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Punctual Torelli principle for non-extremal Klein hypersurfaces

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TÍTULO DE LA TESIS:

PUNTUAL TORELLI PRINCIPLE FOR NON-EXTREMAL KLEIN HYPERSURFACES.

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ABSTRACT

This thesis explores the geometry of Klein hypersurfaces, focusing on their group of regular automorphisms and on the automorphism group of their n-th primitive singular cohomology group with integer coefficients.

The research begins with a review of results characterizing hypersurfaces that have automorphisms of large prime order, showing that a hypersurface with automorphisms of large prime order is a Klein hypersurface. The research continues with the study of the automorphisms of hypersurfaces, and how in most cases these are essentially matrices, reviewing results on these automorphisms before turning to the study of some classic examples, such as the Klein and Fermat hypersurfaces.

The core of this thesis delves into Hodge theory applied to smooth hypersurfaces, focusing on the Hodge structures given by the n-th primitive singular cohomology group with integer coefficients. To this end, we review the concepts of complex varieties, Hermitian varieties, and Kähler varieties, as well as the concepts of Hodge structure and polarized Hodge structure.

By analyzing the action of the automorphism on the cohomology basis, we provide a detailed classification of the eigenvalues and eigenspaces associated with these automorphisms. The study culminates in the identification of the normalizer of σ^* within the automorphism group, showing that under certain conditions (such as $|\Delta| = n + 2$, where Δ is a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ which depends of (n, d)), the normalizer is isomorphic to a semidirect product of cyclic groups. We conjecture that $|\Delta| = n + 2$, and we were able to verify this for a finite number of cases using a Python code.The results obtained provide new evidence in favor of the punctual Torelli principle.

Keywords: *Klein hypersurface, Hodge theory, Regular automorphism, Kähler manifold, Torelli principle, Cohomology*

CONTENIDO

INTRODUCTION

The study of automorphisms of Klein hypersurfaces is a classical subject and begins with the paper by Felix Klein [Kle79b] studying the automorphism group of the quartic Klein curve $X \subseteq \mathbb{P}^2$, where $d = 4$ and $n = 1$, and proving that $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$ in that case. The case of the Klein cubic curve $X \subseteq \mathbb{P}^2$ is also classical. A straightforward computation shows that $\nu : X \to X$, $(x_0 : x_1 : x_2) \mapsto (x_1 : x_2 : x_0)$ is an automorphism of order 3 fixing an inflection point of the elliptic curve and thus it follows from [Sil86, Theorem III.10.1] that $\mathrm{Aut}(X) \simeq \mathbb{Z}/6\mathbb{Z}$. In [Ald78], Alder proved that the automorphism group of the Klein cubic threefold is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{11})$. The automorphism group of the Klein cubic surface was computed by Dolgachev in [Dol12, Theorem 9.5.8], who showed that it is $\mathfrak{S}_5$. The automorphism groups of the remaining Klein curves were computed by Harui in [Har19, Proposition 3.5] analyzing finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$, while the automorphism group of the Klein quintic threefold was computed by Oguiso and Yu in [OY19a, Theorem 3.8]. In order to describe the automorphism group in the latter two cases, let us introduce the following notation: Given $n \geq 1$ and $d \geq 3$ we define

$$m := \frac{(d-1)^{n+2} - (-1)^{n+2}}{d}.$$

A straightforward computation shows that the degree d Klein hypersurface $X \subseteq \mathbb{P}^{n+1}$ admits an automorphism σ of order m and an automorphism τ of order $n+2$ given by

$$\sigma(x_0 : x_1 : \cdots : x_{n+1}) = (\xi_{dm} x_0 : \xi_{dm}^{1-d} : \cdots : \xi_{dm}^{(1-d)^{n+1}} x_{n+1})$$

and

$$\tau(x_0 : x_1 : \cdots : x_{n+1}) = (x_1 : x_2 : \cdots : x_{n+1} : x_0).$$

Hence, the group

$$\mathscr{K}(n,d) := (\mathbb{Z}/m\mathbb{Z}) \rtimes \mathbb{Z}/(n+2)\mathbb{Z}$$

acts faithfully on X, and Harui and Oguiso-Yu showed that $\mathscr{K}(n,d)$ is the full automorphism group in the cases $(n,d) \in \{(1,d),(3,5)\}$, where $d \geq 5$.

In [GALMVL24], it was proven that for $n \geq 2$ and $d \geq 4$, with $(n,d) \neq (2,4)$, the automorphism group of the Klein is also $\mathscr{K}(n,d)$. It was also shown that this holds for the Klein cubic curve, but for dimensions $n \geq 4$.
In [GALMVL24], the authors also proved the following theorem, which provides evidence in favor of the so called Strong Torelli Principle for hypersurfaces.

Theorem 0.0.1. *The middle primitive cohomology group of the Klein hypersurface of Wagstaff type X of dimension $n \geq 3$ and degree $d \geq 3$ is an extremal polarized Hodge structure. Moreover,*

$$\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim})/\{\pm 1\} \simeq \mathrm{Aut}(X)$$

in the following cases:
(a) $d \mid n+3$,
(b) $d = 3$ *and* $n \geq 5$.

Recall that the classical Torelli Theorem [And58, ACGH85] states that for any pair of curves X and X' of genus g, every isomorphism between their poralized Jacobian varieties $J(X) \simeq J(X')$ is induced, possible up to an involution, by a unique isomorphism of curves $X \simeq X'$. Since the Jacobian of a curve X is determined by its Hodge structure $J(X) = (H^{1,0}(X))^*/H_1(X,\mathbb{Z})$, an isomorphism between polarized Jacobian varieties $J(X) \simeq J(X')$ is represented by an isomorphism between the corresponding polarized Hodge structures.
The Torelli Theorem motivated the following conjecture commonly called the Strong Torelli Principle: Let X and X' be two smooth hypersurfaces of degree d and dimension n on $\mathbb{P}^{n+1}$. Then every isomorphism of polarized Hodge structures $\varphi : H^n(X,\mathbb{Z})_{prim} \to H^n(X',\mathbb{Z})_{prim}$ preserving polarizations is induced, possibly up to an involution, by a unique isomorphism $\psi : X \to X'$. The weaker statement where we only ask for the existence of the isomorphisms φ and ψ without requiring the former being induced by the latter is the called the Global Torelli Principle.
The Strong Torelli Principle holds for plane curves by the classical Torelli Theorem, for quartic surfaces [PSS71, DBR75, Fri84], for cubic threefolds [CG72, Bea82] and for cubic fourfolds [Voi86, Loo09, Cha12, HR19]. Furthermore, the verification of the Rational Torelli Principle for generic hypersurfaces of dimension n and degree d in $\mathbb{P}^{n+1}$ has been recently completed by Voisin [Voi22] for all values of (n,d) with the exception of a finite number of cases (complementing previous works of Donagi [Don83] and Cox-Green [CG90]). It is worth noting that in general one can only expect the Rational Torelli Principle to hold for generic hypersurfaces (see [Voi22, Remark 0.3]). For instance, in the case of Hodge structures of level one there are non-isomorphic hypersurfaces with isogenous intermediate Jacobians (e.g. for curves). And for Hodge structures of higher level, whenever the rational Hodge structure splits (for instance in the Noether-Lefschetz locus) we get several non-trivial involutions which in general do not come from actual involutions of the hypersurface (e.g.

for K3 surfaces).
If we take $X = X'$ in the Strong Torelli Principle, we obtain the Punctual Torelli Principle:

Conjecture 0.0.2 (Punctual Torelli Principle)**.** *Let X be a smooth hypersurface of degree d and and dimension n on $\mathbb{P}^{n+1}$. Then every automorphism of polarized Hodge structures $H^n(X, \mathbb{Z})_{prim}$ preserving polarizations is induced, possible up to an involution, by unique automorphism of X.*

In this thesis, we will present new evidence in favor of the punctual Torelli principle for the case of Klein hypersurfaces. Specifically, we will show that for certain values of (n, d), the Klein hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d and dimension n, that is, $X = \{K = 0\}$, where

$$K = x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0$$

satisfies the condition that the subgroup $\mu < \mathrm{Aut}(H^n(X, \mathbb{Z})_{prim}, \langle \cdot, \cdot \rangle)/\{\pm 1\}$ has order $n+2$. To develop all of this, we will first discuss the automorphism group of a hypersurface $X \subseteq \mathbb{P}^{n+1} = \mathbb{P}(V)$, denoted by $\mathrm{Aut}(X)$, and we will examine the theorem by Matsumura and Monsky (Theorem 1.1.2) which tells us that if $(n, d) \neq (1, 3), (2, 4)$, then every automorphism of X is linear, $\mathrm{Aut}(X) = \mathrm{Lin}(X)$, where $\mathrm{Lin}(X)$ is the subgroup formed by the automorphisms of X that come from automorphisms of $\mathbb{P}^{n+1}$(Definition 1.1.1). We will see that if $\varphi \in \mathrm{Lin}(X)$, then $(\varphi')^*(F) = \lambda F$ for some $\lambda \in \mathbb{C}^\times$ (here $\varphi' \in \mathrm{GL}(V)$ is a representative of φ)(Proposition 1.1.4). We will also see that if $\varphi \in \mathrm{Lin}(X)$ has order q, then there exists $\omega \in \mathrm{PGL}(V)$ of order q such that $\omega|_X = \varphi$(Proposition 1.1.5). Moreover, if $d \geq 3$, $(n, d) \neq (1, 3), (2, 4)$, then

