sum_{i=1}^{r} n_i(n_i+1) \quad 2 + \left\lfloor \frac{g}{2} \right\rfloor \quad 3 = \left\lfloor \frac{g}{2} \right\rfloor \quad 1.
$$

Besides each section of $E \quad I$ vanishes in P_j with multiplicity n_r^2, getting

$$
\mathrm{irr}_L(X) \quad \left\lceil \frac{g}{2} \right\rceil + 1 \quad \sum_{i=1}^{r} n_i^2.
$$

□

Notice that we can rewrite the optimization problem as

$$
\max\left\{ \sum_{i=1}^{r} n_i^2 : \sum_{i=1}^{r} n_i^2 + n_i \quad k \right\}
$$

which have the same solution as our original problem. From this, with $f(n_1, \ldots, n_r) = \sum_{i=1}^{r} n_i^2$ and $g(n_1, \ldots, n_r) = \sum_{i=1}^{r} n_i(n_i+1)$, it follows that increasing 1 in previous value n_i gives us

$$
\Delta f = 2n_i + 1 \quad \Delta g = 2n_i + 2
$$

while adding $2n_i + 1$ new variables gives

$$\Delta f = 2n_i + 1 \quad \Delta g = 4n_i + 2.$$

Hence while possible it is better to increase a previous variable than to add a new one. It is therefore possible to compute this manually, getting the following graph comparing the two methods:

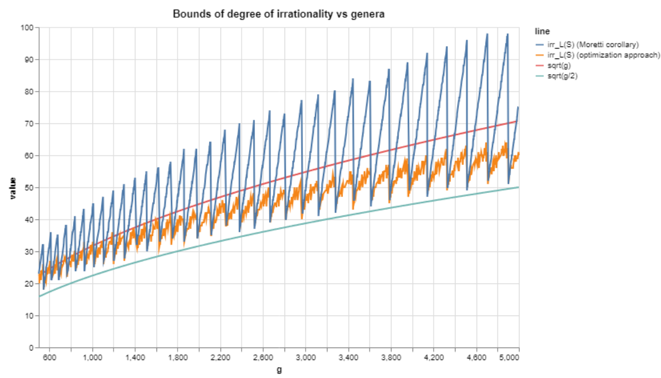


Figura 3.1. Bounds of degree of irrationality vs genera g

Remark 3.3.11.

1. *Although in the general case we get a better bound than Moretti's method, we get the same result when* $g = 2 + 2n(n+1)$.
2. *This asymptotic result supports the conjecture of Stampleton that* $\operatorname{irr}(X)$ *grows as* $\sqrt{g}$.

3.4 Enriques Surfaces

Definition 3.4.1. *An Enriques surface is a complex surface* Y *such that* $h^1(Y, \mathcal{O}_Y) = 0$ *and* $\mathcal{O}_Y(K_Y)^{\otimes 2} \simeq \mathcal{O}_Y$.

Remark 3.4.2. *We can also define Enriques surfaces in positive characteristic where* $p_g(Y) \geq 1$, *but in characteristic zero* $p_g(Y) = h^{0,2} = h^2(Y, \mathcal{O}_Y) = 0$.

Since $K_Y^{\otimes 2} \simeq \mathcal{O}_Y$ we have that for any $L \in H^2(Y, \mathbb{Z})$

$$0 = c_1(\mathcal{O}_Y(K_Y)^{\otimes 2}) = 2c_1(\mathcal{O}_Y(K_Y))$$

then

$$2c_1(\mathcal{O}_Y(K_Y)) \cdot L = 0$$

and hence $c_1(\mathcal{O}_Y(K_Y)) \cdot L = 0$. In particular, $c_1(Y)^2 = c_1(\mathcal{O}_Y(K_Y))^2 = 0$ and therefore by

Noether's formula

$$c_1(Y)^2 + c_2(Y) = 12(p_g(Y) - q(Y) + 1) = 12.$$

Hence $c_2(Y) = 12$. From the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y \to 0$$

we have that

$$\to 0 = H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) = 0 \to$$

Hence $\mathrm{Pic}(Y) \cong H^1(Y, \mathcal{O}_Y) \cong H^2(Y, \mathbb{Z})$.