$$\mathrm{Aut}(X) \cong \{\psi \in \mathrm{PGL}(V) | \psi(X) = X\},$$

that is, for every automorphism $\varphi \in \mathrm{Aut}(X)$, there exists a unique automorphism $\psi \in \mathrm{PGL}(V)$ such that $\psi|_X = \varphi$.
Next, we will introduce the notion of when a prime number p is admissible in dimension n and degree d (Definition 1.2.3) and use it, along with the concept of cyclotomic polynomial (Definition 1.3.2), to prove that if $n \geq 2$ and $d \geq 3$ with $(n, d) \neq (2, 4)$, and $\Phi_{n+2}(1-d)$ is prime, then a smooth hypersurface $X = V(F)$ of dimension n and degree d admits an automorphism φ of prime order $p > (d-1)^n$ if and only if X is isomorphic to the Klein hypersurface, $n = 2$ or $n+2$ is prime, and $p = \Phi_{n+2}(1-d)$.(Theorem 1.3.10).
Next, we introduce the differential method. We will also recall the definitions of generalized permutation matrix and generalized triangular matrix, and we will denote by $\mathrm{PGP}(V, \beta)$ the image of $\mathrm{GP}(V, \beta)$ in $\mathrm{PGL}(V)$, and by $\mathrm{PGT}(V, \beta)$ the image of $\mathrm{GT}(V, \beta)$ in $\mathrm{PGL}(V)$(Definition 1.4.5). And for $F \in S(V^*)$ of degree $d \geq 2$, we endow the set $\beta^* = \{x_0, ..., x_{n+1}\}$ with the

relation $\leq_F$ given by

$$x_i \leq_F x_j \Leftrightarrow \text{Vars}\left(\frac{\partial F}{\partial x_i}\right) \subseteq \text{Vars}\left(\frac{\partial F}{\partial x_j}\right).$$

Then we prove the next theorem

Theorem 0.0.3. *Let $X = V(F) \subset \mathbb{P}(V)$ be a smooth hypersurface of dimension $n \geq 1$ and degree $d \geq 3$ with $(n,d) \neq (1,3),(2,4)$. Let also β and β^* be dual bases of V and V^*, respectively. If $Spars(F) > 4$(Definition 1.4.6) and $(\beta^*, \leq_F)$ is a poset then $Aut(X) \subseteq PGT(V,\beta)$.(Theorem 1.4.10)*

As a corollary of this theorem, we also show that, under the same hypotheses, if additionally $(\beta^*, \leq_F)$ is the trivial poset, then $\text{Aut}(X) \subseteq \text{PGP}(V,\beta)$.

Using the previous theorem, we prove that for the Fermat hypersurface $X = V(F) \subseteq \mathbb{P}^{n+1}$ of degree d and dimension n

$$(F = x_0^d + x_1^d + ... + x_{n+1}^d),$$

we have that if $n \geq 1$, and $d \geq 3$, with $(n,d) \neq (1,3),(2,4)$, then

$\text{Aut}(X) = (\mathbb{Z}/d\mathbb{Z})^{n+1} \rtimes S_{n+2}$ (Proposition 1.5.5).

We also prove that for the Delsarte hypersurface $X = V(T) \subseteq \mathbb{P}^{n+1}$ of degree d and dimension n

$$(T = x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^d),$$

we have that if $n \geq 2$, and $d \geq 4$, with $(n,d) \neq (2,4)$, then

$\text{Aut}(X) = \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}$ (Proposition 1.5.7).

Finally, we prove that for the Klein hypersurface $X = V(K) \subseteq \mathbb{P}^{n+1}$ of degree d and dimension n

$$(K = x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0),$$

we have that if $n \geq 2$, and $d \geq 4$, with $(n,d) \neq (2,4)$, then

$\text{Aut}(X) = (\mathbb{Z}/m\mathbb{Z}) \rtimes \mathbb{Z}/(n+2)\mathbb{Z}$ where $m = \frac{(d-1)^{n+2}-(-1)^{n+2}}{d}$ (Proposition 1.5.9).
In chapter 2, we review the notions of complex manifolds, tangent and cotangent spaces, and differential forms, in order to introduce de Rham and Dolbeault cohomology.

$$H^k_{dR}(X) = H^k_{dR}(X, \mathbb{C}) := \frac{Ker(\Omega^k_{X^\infty}(X) \xrightarrow{d} \Omega^{k+1}_{X^\infty}(X))}{Im(\Omega^{k-1}_{X^\infty}(X) \xrightarrow{d} \Omega^k_{X^\infty}(X))}.$$

$$H^{p,q}_{\overline{\partial}}(X) := \frac{Ker(\Omega^{p,q}_X(X) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}_X(X))}{Im(\Omega^{p,q-1}_X(X) \xrightarrow{\overline{\partial}} \Omega^{p,q}_X(X))}.$$

We will then proceed with the definition of a Hermitian metric on a complex manifold, the Hodge star operator(Definition 2.2.4), and the notions of harmonic forms(Definition 2.2.11) on Kahler manifolds.
We present the following theorem (Theorem 2.2.13)

Theorem 0.0.4 (Hodge Harmonic Forms)**.** *Let X be a compact Hermitian manifold. Then for every pair of integers p, q, there is a natural isomorphism:*

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong H^{p,q}_{\overline{\partial}}(X).$$

And the corollary (Corollary 2.2.15)

Corollary 0.0.5 (Poincaré Duality)**.** *Let X be a compact Hermitian manifold of complex dimension n, and consider de Rham cohomology with complex coefficients. Then, for each k, the space of harmonic k-forms $\mathcal{H}^k_d(X)$ admits a non-degenerate Hermitian pairing*

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\bar{\beta},$$

which induces an isomorphism

$$\mathcal{H}^k_d(X) \cong \left(\mathcal{H}^{2n-k}_d(X)\right)^*.$$

Via the Hodge isomorphism $\mathcal{H}^k_d(X) \cong H^k_{dR}(X; \mathbb{C})$, this yields the complex Poincaré duality:

$$H^k_{dR}(X; \mathbb{C}) \cong \left(H^{2n-k}_{dR}(X; \mathbb{C})\right)^*.$$

We also have the corollary Kodaira-Serre Duality (Corollary 2.3.1), which tells us that

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X) \cong \left(\mathcal{H}^{n-p,n-q}_{\overline{\partial}}(X)\right)^*.$$

and via the Hodge theorem, this induces the corresponding duality on Dolbeault cohomology:

$$H^{p,q}_{\overline{\partial}}(X) \cong \left(H^{n-p,n-q}_{\overline{\partial}}(X)\right)^*.$$

Then we introduce the notion of a Kähler manifold(Definition 2.3.2) and with this, we state the following theorem(Theorem 2.3.3).

Theorem 0.0.6. *If X is a Kähler manifold then we have*

$$\Delta_d = 2\Delta_{\bar{\partial}} : \Omega^{p,q}_{X^\infty} \to \Omega^{p,q}_{X^\infty}.$$

Therefore $\mathcal{H}^{p,q}_d = \mathcal{H}^{p,q}_{\bar{\partial}}$. And how $\mathcal{H}^k_d = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_d$, we have

$$\mathcal{H}^k_d(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X).$$

We will also present foundational results, including the Universal Coefficient Theorem(Theorem 2.3.4) for homology—a fundamental tool in algebraic topology—and de Rham's theorem(Theorem 2.3.5).

Next, we define the notion of a Hodge structure of weight n, which is a fundamental concept (or key definition) to understand the underlying geometry and cohomology of the variety

Definition 0.0.7 (Hodge Structure)**.** *A* Hodge structure of weight n *consists of a free $\mathbb{Z}$-module H of finite rank (i.e., $H \cong \mathbb{Z}^r$) together with a decomposition of its complexification*

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^r$$

into a direct sum of complex subspaces:

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q},$$

such that the decomposition satisfies the symmetry condition

$$\overline{H^{p,q}} = H^{q,p}.$$

We define Lefschetz Operator (Definition 2.3.7) and state the Hard Lefschetz Theorem(Theorem 2.3.8).

Then we define the primitive cohomology group $H^{n-k}(X, \mathbb{C})_{prim}$.

Definition 0.0.8. *We define the the primitive cohomology group $H^{n-k}(X, \mathbb{C})_{prim}$ as*

$$H^{n-k}(X, \mathbb{C})_{prim} := Ker(L^{k+1} : H^{n-k}(X, \mathbb{C}) \to H^{n+k+2}(X, \mathbb{C})).$$

Next, we state the Gysin Exact Sequence and Primitive Cohomology Theorem(Theorem 2.3.10), and as a corollary of this, we obtain

Corollary 0.0.9. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree d. Then the primitive cohomology in middle degree is given by:*

- *If n is odd, then*
$$H^n(X,\mathbb{C})_{\text{prim}} = H^n(X,\mathbb{C}).$$
- *If n is even, then*
$$H^n(X,\mathbb{C})_{\text{prim}} = \left(\mathbb{C}\cdot\omega^{n/2}\right)^{\perp} \subset H^n(X,\mathbb{C}),$$
where $\omega \in H^2(X,\mathbb{C})$ is the Kähler class induced from the ambient projective space, and the orthogonal is taken with respect to the Hodge–Riemann bilinear form.

In particular, the primitive cohomology has codimension

$$\operatorname{codim}\left(H^n(X,\mathbb{C})_{\text{prim}} \subset H^n(X,\mathbb{C})\right) = \begin{cases} 0, & \textit{if } n \textit{ is odd,} \\ 1, & \textit{if } n \textit{ is even.} \end{cases}$$

Next, we state the Griffiths' base theorem(Theorem 2.3.12). From this, we obtain that a basis for $H^{p,q}(X)_{prim}$ is given by

$$\omega_\beta = res\left(\frac{x^\beta\Omega}{F^{q+1}}\right),$$

where $degx^\beta = \beta_0 + \beta_1 + ... + \beta_{n+1} = d(q+1) - n - 2$, with $\beta = (\beta_0, ..., \beta_{n+1})$, and $x^\beta = x_0^{\beta_0} \cdot ... \cdot x_{n+1}^{\beta_{n+1}}$.
The monomials x^β form a basis of the graded component $R^F_{d(q+1)-n-2}$.
We define the concept of a polarized Hodge structure (Definition 2.4.1), and state the Hodge-Riemann Bilinear Relations with the Hodge Index Theorem(Theorem 2.4.3), and define a morphism of Hodge structures and a morphism of polarized Hodge structures.

Definition 0.0.10. *A* morphism of Hodge structures *is a group homomorphism $f\colon H \to H'$ such that the complexified map $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ satisfies*

$$f_{\mathbb{C}}(H^{p,q}) \subseteq H'^{p,q} \quad \textit{for all } p,q.$$

If (H,Q) and (H',Q') are polarized Hodge structures, then f is a morphism of polarized Hodge structures *if, in addition, it satisfies:*

$$Q'\big(f_{\mathbb{C}}(\alpha), f_{\mathbb{C}}(\beta)\big) = Q(\alpha,\beta) \quad \textit{for all } \alpha,\beta \in H_{\mathbb{C}}.$$

Finally, we prove that if (H,Q) is a polarized Hodge structure, then $\operatorname{Aut}(H,Q)$ is finite(Proposition 2.4.8).

In Chapter 3, for the Klein hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d and dimension n, we consider the automorphism $\sigma \in \mathrm{Aut}(X)$ given by

$$\sigma([x_0 : ... : x_{n+1}]) = [\xi_{md} x_0 : \xi_{md}^{1-d} : ... : \xi_{md}^{(1-d)^{n+1}} x_{n+1}],$$

and using Griffiths' base theorem, we analyze how the automorphism $\sigma_{\mathbb{C}}^* : H^n(X, \mathbb{C})_{prim} \to H^n(X, \mathbb{C})_{prim}$, induced by σ acts on the elements of the basis. From this, we obtain the following theorem.

Theorem 0.0.11 (Villaflor et.al.). *Let $X = \{x_0^{d-1} x_1 + ... + x_{n+1}^{d-1} x_0 = 0\}$ be the Klein hypersuperface of dimension n and degree $d \geq 3$. And let $V(\xi_m^j) = \{\omega \in H^n(X, \mathbb{C})_{prim} | \sigma_{\mathbb{C}}^*(\omega) = \xi_m^j \omega\}$ be the eigenspace of $\sigma_{\mathbb{C}}^*$ associated to the eigenvalue ξ_m^j.*
Then, for all $p + q = n$, we have

$$H^{p,q}(X)_{prim} = \bigoplus_{j \in C^{p,q}} V(\xi_m^j).$$

If n is an even number, we have

$$H^n(X, \mathbb{C})_{prim} = \bigoplus_{j=0}^{m-1} V(\xi_m^j),$$

where $\dim_{\mathbb{C}} V(\xi_m^j) = 1$ if $j = 1, ..., m-1$ and $\dim_{\mathbb{C}} V(\xi_m^j) = 2$ if $j = 0$.
If n is an odd number, we have

$$H^n(X, \mathbb{C})_{prim} = \bigoplus_{j=1}^{m-1} V(\xi_m^j),$$

where $\dim_{\mathbb{C}} V(\xi_m^j) = 1$ for all $j = 1, ..., m-1$.
And for $j \in C^{p,q}$, we have

$$V(\xi_m^j) = \langle \{\omega_\beta \,|\, 0 \leq \beta_i \leq d-2, \sum_i \beta_i = d(q+1) - n - 2, \sum_i \frac{(\beta_i + 1)(1-d)^i}{d} = j\} \rangle,$$

where

$$\omega_\beta = res\left(\frac{x^\beta \Omega}{F^{q+1}}\right).$$

Then, we consider the normalizer of σ^*, $N(\langle \sigma^* \rangle)$ in the group $\mathrm{Aut}(H^n(X, \mathbb{Z})_{prim}, \langle \cdot, \cdot \rangle)/\{\pm 1\}$, and for $g \in N(\langle \sigma^* \rangle)$, we define

$$\varphi(g) = i_g \in (\mathbb{Z}/m\mathbb{Z})^\times,$$

where $g \sigma^* g^{-1} = \sigma^{*i_g}$.
then, if we define the set

$$\Delta := \{t \in (\mathbb{Z}/m\mathbb{Z})^{\times} : t \cdot C^{p,q} = C^{p,q}, \forall p+q=n\},$$

we have that $\varphi(N(\langle\sigma^*\rangle)) < \Delta$
Then, if we have that $\mathrm{Ker}(\varphi) = \langle\sigma^*\rangle$, and $|\Delta| = n+2$, then

$$N(\langle\sigma^*\rangle) = \langle\sigma^*\rangle \rtimes \langle\tau^*\rangle = (\mathbb{Z}/m\mathbb{Z}) \rtimes \mathbb{Z}/(n+2)\mathbb{Z} \simeq \mathrm{Aut}(X),$$

and if additionally we have that $N(\langle\sigma^*\rangle) = \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim}, \langle\cdot,\cdot\rangle)/\{\pm 1\}$, then we would have what we are looking for.

We conjecture that $|\Delta| = n+2$.

We developed a Python code in which we implemented a function to compute the groups Δ for given ordered pairs (n,d). In all cases, we verify that Δ is cyclic and that its order is $n+2$ (Tables 3.1,3.2,3.3,3.4,3.5,3.6,3.7,3.8).

Chapter 1

AUTOMORPHISMS OF KLEIN HYPERSURFACES

In this chapter, we will study the automorphisms of hypersurfaces in projective space, with special interest in the Klein hypersurfaces. To this end, we will first define the set of linear automorphisms $\mathrm{Lin}(X)$, which consists of the automorphisms of a hypersurface X that can be extended to the entire projective space. By a theorem of Matsumura and Monsky (Theorem 1.1.2), when X is not a cubic curve nor a quartic surface, all automorphisms of X are linear. By analyzing the properties of the elements of $\mathrm{Lin}(X)$ we show that in most cases (degree greater than 1), these automorphisms are the restriction of a unique automorphism of the whole projective space (Proposition 1.1.6).

In section 1.2 we give a characterization of the prime numbers p that can occur as the order of an automorphism of a smooth hypersurface, by introducing the notion of admissibility in dimension n and degree d (Definition 1.2.3).

Section 1.3 is devoted to study upper bounds for the prime numbers that can occur as the order of an automorphism of a smooth hypersurface. We show that, under a certain condition, a smooth hypersurface admits an automorphism of sufficiently large prime order if and only if the hypersurface is the Klein hypersurface (Theorem 1.3.10).

In section 1.4 we introduce the differential method to show that, for certain hypersurfaces, their automorphisms can be represented as generalized triangular or generalized permutation matrices(Theorem 1.4.10). Finally, in section 1.5, we apply these results to explicitly compute the automorphism groups for classical hypersurfaces: the Fermat, Delsarte, and Klein hypersurfaces, as well as for the so-called simple hypersurfaces, which are defined as the zero sets of polynomials that are sums of Klein and Delsarte polynomials. In particular, we will obtain that for X the Klein hypersurface of degree $d \geq 4$ and dimension $n \geq 2$ with $(n, d) \neq (2, 4)$

$$\mathrm{Aut}(X) = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/(n+2)\mathbb{Z}.$$

1.1 Liftability of automorphisms of hypersurfaces

Let V be a vector space over $\mathbb{C}$ of dimension $n+2$, $n \geq 2$ with a fixed basis $\{v_0, ..., v_{n+1}\}$, and let $\mathbb{P}^{n+1} = \mathbb{P}(V)$ be the corresponding projective space. We also let $\{x_0, ..., x_{n+1}\}$ be the dual basis of the linear forms on V, i.e. $x_i(v_j) = \delta_{ij}$ for all $0 \leq i, j \leq n+1$. And then $\{x_{i_1} \cdots x_{i_d} : 0 \leq i_1 \leq ... \leq i_d \leq n+1\}$ is a basis of the vector space $S^d(V^*)$ of symmetric forms of degree d on V.
Let $GL(V)$, and $PGL(V)$ be the general linear group and the projective linear group, respectively. We denote by $\pi : GL(V) \to PGL(V)$ the canonical projection. Recall that an automorphism of an algebraic variety X is a regular map $X \to X$ having a regular inverse map. The group of all automorphisms of X is denoted by $Aut(X)$. The automorphism group of $\mathbb{P}^{n+1}$ is the projective linear group $PGL(V)$. For an irreducible form $F \in S^d(V^*)$, we denote by $X = V(F) \subseteq \mathbb{P}^{n+1}$ the corresponding hypersurface of dimension n and degree d. Then, we have the following definition.

Definition 1.1.1. *Let $X = V(F) \subseteq \mathbb{P}^{n+1}$ be a hypersurface of dimension n and degree d. We define the group of linear automorphisms of X, denoted by* $\mathrm{Lin}(X)$*, as the subgroup of* $\mathrm{Aut}(X)$ *of automorphisms that extend to an automorphism of the ambient space $\mathbb{P}^{n+1}$, i.e.,*

$$\mathrm{Lin}(X) = \{\varphi \in \mathrm{Aut}(X) \,|\, \exists \omega \in \mathrm{PGL}(V) : \varphi = \omega|_X\}.$$

In general, the group $Lin(X)$ is a proper subgroup of $Aut(X)$, nevertheless for smooth hypersurfaces, we have the following classical theorem [MM63], see also [Kol19] and the references therein.