It is a well know fact that the universal covering of an Enriques surface is a K3 surface, in fact we have the following classical Proposition

Proposition 3.4.3. *Let Y be an Enriques surface. Then the fundamental group of Y is $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ and the universal covering of Y is a K3 surface. Conversely, let X be a K3 surface with fixed-point-free automorphism σ of order 2. Then the quotient surface $X/\langle\sigma\rangle$ is an Enriques surface.*

Moreover, $H_1(Y, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and by universal coefficient Theorem

$$H^1(Y, \mathbb{Z}) \cong \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}(H_1(Y, \mathbb{Z}), \mathbb{Z}) \cong \{0\} \oplus \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \{0\}.$$

By Hirzebruch's index Theorem

$$b^+(Y) - b^-(Y) = \frac{1}{3}(c_1^2(Y) - c_2(Y)) = 8$$

where $(b^+(Y), b^-(Y))$ is the signature of $H^2(Y, \mathbb{R}) \times H^2(Y, \mathbb{R})$. Then, since $q(Y) = 0$ we have $b^+(Y) = 1$ and then $H^2(Y, \mathbb{R}) \cong \mathbb{R}^{10}$. Again by universal coefficient Theorem

$$\begin{aligned} \mathbb{R}^{10} \cong H^2(Y, \mathbb{R}) &\cong \mathrm{Ext}(H_1(Y, \mathbb{Z}), \mathbb{R}) \oplus \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{R}) \\ &\cong \{0\} \oplus \mathrm{Hom}(\mathbb{Z}^{b_2(Y)}, \mathbb{R}) \\ &\cong \mathbb{R}^{b_2(Y)}. \end{aligned}$$

Concluding $\mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by $\mathcal{O}_Y(K_Y)$.

By deformation theory (c.f., [Kon20]) the moduli space of Enriques surfaces is 10 dimensional.

Remark 3.4.4. *Let Y be an Enriques surface and $\pi : X \to Y$ its covering K3 surface.*

1. *Let G be the group of deck deformations of X, then (c.f. [CDL24, §1.3]) there is an*

exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \operatorname{Pic}(Y) \xrightarrow{f} \operatorname{Pic}(X)^G \to 0$$

where $\ker(f)$ *is generated by* $\mathcal{O}_Y(K_Y)$*. It follows from the isomorphism Theorem that*

$$\mathbb{Z}^{10} \simeq \operatorname{Pic}(Y)/\ker(f) \simeq \operatorname{Pic}(X)^G$$

In particular $\operatorname{rk}(\operatorname{Pic}(X)) \geq 10$*.*

2. X *is birational to a complete intersection of three quadrics in* $\mathbb{P}^5$*. In particular (see Chapter 2) there is a polarization* L *with* $L^2 = 8$*.*

Definition 3.4.5. *An Enriques surface is called a* ***nodal surface*** *if contains a smooth rational curve* R *(*$R^2 = -2$*), otherwise is called* ***unnodal surface****.*

Remark 3.4.6. *In the 10 dimensional moduli space of Enriques surfaces, the nodal surfaces form a 9 dimensional variety, while the generic one is an unnodal surface.*

Following Mukai's work, Kim showed in [Kim98] that the pull back of the moduli space of stable vector bundles on an Enriques surface is a Lagranian subvariety of the moduli space of stable bundles on its covering space and studied the singularities. Later on, Nuer in [Nue16] gave conditions for the existence of stable bundles, that is sumarize in the following Theorem.

Theorem 3.4.7. *Let* $v = mv_0$ *be a Mukai vector with* v_0 *primitive and* $m > 0$ *with* H *generic with respect to* v*. Assume* Y *is unnodal. Then*

1. *The moduli space of Gieseker-semistable sheaves* $M_{H,Y}(v) \neq \emptyset$ *if and only if* $v_0^2 \geq -1$*.*
2. *Either* $\dim M_{H,Y}(v) = v^2 + 1$ *and* $M^s_{H,Y}(v) \neq \emptyset$*, or* $m > 1$ *and* $v_0^2 \leq -1$*.*
3. *If* $M_{H,Y}(v) \neq M^s_{H,Y}(v)$ *and* $M^s_{H,Y}(v) \neq \emptyset$*, the codimension of the semistable locus is at least 2 if and only if* $v_0^2 > 1$ *or* $m > 2$*. Moreover, in the case* $M_{H,Y}(v)$ *is normal with torsion canonical divisor.*