Theorem 1.1.2 (Matsumura-Monsky)**.** *Let X and Y be smooth hypersurfaces of dimension $n \geq 1$ and degree $d \geq 3$ in the complex projective space $\mathbb{P}(V)$ and let $\tau : X \to Y$ be an ismorphism. If $(n, d) \neq (1, 3), (2, 4)$, then τ is the restriction of a linear automorphism $\mathbb{P}(V) \to \mathbb{P}(V)$. In particular, every automorphism of X is linear. Moreover, $Aut(X)$ is a finite group.*

We now present a result, which shows that the group of linear automorphisms of X can be expressed as a quotient group. To establish this, we first introduce two subgroups of $\mathrm{PGL}(V)$, as well as the map that realizes the isomorphism between their quotient and the group of linear automorphisms of X.

Let B be the subgroup of $\mathrm{PGL}(V)$ whose elements are the automorphisms of $\mathbb{P}^{n+1}$ that preserve X, i.e., $B = \{\psi \in \mathrm{PGL}(V) : \psi(X) = X\}$, then for $\phi \in \mathrm{Lin}(X)$ there exists an automorphism $\psi \in \mathrm{PGL}(V)$ such that $\phi = \psi|_X$; moreover $\psi \in B$. In other words, we have an epimorphism of groups

$$B \to \mathrm{Lin}(X)$$
$$\psi \mapsto \psi|_X$$

whose kernel is $A := \{\psi \in \mathrm{PGL}(V) : \psi|_X = \mathrm{id}_X\}$. By the isomorphism theorem we conclude the following result.

Proposition 1.1.3. *Let $\Psi : B/A \to \mathrm{Lin}(X)$ the function defined by $\Psi([\omega]) = \omega|_X$. Then Ψ is well-defined and, moreover, it is a group isomorphism. In particular,*

$$\mathrm{Lin}(X) \cong B/A.$$

If we have $d \geq 3$, then $(n, d) \neq (1, 3), (2, 4)$, therefore by the Theorem 1.1.2 if X is smooth then

$$\mathrm{Aut}(X) = \mathrm{Lin}(X) \quad \text{and} \quad |\mathrm{Aut}(X)| < \infty.$$

In this setting, for an automorphism φ in $\mathrm{Aut}(X)$ there exists a unique $\Omega \in B/A$ such that $\varphi = \Psi(\Omega)$. Then, we can choose an element $\omega \in \mathrm{PGL}(V)$ such that $\omega|_X = \varphi$. We can also choose a representative $\varphi' \in \mathrm{GL}(V)$ for ω, i.e., $\pi(\varphi') = \omega$. The linear map φ' induces an automorphism of $S^d(V^*)$, which we denote by $(\varphi')^*$, given by $(\varphi')^*(G) = G \circ \varphi'$ for all $G \in S^d(V^*)$. With this notation in mind, we have the following proposition.

Proposition 1.1.4. *Let $\varphi \in \mathrm{Lin}(X)$. Then $(\varphi')^*(F) = \lambda F$ for some $\lambda \in \mathbb{C}^*$.*

Proof. Since $\varphi \in Lin(X)$, if $x \in X$, we have $\varphi(x) \in X = V(F)$, then $F(\varphi'(x)) = 0$, so $(\varphi')^*(F) \in I(V(F)) = \sqrt{< F >}$ by the Hilbert Nullstellensatz. Since F is irreducible, $(\varphi')^*(F) \in < F >$, therefore there exists a nonzero polynomial p such that $(\varphi')^*(F) = p \cdot F$. Since $\varphi' \in GL(V)$, it is linear, therefore $deg((\varphi')^*(F)) = d$, so $d = deg((\varphi')^*(F)) = deg(p \cdot F) = deg(p) + deg(F) = deg(p) + d$, then $deg(p) = 0$, therefore $(\varphi')^*(F) = \lambda F$ for some $\lambda \in \mathbb{C}^*$.

□

Therefore, in the case where $d \geq 3$, we have that for an automorphism $\varphi \in \mathrm{Aut}(X)$, the representative $\varphi' \in \mathrm{GL}(V)$ $(\Psi([\pi(\varphi')]) = \pi(\varphi')|_X = \varphi)$ is such that $(\varphi')^*(F) = \lambda F$ for some $\lambda \in \mathbb{C}^*$.

We next state a result asserting that any linear automorphism of X of order q can be extended to an automorphism of the ambient projective space $\mathbb{P}^{n+1}$ of the same order.

Proposition 1.1.5. *Let $\varphi \in \mathrm{Lin}(X)$ be an automorphism of order q. Then there exists $\omega \in \mathrm{PGL}(V)$ of order q such that $\Psi([\omega]) = \omega|_X = \varphi$.*

Proof. Suppose there exists a linear subspace $W \subseteq V$ of dimension $n+1$ such that $X \subseteq \mathbb{P}(W)$ then, $X = \mathbb{P}(W)$. Indeed, assume by contradiction that $X \subsetneq \mathbb{P}(W)$, since the krull dimension of X is n, there exists a chain of irreducible closed subsets of X such that

$$X_0 \subsetneq X_1 \subsetneq ... \subsetneq X_n = X, \ (X_n = X \text{ since } X \text{ is irreducible})$$

so

$$X_0 \subsetneq X_1 \subsetneq ... \subsetneq X_n = X \subsetneq X_{n+1} = \mathbb{P}(W)$$

is a chain of irreducible closed subsets of $\mathbb{P}(W)$, but $\mathbb{P}(W)$ is also irreducible with Krull dimension n, which is a contradiction, therefore $X = \mathbb{P}(W)$.
In the case where such $W \subseteq V$ of dimension $n+1$ exists, we have that $X = \mathbb{P}(\langle w_0, ..., w_n\rangle)$ where $\{w_0, ..., w_n\}$ is a basis of W. Let $w_{n+1} \in V$ be a vector such that $\{w_0, ..., w_n, w_{n+1}\}$ is a basis of V, and let $\omega \in PGL(V)$ be such that $\Psi([\omega]) = \omega|_X = \varphi$. Let $\varphi' \in GL(V)$ be a representative of ω. Then, since $\omega^q|_X = \varphi^q = Id_X$, we have

$$\begin{aligned}[w_0 + w_1 + ... + w_n] &= \omega^q([w_0 + w_1 + ... + w_n]) \\ &= [\varphi'^q(w_0 + w_1 + ... + w_n)],\end{aligned}$$

therefore, there exists $\alpha \in \mathbb{C}^*$ such that

$$\varphi'^q(w_0 + w_1 + ... + w_n) = \alpha(w_0 + w_1 + ... + w_n) \Rightarrow \sum_{i=0}^{n} \varphi'^q(w_i) = \alpha \sum_{i=0}^{n} w_i,$$

and since $[\varphi'^q(w_i)] = \omega^q([w_i]) = [w_i]$, for all $i = 0, ..., n$, then

$$\exists \lambda_0, ..., \lambda_n \in \mathbb{C}^* : \quad \varphi'^q(w_i) = \lambda_i w_i, \quad \forall i = 0, ..., n,$$

then we have

$$\sum_{i=0}^{n} \lambda_i w_i = \alpha \sum_{i=0}^{n} w_i \Rightarrow \sum_{i=0}^{n} (\lambda_i - \alpha) w_i = 0,$$

and since $w_0, ..., w_n$ are l.i., $\lambda_i - \alpha = 0$, for all $i = 0, ..., n$, therefore $\varphi'^q(w_i) = \alpha w_i$, for all $i = 0, ..., n$.
Let $\varphi'' \in GL(V)$ given by $\varphi''(w_i) = \varphi'(w_i)$, for all $i = 0, ..., n$ and $\varphi''(w_{n+1}) = \xi w_{n+1}$, where ξ is a primitive q-th root of α, then $\varphi''|_W = \varphi'|_W$. Clearly $\Psi([\pi(\varphi'')]) = \varphi$.
Let $\omega'' := \pi(\varphi'')$, then $\Psi([\omega'']) = \varphi$. We claim that $\omega''^q = Id$.

Indeed

$$\begin{aligned}\omega''^q &= \pi(\varphi'')^q \\ &= \pi(\varphi''^q) \\ &= \pi(\alpha Id_V) \\ &= \pi(Id_V) \\ &= Id \in PGL(V).\end{aligned}$$

Hence, the order of ω'' is q ($\omega''^l|_X = \varphi^l \neq Id_X$, for all $1 \leq l < q$, so $\omega''^l \neq Id \in PGL(V)$).

If no such subspace W exists, then there exists a basis $\{w_0, ..., w_{n+1}\}$ of V such that $[w_i] \in X$ for all $i = 0, ..., n+1$. Let $\varphi' \in GL(V)$ be such that $\pi(\varphi')|_X = \varphi$, then

$$\begin{aligned}[][w_i] &= \varphi^q([w_i]) \\ &= \pi(\varphi')^q([w_i]) \\ &= \pi(\varphi'^q)([w_i]) \\ &= [\varphi'^q(w_i)],\ \forall i = 0, ..., n+1,\end{aligned}$$

so there exists $\lambda_0, ..., \lambda_{n+1} \in \mathbb{C}^*$ such that $\varphi'^q(w_i) = \lambda_i w_i$, for all $i = 0, ..., n+1$. If $\lambda_0 = \lambda_1 = ... = \lambda_n = \lambda_{n+1}$, then $\pi(\varphi')^q = Id \in PGL(V)$. Otherwise let

$$G_j := \{w_i : \lambda_i = \lambda_j\}.$$

Note that if $\lambda_k = \lambda_{k'}$, then $G_k = G_{k'}$.
We claim that $X \subseteq \bigcup_{i=0}^{n+1} \mathbb{P}(< G_i >)$. Suppose, by contradiction, that

$$X \nsubseteq \bigcup_{i=0}^{n+1} \mathbb{P}(\langle G_i \rangle),$$

then there exists $x \in X$ such that $x \notin \bigcup_{i=0}^{n+1} \mathbb{P}(\langle G_i \rangle)$. Since $x \in X$, there exists $x' \in V$, such that $x = [x']$ and since $\{w_0, ..., w_{n+1}\}$ is a basis of V, we have

$$x' = \sum_{i=0}^{n+1} u_i w_i, \text{ where } u_0, ..., u_{n+1} \in \mathbb{C}.$$