In the Theorem, $M^s_{H,Y}(v)$ is the moduli space of Gieseker-stable bundles. It is important to distinguish that, in the case of Enriques surfaces, Mukai vectors are defined in the space

$$H_{\text{alg}}(Y,\mathbb{Z}) := H^0(Y,\mathbb{Z}) \oplus \operatorname{NS}(Y) \oplus \frac{1}{2}\mathbb{Z}\rho_Y \subset H^0(Y,\mathbb{Z}) \oplus H^2(Y,\mathbb{Z}) \oplus H^4(Y,\mathbb{Q})$$

where ρ_Y is the fundamental class of Y. For a vector bundle $E \to Y$, the mukai vector becomes

$$v(E) := \operatorname{ch}(E)\sqrt{\operatorname{td}(Y)} = \left(r(E), c_1(E), \frac{1}{2}r(E) + \operatorname{ch}_2(E)\right)$$

where $\operatorname{ch}_2(E) = 1/2c_1(E)^2 - c_2(E)$ is the second chern character. Again we define the pairing

$$\langle v_1, v_2 \rangle = c_1 c_2 - r_1 s_2 - r_2 s_1$$

hence for a vector bundle of rank 2

$$v(E)^2+1=c_1(E)^2-2r(E)\left(\frac{1}{2}r(E)+\frac{1}{2}c_1(E)^2-c_2(E)\right)+1=4c_2(E)-c_1(E)^2-3.$$

Since $c_1(E)^2$ is even, $v(E)^2 \neq -1$, therefore the existence condition for stable vector bundles of rank 2 is $v(E)\geq 0$.

Proposition 3.4.8. *Let (Y,L) a polarized Enriques Surface and $E\to Y$ a L-semi-stable vector bundle on Y with $c_1(E)=L$. Then $h^2(Y,E)=0$.*

Proof. Similar to the case of K3 surfaces, since E is L-semi-stable, by [Tak72, Proposition 1.4] $E\otimes\omega_Y$ is L-stable and so is $E^{\vee}\otimes\omega_Y$, hence by Serre's duality

$$h^2(Y,E)=h^0(Y,E^{\vee}\otimes\omega_Y)=\dim\operatorname{Hom}(\mathcal{O}_Y,E^{\vee}\otimes\omega_Y)$$

where $\mathcal{O}_Y$ and $E^{\vee}\otimes K_Y$ are L-semi-stable. Moreover

$$\mu_L(E^{\vee}\otimes\mathcal{O}_Y(K_Y))=\frac{c_1(E^{\vee}\otimes\mathcal{O}_Y(K_Y))\cdot L}{\operatorname{rank}(E)}=\frac{c_1(E^{\vee})\cdot L+2\mathcal{O}_Y(K_Y)\cdot L}{\operatorname{rank}(E)}=\frac{-L^2}{\operatorname{rank}(E)}<0=\mu_L(\mathcal{O}_Y)$$

then $\mu_L(\mathcal{O}_Y)>\mu_L(E^{\vee}\otimes\mathcal{O}_Y(K_Y))$ and hence by Proposition 3.2.4

$$h^2(Y,E)=\dim\operatorname{Hom}(\mathcal{O}_Y,E^{\vee}\otimes\mathcal{O}_Y(K_Y))=0.$$

$\square$

Remark 3.4.9. *If Y is an Enriques surface, then Riemann-Roch Theorem for vector bundles states*

$$\chi(Y,E)=r\chi(Y,\mathcal{O}_Y)+\frac{1}{2}(c_1(E)^2-c_1(E)\cdot c_1(\omega_Y))-c_2(E)=r+\frac{1}{2}c_1(E)^2-c_2(E).$$

Following the construction of Moretti, we have the following result for Enriques Surfaces

Theorem 3.4.10. *Let (Y,L) a polarized unnodal Enriques surface such that L is idecomposable. Let $g\geq 5$ be the genus of Y. Then for each choice of $n_1,\dots,n_r\in\mathbb{Z}_{\geq 0}$ such that*

$$\sum_{i=1}^{r}\binom{n_i}{2}\leq\left\lfloor\frac{g-5}{2}\right\rfloor$$

we have an upper bound

$$\operatorname{irr}_L(Y)\leq\left\lceil\frac{g-1}{2}\right\rceil+1-\sum_{i=1}^{k}n_i^2.$$