Since $x \notin \bigcup_{i=0}^{n+1} \mathbb{P}(\langle G_i \rangle)$, we have $x' \notin \bigcup_{i=0}^{n+1} \langle G_i \rangle$, then there exists i, j such that $G_i \neq G_j$ and k, k' such that $w_k \in G_i$, $w_{k'} \in G_j$, and $u_k, u_{k'} \neq 0$, then

$$\begin{aligned}
[\sum_{i=0}^{n+1} u_i w_i] &= [x'] \\
&= x \\
&= \varphi^q(x) \\
&= \varphi^q([x']) \\
&= [\varphi'^q(x')] \\
&= [\sum_{i=0}^{n+1} u_i \varphi'^q(w_i)] \\
&= [\sum_{i=0}^{n+1} u_i \lambda_i w_i], \\
\Rightarrow \exists \alpha \in \mathbb{C}^* : \alpha \sum_{i=0}^{n+1} u_i w_i &= \sum_{i=0}^{n+1} u_i \lambda_i w_i \\
\Rightarrow \sum_{i=0}^{n+1} (\alpha u_i - u_i \lambda_i) w_i &= 0 \\
\Rightarrow \alpha u_i - u_i \lambda_i &= 0, \forall i = 0, ..., n+1,
\end{aligned}$$

and since $u_k, u_{k'} \neq 0$, we have $\alpha - \lambda_k = \alpha - \lambda_{k'} = 0$, then $\lambda_k = \alpha = \lambda_{k'}$, then $G_i = G_k = G_{k'} = G_j$, wich provides a contradiction. Therefore $X \subseteq \bigcup_{i=0}^{n+1} \mathbb{P}(\langle G_i \rangle)$.

Let $\varphi'' \in GL(V)$ given by $\varphi''(w_i) = \xi_i \varphi'(w_i)$, where ξ_i is a primitive q-th root of $\frac{\lambda_0}{\lambda_i}$.
We claim that $\pi(\varphi'')|_X = \varphi$. It is easy to see that $\varphi''|_{\langle G_i \rangle} = \xi_i \varphi'|_{\langle G_i \rangle}$ and therefore , if $y \in \mathbb{P}(\langle G_i \rangle)$, $\pi(\varphi'')(y) = \pi(\varphi')(y)$. Let $x \in X$, then $x \in \mathbb{P}(< G_i >)$ for some $i = 0, ..., n+1$, and then

$$\begin{aligned}
\pi(\varphi'')|_X(x) &= \pi(\varphi'')(x) \\
&= \pi(\varphi')(x) \\
&= \pi(\varphi')|_X(x) \\
&= \varphi(x),
\end{aligned}$$

therefore $\pi(\varphi'')|_X = \varphi$.
We now verify that $\pi(\varphi'')^q = Id \in PGL(V)$. Let $v \in V$, then $v = v_{i_0} + ... + v_{i_l}$, where

$v_{i_k} \in < G_{i_k} >$, then

$$\begin{aligned}\varphi''^q(v) &= \varphi''^q(v_{i_0} + ... + v_{i_l})\\ &= \varphi''^q(v_{i_0}) + ... + \varphi''^q(v_{i_l})\\ &= \xi^q_{i_0}\varphi'^q(v_{i_0}) + ... + \xi^q_{i_l}\varphi'^q(v_{i_l})\\ &= \frac{\lambda_0}{\lambda_{i_0}}\lambda_{i_0}v_{i_0} + ... + \frac{\lambda_0}{\lambda_{i_l}}\lambda_{i_l}v_{i_l}\\ &= \lambda_0 v\\ \Rightarrow \varphi''^q &= \lambda_0 Id_V\\ \Rightarrow \pi(\varphi'')^q &= Id \in PGL(V),\end{aligned}$$

and then, we have $\Psi(\pi(\varphi'')) = \varphi$ with $\pi(\varphi'') \in PGL(V)$ be an automorphism of order q ($\pi(\varphi'')^l|_X = \varphi^l \neq Id_X$ if $1 \leq l < q$, then $\pi(\varphi'')^l \neq Id \in PGL(V)$).

□

In the case where $d \geq 3$, we have that $\mathrm{Aut}(X) = \mathrm{Lin}(X)$, and X is not an hyperplane, since $d \neq 1$, then $X \neq \mathbb{P}(W)$ for all $W \subseteq V$ proper linear subspace of V of dimension $n+1$. In this case we have that $A = \{Id\}$, we show that in the following proposition.

Proposition 1.1.6. *Let $X = V(F) \subseteq \mathbb{P}^{n+1}$ an irreducible hypersurface of dimension n and degree $d \geq 2$. Then we have that de subgroup*

$$A = \{\varphi \in \mathrm{PGL}(V) | \varphi|_X = Id_X\} < \mathrm{Aut}(X),$$

is equal to $\{Id\}$.

Proof. Let $\varphi \in \mathrm{PGL}(V)$ be such that $\varphi|_X = Id_X$. Since X have degree $d \geq 2$, we have that is not an hyperplane, therefore $\exists\{w_0, ..., w_{n+1}\} \subseteq V$ basis of V, such that $[w_i] \in X$, for all $i = 0, ..., n+1$. Let $\varphi' \in \mathrm{GL}(V)$ be a representative of φ, i.e., $\pi(\varphi') = \varphi$, then we have that $\exists \lambda_0, ..., \lambda_{n+1} \in \mathbb{C}^*$ such that

$$\varphi'(w_i) = \lambda_i w_i, \forall i = 0, ..., n+1.$$

How in the previous proposition, let define the sets

$$G_j := \{w_i | \lambda_i = \lambda_j\},$$

then as in the previous proposition, we have that $X \subseteq \bigcup_{i=0}^{n+1} \mathbb{P}(\langle G_i \rangle)$.

$\mathbb{P}^{n+1}(\langle G_j \rangle)$ is a Zariski closed subset of $\mathbb{P}^{n+1}$(it is a linear subvariety).

We have

$$X = X \cap \left(\bigcup_{i=0}^{n+1} \mathbb{P}(\langle G_i \rangle) \right) = \bigcup_{i=0}^{n+1} X \cap \mathbb{P}(\langle G_i \rangle),$$

and $X \cap \mathbb{P}(\langle G_i \rangle)$ is a closed subset of X, then since X is irreducible, we have that

$$X = X \cap \mathbb{P}(\langle G_{i^*} \rangle),$$

for some i^*, and then

$$X \subseteq \mathbb{P}(\langle G_{i^*} \rangle).$$

By the definition of the sets G_j, we have that if $\langle G_j \rangle \cap \langle G_l \rangle \neq \emptyset$, then $\langle G_j \rangle = \langle G_i \rangle$, therefore, if $\mathbb{P}(\langle G_j \rangle) \cap \mathbb{P}(\langle G_l \rangle) \neq \emptyset$, then $\mathbb{P}(\langle G_j \rangle) = \mathbb{P}(\langle G_i \rangle)$, and since $\{w_0, ..., w_{n+1}\} \subseteq X$, we have that for all $i = 0, ..., n+1$, $[w_i] \in X$, and since $X \subseteq \mathbb{P}(\langle G_{i^*} \rangle)$, we have $[w_i] \in \mathbb{P}(\langle G_{i^*} \rangle)$, therefore $\mathbb{P}(\langle G_{i^*} \rangle) = \mathbb{P}(\langle G_i \rangle)$. Then we have that $\exists \lambda \in \mathbb{C}^*$ such that

$$\varphi'(w_i) = \lambda w_i, \forall i = 0, ..., n+1,$$

Therefore $\varphi = \pi(\varphi') = \pi(\lambda Id_V) = Id$.

□

Therefore, in the case where $d \geq 3$, $(n, d) \neq (1, 3), (2, 4)$ we have

$$\mathrm{Aut}(X) \cong \{\psi \in \mathrm{PGL}(V) | \, \psi(X) = X\},$$

then for an automorphism $\varphi \in \mathrm{Aut}(X)$, there exists an unique automorphism $\psi \in \mathrm{PGL}(V)$ such that $\varphi = \psi|_X$.

1.2 Admissible orders of an automorphism

Recapitulating, we have that $(\varphi')^*(F) = \lambda F$ for some $\lambda \in \mathbb{C}^*$, and that there exists $\omega \in PGL(V)$ (unique in our case, since $d \geq 3$) such that $\Psi([\omega]) = \varphi$, and the order of ω is also q, then if φ' is a representative of ω, we have $\pi(\varphi'^q) = \pi(\varphi')^q = \omega^q = Id \in \mathrm{PGL}(V)$, and then, multiplying by an appropriate constant, we can assume that $\varphi'^q = Id_V$, so that φ' is also a linear automorphism of order q of V and $(\varphi')^*(F) = \xi^a F$ where ξ is a primitive q-th root of unity. Furthermore, we can apply a linear change of coordinates on V to diagonalize φ', so that

$$\varphi' : V \to V, \quad (\alpha_0, ..., \alpha_{n+1}) \longmapsto (\xi^{\sigma_0} \alpha_0, ..., \xi^{\sigma_{n+1}} \alpha_{n+1}), \quad 0 \leq \sigma_i < q.$$

Definition 1.2.1. *: Letting $\mathbb{Z}/q\mathbb{Z}$ be the ring of integers modulo q, we define the signature*

σ of an automorphism φ' as above by

$$\sigma = (\sigma_0, ..., \sigma_{n+1}) \in (\mathbb{Z}/q\mathbb{Z})^{n+2},$$

where we identify σ_i with its class in the ring $\mathbb{Z}/q\mathbb{Z}$. We also denote $\varphi' = \mathrm{diag}(\sigma)$ and we say that φ' is a diagonal automorphism.

For every $F \in S^d(V^*)$ and $i \in \{0, ..., n+1\}$, we let $\deg_i(F)$ denote the degree of F seen as a polynomial in x_i. The following simple lemma is a key ingredient in the proof of the next proposition.

Lemma 1.2.2. *Let X be a hypersuperface of dimension n and degree d, given by the homogeneous form $F \in S^d(V^*)$. If $\deg_i(F) \leq d-2$, for some $i \in \{0, ..., n+1\}$, then X is singular.*

Proof. After a linear change of coordinates, we may and will assume that $deg_0(F) \leq d-2$ so that

$$F = x_0^{d-2}L_2 + x_0^{d-3}L_3 + ... + x_0 L_{d-1} + L^d,$$

where L_j is a form of degree j in the variables $\{x_1, ..., x_{n+1}\}$. Hence,

$$\frac{\partial F}{\partial x_0} = (d-2)x_0^{d-3}L_2 + (d-3)x_0^{d-4}L_3 + ... + L_{d-1},$$

$$\frac{\partial F}{\partial x_i} = x_0^{d-2}\frac{dL_2}{dx_i} + x_0^{d-3}\frac{dL_3}{dx_i} + ... + \frac{dL_d}{dx_i}, \quad i \in \{1, ..., n+1\}.$$

Now, the Jacobian criterion shows that the point $(1:0:...:0)$ is singular, since L_j is a form of degree $j \geq 2$ in the variables $\{x_1, ..., x_{n+1}\}$ and then, $L_j(1:0:...:0) = 0$ and then $F(1:0:...:0) = 0$ so indeed the point $(1:0:...:0) \in X$, and for the same reason $\frac{\partial F}{\partial x_0}(1:0:...:0) = 0$, and $\frac{dL_j}{dx_i}$ is a form of degree greater than or equal to 1 and then $\frac{dL_j}{dx_i}(1:0:...:0) = 0$, and then $\frac{\partial F}{\partial x_i}(1:0:...:0) = 0$.

□

In this part we study the particular case of automorphisms of prime order p. In this case we are able to give a full characterization of the prime numbers that appear as the order of an automorphism of some smooth hypersurface of dimension n and degree d. We also show that the order of such an automorphism is bounded by $(d-1)^{n+1}$.

Definition 1.2.3. *We say that a prime number p is* ***admissible in dimension*** *n* ***and degree*** *d if either p divides $(d-1)$ or there exist $l \in \{1, ..., n+2\}$ such that*

$$(1-d)^l \equiv 1 \quad mod\ p.$$

This definition is justified by the following proposition.

Proposition 1.2.4. *Let $n \geq 2$ and $d \geq 3$ be integers, and $(n, d) \neq (2, 4)$. A prime number p is the order of an automorphism of a smooth hypersurface of dimension n and degree d if and only if p is admissible in dimension n and degree d.*

Proof. Let's divide the proof into three cases.
Case 1. p does not divide d or $d-1$. In this case, we have $\gcd(p, d) = \gcd(p, d-1) = 1$. To prove the "only if" direction, suppose that $F \in S^d(V^*)$ is a form of degree d such that the hypersurface $X = V(F) \subseteq \mathbb{P}^{n+1}$ is smooth and admits an automorphism φ of order p. Since $\varphi \in \mathrm{Aut}(X) = \mathrm{Lin}(X)$, it follows from Proposition 1.1.3 and Proposition 1.1.5 that there exists $\Omega \in B/A$ such that $\Psi(\Omega) = \varphi$ and we can choose a representative of Ω, denoted by $\omega \in \mathrm{PGL}(V)$ such that ω is also an automorphism of order p of $PGL(V)$, and also we can choose a representative of ω, denoted by $\varphi' \in \mathrm{GL}(V)$ such that the order of this automorphism is also p ($\varphi'^p = Id_V$). Hence φ' is diagonalizable, assume the basis for V has been chosen such that φ' is diagonal, and let $\sigma = (\sigma_0, ..., \sigma_{n+1}) \in (\mathbb{Z}/p\mathbb{Z})^{n+2}$ be such that $\varphi' = \mathrm{diag}(\xi_p^{\sigma_0}, ..., \xi_p^{\sigma_{n+1}})$.
We have that $(\varphi')^*(F) = \xi^a F$, where ξ is a primitive p-th root of unity. Let b be such that $d \cdot b \equiv -a \bmod p$; such a b always exists since $\gcd(p, d) = 1$, and therefore d is invertible in $\mathbb{Z}/p\mathbb{Z}$. Consider the automorphism $\psi = \xi^b \varphi' \in \mathrm{GL}(V)$. Clearly, ψ and φ induce the same automorphism in $\mathbb{P}^{n+1}$. Furthermore, for the form F of degree d we have

$$(\psi)^*(F) = (\xi^b\varphi')^*(F) = F \circ (\xi^b\varphi') = \xi^{db} F \circ \varphi' = \xi^{db}(\varphi')^*(F) = \xi^{db}\xi^a F = \xi^{db+a}F,$$

and since $d \cdot b \equiv -a \bmod p$, we have $(\psi)^*(F) = F$. Hence, we may and will assume that $\varphi'(F) = F$. Let k_0 be such that $\sigma_{k_0} \not\equiv 0 \bmod p$ (since the order of φ' is a prime number, then $\varphi' \neq Id_V$, and therefore not all σ_i can be equal to 0 mod p). By the previous lemma, F contains a monomial $x_{k_0}^{d-1}x_{k_1}$ for some $k_1 \in \{0, ..., n+1\}$ (not necessarily with the coefficient 1). The form F is invariant by the diagonal automorphism φ', hence the monomial $x_{k_0}^{d-1}x_{k_1}$ is also invariant by φ', then $(d-1)\sigma_{k_0} + \sigma_{k_1} \equiv 0 \bmod p$, and then

$$\sigma_{k_1} \equiv (1-d)\sigma_{k_0} \mod \text{p}.$$

Furthermore, since $\gcd(p, d-1) = 1$, $d-1$ is invertible in $\mathbb{Z}/p\mathbb{Z}$, and hence

$$[\sigma_{k_1}][(d-1)]^{-1} = [\sigma_{k_0}] \neq [0].$$

Therefore $[\sigma_{k_1}] \neq [0]$, i.e., $\sigma_{k_1} \not\equiv 0 \pmod p$, and since $\gcd(p, d) = 1$ we have $k_1 \neq k_0$, since if $k_1 = k_0$, then $d\sigma_{k_0} = (d-1)\sigma_{k_0} + \sigma_{k_1} \equiv 0 \pmod p$, and then $d \equiv 0 \pmod p$(σ_{k_0} is invertible in $\mathbb{Z}/p\mathbb{Z}$), which is a contradiction. Applying the above argument with k_0 replaced by k_1, we let k_2 be such that the monomial $x_{k_1}^{d-1}x_{k_2}$ is invariant by φ' and is contained in F (not necessarily with coefficient 1). Iterating this process, for all $i \in \{3, ..., n+2\}$ we let

$k_i \in \{0, ..., n+1\}$ be such that $x_{k_{i-1}}^{d-1} x_{k_i}$ is a monomial in F (not necessarily with coefficient 1) invariant by φ'. And then, we have

$$\sigma_{k_i} \equiv (1-d)\sigma_{k_{i-1}} \equiv (1-d)^2 \sigma_{k_{i-2}} \equiv (1-d)^i \sigma_{k_0} \ (\text{mod p}), \forall i \in \{2, ..., n+2\},$$

and all of the σ_{k_i} are non-zero.

Since $k_i \in \{0, ..., n+1\}$ there are at least two $i, j \in \{0, ..., n+2\}$, $i > j$ such that $k_i = k_j$. Thus $\sigma_{k_i} = \sigma_{k_j}$, and since $\sigma_{k_i} \equiv (1-d)^i \sigma_{k_0} \pmod p$ and $\sigma_{k_j} \equiv (1-d)^j \sigma_{k_0} \pmod p$, we have

$$(1-d)^{i-j} \equiv 1 \ (\text{mod p}),$$

and so, the "only if" part of the proposition follows, and p is admissible in dimension n and degree d, since exists $l = i - j$. To prove the converse statement, suppose that p is admissible in dimension n and degree d. Since p does not divide d and $d-1$, there exists some $l \in \{1, ..., n+2\}$ such that $(1-d)^l \equiv 1 \pmod p$.

We let $F \in S^d(V^*)$ be the form

$$F = \sum_{i=1}^{l-1} x_{i-1}^{d-1} x_i + x_{l-1}^{d-1} x_0 + \sum_{i=l}^{n+1} x_i^d.$$

By construction. the form F admits the automorphism $\varphi = \text{diag}(\xi_p^{\sigma_0}, ..., \xi_p^{\sigma_{n+1}})$, where

$$\sigma = (1, 1-d, (1-d)^2, ..., (1-d)^{l-1}, \overbrace{0, ..., 0}^{n+2-l \text{ times}}) \in (\mathbb{Z}/p\mathbb{Z})^{n+2}.$$

Let's see that $X = V(F)$ is smooth. Assume that $\alpha = (\alpha_0 : ... : \alpha_{n+1}) \in X$ is a singular point, i.e.,

$$F(\alpha) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x_i}(\alpha) = 0.$$

Then $\alpha_i = 0$, for all $i \in \{l, ..., n+1\}$. Suppose that also $\alpha_j = 0$ for some $j \in \{0, ..., l-1\}$, if $j = 0$ then, since $\frac{\partial F}{\partial x_0}(\alpha) = 0$, we have $\alpha_{l-1} = 0$, and since $\frac{\partial F}{\partial x_{l-1}}(\alpha) = 0$, then α_{l-2} is also zero, and using the same argument, we have that $\alpha_k = 0$ implies that $\alpha_{k-1} = 0$, so if there exists some $j \in \{0, ..., l-1\}$ such that $\alpha_j = 0$, then $\alpha_k = 0$ for all $k \in \{0, ..., l-1\}$ which is a contradiction, therefore $\alpha_j \neq 0$ for all $j \in \{0, ..., l-1\}$.