Proof. Let $v = \left(2, L, \left\lfloor \frac{g-1}{2} \right\rfloor\right) \in H_{\mathrm{alg}}(Y, \mathbb{Z})$, then

$$v^2 = L^2 - 2 \cdot 2 \cdot 2\left\lfloor \frac{g-1}{2} \right\rfloor = 2g - 2 - 4\left\lfloor \frac{g-1}{2} \right\rfloor \geq 0$$

then $M_{H,Y}(v) \neq \emptyset$, moreover, $M^{s}_{H,Y}(v) \neq \emptyset$, therefore there is an stable vector bundle E such that $v(E) = v$. Hence

$$c_2(E) = \frac{1}{2}2 + \frac{L^2}{2} - \left\lfloor \frac{g-1}{2} \right\rfloor = g - \left\lfloor \frac{g-1}{2} \right\rfloor = \left\lceil \frac{g-1}{2} \right\rceil + 1.$$

By Riemann-Roch Theorem

$$\chi(E) = 2 + \frac{1}{2}c_1(E)^2 - c_2(E) = \left\lfloor \frac{g-1}{2} \right\rfloor + 1 = \left\lfloor \frac{g+1}{2} \right\rfloor.$$

By Lemma 3.4.8 we have that $h^0(Y, E) \geq \chi(E) \geq 3$. By Lemma 3.3.3 we have that for any choices of $n_1, \ldots, n_r \in \mathbb{Z}^{>0}$ such that

$$h^0(Y, E \otimes I) \geq 3$$

from where it follows

$$\sum_{i=1}^{r} \binom{n_i}{2} \leq \left\lfloor \frac{g-5}{2} \right\rfloor$$

we have that every section of $E \otimes I$ vanishes with order $\sum_{i=1}^{r} n_i^2$ and then we conclude by Lemma 3.3.2.

□

We get an optimal bound for this method in the following cases.

Corollary 3.4.11. *Let* (Y, L) *an unnodal polarized Enriques surface of genus* $g = 5+2n(n+1)$ *with* L *idecomposable. Then*

$$\mathrm{irr}_L(Y) \leq 3 + n.$$

Proof. In the previous Theorem, if $g = 5 + 2n(n+1) \geq 5$ then

$$\frac{g-5}{2} = n(n+1)$$

and hence

$$c_2(E) = 2 + n(n+1) + 1 = 3 + n(n+1)$$

Finally,

$$\mathrm{irr}_L(Y) \leq 3 + n(n+1) - n^2 = n + 3.$$

□

Remark 3.4.12. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two algebraic varieties X and Y of the same dimension n. Then, for each dominant rational map $\varphi : Y \dashrightarrow \mathbb{P}^n$, the composition $\psi := \pi \circ \varphi : X \dashrightarrow \mathbb{P}^n$ is a dominant rational map. Hence, we have that*

$$\operatorname{irr}(X) \leq \deg(\pi) \operatorname{irr}(Y).$$

In particular, from the remark, we have that if Y is an Enriques surface and X its universal covering space (K3 surface), then the map $\pi : X \to Y$ has degree 2 and hence

$$\operatorname{irr}(X) \leq 2 \operatorname{irr}(Y).$$

From Remark 3.4.4 there is not a lot we can say for $\operatorname{irr}(X)$, since its computation when $L^2 = 8$ uses the fact that $\operatorname{Pic}(X) = \mathbb{Z}[L]$ (c.f. [Mor23, §2.2]). Nevertheless, if we consider a genus 5 Enriques surface Y, then

$$\operatorname{irr}(X) \leq 2 \operatorname{irr}(Y) \leq 6$$

getting an upper bound for the degree of irrationality of complete intersection of three quadrics.

Chapter 4

CONCLUSION AND FUTURE WORKS

Throughout this work we study the linear systems, moduli space and degree of irrationality of complex K3 surfaces.

The first chapter was review of literature for linear systems in K3 surface, getting a classification of line bundle as hyperelliptic and non-hyperelliptic. These results were shown for complex K3 surfaces, but can also be obtained for algebraic K3 surfaces over an algebraically closed field k of characteristic different from 2. In particular, we follow [SD74] which shows everything for algebraic K3 surfaces. This study of linear systems allows us to have a first understanding of models for K3 surfaces and its moduli space.

In the second chapter we review in detail the construction of the (analytic) moduli space of K3 surfaces, showing that there are models for K3 surfaces of low degree and using the Torelli type theorems to show that there is a (set theoretical) bijection between the set of isomorphism classes of K3 surfaces and the period domain. Although this construction only works for complex K3 surfaces, at the end of the chapter we construct (without the details of geometric invariant theory) a coarse quasi-projective moduli space. For this second construction, we did not use the fact that the field was the complex numbers, and so it can be constructed for algebraically closed fields of characteristic different from 2.