Since $\alpha_1 \frac{\partial F}{\partial x_1}(\alpha) = 0$, we have

$$\alpha_0^{d-1} \alpha_1 = (1-d) \alpha_1^{d-1} \alpha_2,$$

and since also $\alpha_2 \frac{\partial F}{\partial x_2}(\alpha) = 0$, we have

$$\alpha_1^{d-1} \alpha_2 = (1-d) \alpha_2^{d-1} \alpha_3,$$

so

$$\alpha_0^{d-1}\alpha_1 = (1-d)\alpha_1^{d-1}\alpha_2 = (1-d)^2\alpha_2^{d-1}\alpha_3.$$

Repeating the same argument, we obtain that

$$\alpha_0^{d-1}\alpha_1 = (1-d)^i\alpha_i^{d-1}\alpha_{i+1}, \forall i \in \{0, ..., l-2\}$$

and so, we have

$$\sum_{j=1}^{l-1}\alpha_{j-1}^{d-1}\alpha_j = \sum_{i=0}^{l-2}\alpha_i^{d-1}\alpha_{i+1} = \sum_{i=0}^{l-2}\frac{\alpha_0^{d-1}\alpha_1}{(1-d)^i},$$

and also $\alpha_{l-1}\frac{\partial F}{\partial \alpha_{l-1}}(\alpha) = 0$, so

$$\alpha_{l-2}^{d-1}\alpha_{l-1} = (1-d)\alpha_{l-1}^{d}\alpha_0,$$

then $\alpha_{l-1}^d\alpha_0 = \frac{\alpha_0^{d-1}\alpha_1}{(1-d)^{l-1}}$ and since $F(\alpha) = 0$, and $\alpha_i = 0, \forall i \in \{l, ..., n+1\}$, we have

$$\sum_{i=0}^{l-1}\frac{\alpha_0^{d-1}\alpha_1}{(1-d)^i} = 0.$$

Hence, either $\alpha_0 = 0$ or $\alpha_1 = 0$ which contradicts what we have already shown. Therefore, $X = V(F)$ is smooth.

Case 2. p divides d. Then p is admissible with $l = 1$. Indeed, $(1-d)^1 = 1-d \equiv 1 \pmod{p}$. On the other hand, for every $n \geq 2$ and $d \geq 3$, let X be the Fermat hypersurface, i.e., $X = V(F)$ with

$$F = x_0^d + x_1^d + ... + x_n^d + x_{n+1}^d.$$

The hypersurface is smooth and admits the automorphism or order d given by

$$\varphi' : V \to V, \quad (\alpha_0, \alpha_1, ..., \alpha_{n+1}) \longmapsto (\xi\alpha_0, \alpha_1, ..., \alpha_{n+1}),$$

where ξ is a primitive d-th root of unity. Hence, X also admits an automorphism of order p.

Case 3. p divides $d-1$. The prime number p is admissible by definition. On the other hand, for $n \geq 2$ and $d \geq 3$, let $X = V(F)$ be the hypersurface given by

$$F = x_0^{d-1}x_1 + x_1^d + ... + x_n^d + x_{n+1}^d.$$

A routine computation shows that the hypersurface X is smooth and admits the automorphism or order $d-1$ given by

$$\varphi' : V \to V, \quad (\alpha_0, \alpha_1, ..., \alpha_{n+1}) \longmapsto (\xi\alpha_0, \alpha_1, ..., \alpha_{n+1}),$$

where ξ is a primitive $(d-1)$-th root of unity. Hence, X also admits an automorphism of order p.

□

1.3 Hypersurfaces with automorphisms of high order

Corollary 1.3.1. *: Let $n \geq 2$ and $d \geq 3$ be integers, and $(n,d) \neq (2,4)$. If a prime number p is the order of an automorphism of a smooth hypersurface of dimension n and degree d, then $p < (d-1)^{n+1}$.*

Proof. Since p is a prime, and $n \geq 2$, we have $p \neq (d-1)^{n+1}$. Suppose that $p > (d-1)^{n+1}$. By the previous proposition, p is admissible in dimension n and degree d, so

$$(1-d)^{n+1} \equiv 1 \pmod{p}, \quad \text{or} \quad (1-d)^{n+2} \equiv 1 \pmod{p}.$$

This yields

$$p = (d-1)^{n+1} - (-1)^{n+1}, \quad \text{or} \quad kp = (d-1)^{n+2} - (-1)^{n+2}, \quad k \in \{1, ..., d-1\}.$$

Since $d-1 \equiv -1 \pmod{d}$, we have that d is a divisor of p or kp. In both cases, this yields $\gcd(p,d) \neq 1$, wich provides a contradiction since $p > d$.

□

Definition 1.3.2. *For every $m \in \mathbb{Z}_{>0}$, the m-th cyclotomic polynomial is defined as*

$$\Phi_m(t) = \prod_{\xi} (t - \xi),$$

where the product is over all primitive m-th roots of unity ξ.

It is well known that $\Phi_m(t)$ is irreducible over $\mathbb{Q}$ and has integers coefficients. Furthermore, a routine computation shows that $\Phi_1(t) = t - 1$ and for every q prime and $r \geq 1$

$$\Phi_q(t) = t^{q-1} + t^{q-2} + ... + 1, \quad \text{and} \quad \Phi_{q^r}(t) = \Phi_q(t^{q^{r-1}}).$$

The main result about cyclotomic polynomials that we will need in the sequel is the following factorization

$$t^n - 1 = \prod_{r|n} \Phi_r(t),$$

where $r|n$ means r is a divisor of n.
Our next theorem gives a criterion for the existence of a smooth hypersurface of dimension n and degree d admitting an automorphism of prime order $p > (d-1)^n$. For the proof we need the following simple inequalities:

$$\Phi_1(1-d) = -d, \quad \Phi_2(1-d) = 2-d, \quad \Phi_4(1-d) = d^2 - 2d + 2 > (d-1)^2,$$

and

$$(d-1)^{q-2} < \Phi_q(1-d) = (1-d)^{q-1} + ... + 1 < (d-1)^{q-1},$$

for all $q \geq 3$ prime.

Lemma 1.3.3. *Let $n \geq 2$ and $d \geq 3$ be integers, and $(n,d) \neq (2,4)$. If there exists a smooth hypersurface of dimension n and degree d admitting an automorphism of prime order $p > (d-1)^n$, then either $n = 2$ or $n+2$ is prime, and p divides $\Phi_{n+2}(1-d)$.*

Proof. Assume that there exists a smooth hypersurface of dimension n and degree d admitting an automorphism of prime order $p > (d-1)^n$. By the previous proposition, p is admissible in dimension n and degree d and, by the previous corollary and since $p > (d-1)^n$, it follows that p is not admissible in dimension $n-1$ and degree d. Hence $(1-d)^{n+2} \equiv 1 \pmod{p}$ $(l = n+2)$, and so

$$(1-d)^{n+2} - 1 = k \cdot p, \qquad \text{for some} \quad k \in \{-(d-1)^2, ..., (d-1)^2\}.$$

If $n = 2$, then

$$\begin{aligned}(1-d)^{n+2} - 1 &= (1-d)^4 - 1 \\ &= \prod_{r|4} \Phi_r(1-d) \\ &= \Phi_1(1-d)\Phi_2(1-d)\Phi_4(1-d) \\ &= -d \cdot (2-d) \cdot (d^2 - 2d + 2) \\ &= (d^2 - 2d)(d^2 - 2d + 2)\end{aligned}$$

so, since p is prime, $p|(d^2 - 2d)$ or $p|(d^2 - 2d + 2)$, and since $p > (d-1)^n = (d-1)^2 = d^2 - 2d + 1 > (d^2 - 2d)$, p divides $(d^2 - 2d + 2)$, and since $(d^2 - 2d + 2) = (d-1)^2 + 1$, we have that $p = (d^2 - 2d + 2) = \Phi_4(1-d) = \Phi_{n+2}(1-d)$.
If $n+2$ is prime, then

$$(1-d)^{n+2}-1=\prod_{r|n+2}\Phi_r(1-d)=\Phi_1(1-d)\cdot\Phi_{n+2}(1-d)$$

so, since $p>|\Phi_1(1-d)|$, we have that p divides $\Phi_{n+2}(1-d)$.

To finish the proof, we need to show that these two are the only possible cases. If $n\neq 2$ and $n+2$ is not prime, then

$$n+2=q\cdot n', \quad \text{or} \quad n+2=2^i,$$

where $q\geq 3$ is a prime number, $n'\geq 2$, and $i\geq 3$.
Assume first that $n+2=q\cdot n'$. In this case

$$(1-d)^{n+2}-1=\Phi_1(1-d)\cdot\Phi_q(1-d)\cdot P(1-d),$$

for some polynomial $P(t)$. Let $k':=\Phi_1(1-d)\cdot\Phi_q(1-d)$. Since $|k'|<d(d-1)^{q-1}<(d-1)^n<p$, then p divides $P(1-d)$, and then k is a multiple of k'. However, we also have $|k'|>d(d-1)^{q-2}>(d-1)^2$, so $|k|>(d-1)^2\Rightarrow p|k|=|(1-d)^{n+2}-1|\geq(d-1)^{n+2}+(d-1)^2$, which provides a contradiction.
Finally, assume that $n+2=2^i$, $i\geq 3$. In this case

$$(1-d)^{n+2}-1=\Phi_4(1-d)\cdot P(1-d),$$

for some polynomial $P(t)$. Let $k'=\Phi_4(1-d)=(d-1)^2+1$. Since $k'<(d-1)^3<p$, k is a multiple of k'. But $k'>(d-1)^2$ which provides a contradiction.