In the third chapter, we reconstructed the work of Moretti in [Mor23] and got an upper bound for the degree of irrationality of K3 surfaces of a low genus. Moreover, we got a better asymptotic result for this upper bound just by solving an optimization problem. This result, as mentioned in chapter 3, goes in the same direction as Stapleton conjecture in [Sta17], who said that $\mathrm{irr}(X)$ grows as $\sqrt{g}/2$. We also use these results to get an upper bound for the degree of irrationality of Enriques surfaces, and later on use this result to get another bound for K3 surfaces. Although it is a better bound, this work doesn't get close enough to solve this conjecture, so more work in this area is needed.

As future work, in terms of computing the degree of irrationality, Moretti and Rojas construct in [Mor23] and [MR24] a Brill-Noether theory, but in this later work, instead of considering stable vector bundles (in the sense of slope or Gieseker), they consider Bridgeland stability conditions, studied in [MYY14], and they compute using this technique the degree of irrationality for small genus. The same techniques might be applied to Enriques surfaces. For example, in [Yos16] Yoshioka study the moduli space of Briedgeland stable bundles on Enriques surfaces. Since the results of Moretti and Rojas depended on the existence of this kind of moduli space, there might be a chance to use Brill-Noether theory on Enriques surfaces.

Given the last remark on chapter 3, we may consider to study the quotient of a K3 surfaces X by the action of subgroups $G \subseteq \mathrm{Aut}(X)$. The group of automorphisms of a K3 surface can be study, using the Torelli type theorems, by studying automorphisms of $H^2(X,\mathbb{Z}) \simeq L_{K3}$.

There might be also possible to use the same technique with other varieties, for example, hyperkähler and Calabi-Yau manifolds. This manifolds are also very important and so the computation of their degree of irrationality is very interesting. Varieties with $\mathcal{O}_X(K_X)^{\otimes m} \simeq \mathcal{O}_X$ for some $m \in \mathbb{Z}^{>0}$ still satisfy most of the hypothesis of the lemmas, so the study of the moduli space of stable bundles on these varieties could lead to upper bounds to the degree of irrationality.

Appendix A

AUXILIARY RESULTS

A.1 Wu formula

This section is dedicated to recall some definitions and propositions needed to state Wu formula and other topological results. The main references to this section will be [HBS66] and [Ive12].

Let X be a complex projective variety. Recall that from the exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

we get a long exact sequence in cohomology

$$\to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to$$

so identifying $\mathrm{Pic}(X) \simeq H^1(X,\mathcal{O}_X^*)$ we define the first **chern class** of a line bundle $[L] \in \mathrm{Pic}(X)$ as $c_1(L)$. In general chern classes may be defined axiomatically. In the same spirit, for a differentiable manifold X, we can instead consider the exact sequence

$$0 \to SO(r) \hookrightarrow O(r) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

which induces a long exact sequence

$$\to H^0(X,\mathbb{Z}/2\mathbb{Z}) \to H^1(X,SO(r)) \to H^1(X,O(r)) \to H^1(X,\mathbb{Z}/2\mathbb{Z}) \to$$

and so we define the first **Whitney class** as $w_1(\xi) = \rho_*(\xi)$. Here we identify $H^1(X,O(r))$

as the group of (isomorphism classes of) $O(r)$-bundles, i.e., vector bundles with structure group $O(r)$. Axiomatically, the **total Whitney class** $w(\xi) = \sum_{j=0}^{n} w_j(\xi)$ satisfies:

1. For every $O(r)$-bundle ξ and $j \geq 0$ there is a Whitney class $w_j(\xi) \in H^j(X, \mathbb{Z}/2\mathbb{Z})$. $w_0(\xi) = 1$ is the identity element.
2. $w(f^* \xi) = f^* w(\xi)$.
3. $w(\xi \oplus \eta) = w(\xi)w(\eta)$.
4. $w(\eta_n) = 1 + h_n$, where η_n is the $O(1)$-bundle over the n-dimensional real projective space $\mathbb{P}^n(\mathbb{R})$ and h_n is the non-zero element of $H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$.

It is clear from the definition that X is orientable if only if $w_1(X) := \rho(T_X) = 0$. From [MS74, proposition 9.8] we know that $w_{2i}(\rho(\xi))$ is the reduction of $c_i(\xi)$ modulo 2.
Following [MS74] we have the construction: Let X be a closed, smooth n-dimensional manifold. Then by the universal coefficient theorem $H_n(M, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}$, we denote $\mu \in H_n(X, \mathbb{Z}/2\mathbb{Z})$ the unique fundamental homology class (i.e., $\mu \neq 0$). Hence for any cohomology class $v \in H^n(X, \mathbb{Z}/2\mathbb{Z})$, the Kronecker index $(v, \mu) \in \mathbb{Z}/2\mathbb{Z}$ is defined. An important family of operators related to this are the *Steenrod squaring operators* $Sq^i : H^n(X, \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X, \mathbb{Z}/2\mathbb{Z})$ which satisfy for $a \in H^n(X, \mathbb{Z}/2\mathbb{Z})$

$$Sq^0(a) = 0; \qquad Sq^n(a) = a \cup a \qquad Sq^i(a) = 0 \quad \text{for } i > n.$$

By Poincaré duality, there is a $v_k \in H^k(X, \mathbb{Z}/2\mathbb{Z})$ such that

$$(v_k \cup x, \mu) = (Sq^k(x), \mu).$$

The Wu formula (see [MS74, Theorem 11.14]) relates this to the Whitney class in the following way:

$$w_k = \sum_{i+j=k} Sq^k(v_j).$$

A.2 Leray spectral sequence

The aim of this Section is to give a brief introduction to spectral sequences that are used in Chapter 1. For more detail one might read for example [Vak17, Chapter 1, §1.7].

A **filtered complex** is a complex of abelian groups $A^\bullet$ together with a decreasing filtration od complexes

$$F^\bullet A^\bullet : \quad \cdots \hookrightarrow F^2A^\bullet \hookrightarrow F^1A^\bullet \hookrightarrow F^0A^\bullet = A^\bullet$$

such that for every $k \geq 0$ there exists an $\ell \geq 0$ such that $F^\ell A^k = 0$.

Any filtered complex induces a filtration in the cohomology groups of the complexes

$$F^p H^k(A^\bullet) := \mathrm{im}(H^k(F^p A^\bullet) \to H^k(A^\bullet)).$$

Theorem A.2.1. *Given a filtered complex of abelian groups $F^\bullet A^\bullet$ there exists a complex of abelian groups*

$$(E_r^{p,q}, d_r), \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that

- $E_0^{p,q} := Gr_F^q(A^{p+q}) = F^q A^{p+q}/F^{p+1}A^{p+q}$ *and* d_0 *is induced by* d.
- $E_{r+1}^{p,q}$ *can be identified with the cohomology of* (E_r^{p+q}, d_r)*, i.e.,*

$$E_{r+1}^{p+q} \simeq \frac{\ker(E_r^{p+q} \xrightarrow{d_r} E_r^{p+r,q-r+1})}{\mathrm{im}(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q})}.$$

- *For fixed* $p+q$ *and* $r >> 0$,

$$E_r^{p+q} = Gr_F^p H^{p+q}(A^\bullet) =: E_\infty.$$

This sequence of complexes is called ***spectral sequence*** *associated to* $F^\bullet A^\bullet$. *We say that* $(E_r^{p,q}, d_r)$ *is the* r*-th page of the spectral sequence. We say that the spectral sequence abupts with* $H^{p+q}(A^\bullet)$ *and we denote this by*

$$E_r^{p,q} \Rightarrow H^{p+q}(A^\bullet).$$

Definition A.2.2. *We say that a spectral sequence* ***degenerates*** *at* E_r *if* $d_k = 0$ *for all* $k \geq r$. *This is equivalent to*

$$E_r^{p,q} = E_\infty^{p,q} = Gr_F^p H^{p+q}(A^\bullet).$$

Theorem A.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. Then for every left bounded complex $(A^\bullet, d)$ in $\mathcal{A}$ there exists a canonical spectral sequence (canonical starting from page E_2)*

$$E_2^{p,q} = R^p\mathsf{F}(H^q(A^\bullet)) \Rightarrow R^{p+q}(A^\bullet).$$

This spectral sequence is functorial from page E_2 *and is called the* ***hypercohomology spectral sequence.***

Definition A.2.4. *The spectral sequence associated to the composed functor* $\Gamma_X = \Gamma_Y \circ f$ *for* $f : X \to Y$ *is called* ***Leray spectral sequence.*** *It induces a Leray filtration* L *on*

$H^k(X, \mathcal{F})$ *and satisfies*

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{F}).$$

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