□

Lemma 1.3.4. *Let $n\geq 2$ and $d\geq 3$ be integers, and $(n,d)\neq(2,4)$. If $n=2$ or $n+2$ is prime, and $p=\Phi_{n+2}(1-d)$ is also prime, then there exists a smooth hypersurface of dimension n and degree d admitting an automorphism of prime order p.*

Proof. If $n+2$ and $\Phi_{n+2}(1-d)$ are prime, then

$$(1-d)^{n+2}-1=\Phi_1(1-d)\cdot\Phi_{n+2}(1-d)\equiv 0\,(\mathrm{mod}\ \Phi_{n+2}(1-d)),$$

and so $\Phi_{n+2}(1-d)$ is admissible in dimension n and degree d, then by the previous proposition, $p=\Phi_{n+2}(1-d)$ is the order of an automorphism of a smooth hypersurface of dimension n and degree d. If $n=2$, and $\Phi_{n+2}(1-d)$ is prime, then

$$(1-d)^4-1=\Phi_1(1-d)\cdot\Phi_2(1-d)\cdot\Phi_4(1-d)\equiv 0\,(\mathrm{mod}\ \Phi_4(1-d)).$$

Hence, $\Phi_4(1-d) = \Phi_{n+2}(1-d)$ is admissible in dimension n and degree d, so $p = \Phi_{n+2}(1-d)$ is the order of an automorphism of a smooth hypersurface of dimension n and degree d. $\square$

So, taking into account the two previous lemmas, if $\Phi_{n+2}(1-d)$ is a prime number, we obtain the following result.

Theorem 1.3.5. *Let $n \geq 2$ and $d \geq 3$ be integers, and $(n,d) \neq (2,4)$, and such that $\Phi_{n+2}(1-d)$ is prime. There exists a smooth hypersurface of dimension n and degree d admitting an automorphism of prime order $p > (d-1)^n$ if and only if $n = 2$ or $n+2$ is prime, and $p = \Phi_{n+2}(1-d)$.*

Proof. The proof follows directly from the two previous lemmas. $\square$

In the following corollary, which follows directly from the previous theorem and Lemma 1.1.12, we provide a sharp bound for the order of an automorphism of a smooth hypersurface of dimension n and degree d.

Corollary 1.3.6. *Let $n \geq 2$ and $d \geq 3$ be integers, and $(n,d) \neq (2,4)$. Assume that a smooth hypersurface of dimension n and degree d admits an automorphism of prime order p.*

(i) If $n = 2$ or $n+2$ is prime, and $\Phi_{n+2}(1-d)$ is prime, then $p \leq \Phi_{n+2}(1-d)$. This bound is sharp.

(ii) In any other case, $p < (d-1)^n$.

Remark 1.3.7. *Assume that (n,d) is such that $\Phi_{n+2}(1-d)$ is prime and $n \neq 2$. Then*

$$p = \frac{(1-d)^{n+2}-1}{(1-d)-1}.$$

Prime numbers of this form are usually known as generalized Mersenne primes or repunit primes. For $d = -1$ they correspond to the classical Mersenne primes and for $d = 3$ they are usually called Wagstaff primes. It is conjectured that there are infinitely many such primes ([WS79], [Mel08]).

In the following example we define the classical Klein hypersurface that will be the subject of the remainder of this section.

Example 1.3.8. *For any $n \geq 1$ and $d \geq 2$, we define the Klein hypersurfarce of dimension n and degree d as $X = V(F) \subseteq \mathbb{P}^{n+1}$, where*

$$F = x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0.$$

It is well known that X is smooth except in the case where $d = 2$ and $n \equiv 2 \pmod 4$ (A proof can also be obtained using the argument from Case 1 of Proposition 1.1.7.).

These hypersurfaces were first introduced by Klein who studied the automorphism group of the Klein hypersurface of dimensions 1,3 and degree 3 [Kle79a]. For the proof of the theorem below, we need the following simple lemma that follows from the uniqueness of the descomposition of an integer in base $(d-1)$.

Lemma 1.3.9. *Let $d \geq 3$ and $a_0, a_1, ..., a_N \in \mathbb{Z}$ with $|a_i| \in \{0, ..., d-2\}$ for all $i = 0, ..., N$. If $\sum_i a_i(1-d)^i = 0$, then $a_i = 0$ for all $i = 0, ..., N$.*

Proof. Note that

$$\sum_i a_i(1-d)^i = 0 \Rightarrow \sum_{i \text{ even}} a_i(d-1)^i = \sum_{j\ odd} a_j(d-1)^j$$

by uniqueness in base $(d-1)$, we have $a_i = 0$ for all i even and $a_j = 0$ for all j odd.

□

The following is the main result of this section.

Theorem 1.3.10. *Let $n \geq 2$ and $d \geq 3$ be integers, and $(n, d) \neq (2, 4)$, and suppose that $\Phi_{n+2}(1-d)$ is prime. A smooth hypersurface $X = V(F)$ of dimension n and degree d admits an automorphism φ of prime order $p > (d-1)^n$ if and only if X is isomorphic to the Klein hypersurface, $n = 2$ or $n+2$ is prime, and $p = \Phi_{n+2}(1-d)$.*

Proof. Since $p > (d-1)^n$, by corollary 1.1.8, p is not admissible in dimension $n-1$ and degree d. Also, we can choose a representative $\varphi' \in \mathrm{GL}(V)$ $(\varphi = \Psi([\pi(\varphi')]))$ such that $(\varphi')^*(F) = F$ and $\varphi' = \mathrm{diag}(\xi_p^{\sigma_0}, ..., \xi_p^{\sigma_{n+1}})$, where

$$\sigma = (\sigma_0, ..., \sigma_{n+1}) = (1, (1-d), (1-d)^2, ..., (1-d)^{n+1}).$$

The Klein hypersurface previously defined admits the automorphism φ above. This together with theorem 1.1.14 proves the "if" part.
Assume now that $X = V(F)$ is a smooth hypersurface of dimension n and degree d admitting the automorphism φ of prime order $p > (d-1)^n$. Let $\mathcal{E} \subseteq S^d(V^*)$ be the eigenspace associated to the eigenvalue 1 of the linear automorphism $(\varphi')^* : S^d(V^*) \to S^d(V^*)$, so that $F \in \mathcal{E}$. In the following we compute a basis for $\mathcal{E}$. Let $\mathbf{x}^\alpha$ be a monomial in $S^d(V^*)$, i.e.,

$$\mathbf{x}^\alpha := x_0^{\alpha_0} \cdots x_{n+1}^{\alpha_{n+1}}, \quad \sum_{i=0}^{n+1} \alpha_i = d, \quad \text{and} \quad \alpha_i \geq 0.$$

We have

$$\mathbf{x}^\alpha \in \mathcal{E} \Leftrightarrow (\varphi')^*(\mathbf{x}^\alpha) = \mathbf{x}^\alpha \Leftrightarrow \mathbf{x}^\alpha \circ \varphi' = \mathbf{x}^\alpha \Leftrightarrow \mathbf{x}^\alpha \circ diag(1, (1-d), ..., (1-d)^{n+1}) = \mathbf{x}^\alpha$$

$$\Leftrightarrow \prod_{i=0}^{n+1} x_i^{\alpha_i} = \prod_{i=0}^{n+1} (x_i \xi^{(1-d)^i})^{\alpha_i}, \quad \forall x = (x_0, ..., x_{n+1}) \in V$$

$$\Leftrightarrow 1 = \prod_{i=0}^{n+1} (\xi^{(1-d)^i})^{\alpha_i} = \xi^{\sum_{i=0}^{n+1}(1-d)^i \alpha_i} \Leftrightarrow \sum_{i=0}^{n+1} (1-d)^i \alpha_i \equiv 0 \bmod \quad p.$$

Since $\alpha_{n+1} = d - \sum_{i=0}^{n} \alpha_i$, we have

$$\begin{aligned} L :&= \sum_{i=0}^{n+1} \alpha_i (1-d)^i \\ &= \sum_{i=0}^{n} \alpha_i (1-d)^i + \alpha_{n+1}(1-d)^{n+1} \\ &= \sum_{i=0}^{n} \alpha_i (1-d)^i + (d - \sum_{i=0}^{n} \alpha_i)(1-d)^{n+1} \\ &= \sum_{i=0}^{n} \alpha_i ((1-d)^i - (1-d)^{n+1}) + d(1-d)^{n+1} \\ &= d \sum_{i=0}^{n} \alpha_i ((1-d)^i + ... + (1-d)^n) + d(1-d)^{n+1}. \end{aligned}$$

Letting $\beta_i = \sum_{j=0}^{i} \alpha_j$, for all $0 \le i \le n$, we have $0 \le \beta_i \le \beta_j \le d$, for all $i < j$, and

$$L = d \cdot M, \quad \text{where} \quad M = \beta_0 + \beta_1(1-d) + ... + \beta_n(1-d)^n + (1-d)^{n+1}.$$

Since d is invertible in $\mathbb{Z}/p\mathbb{Z}$ we have

$$\mathbf{x}^\alpha \in \mathcal{E} \Leftrightarrow L \equiv 0 \mod p \Leftrightarrow M \equiv 0 \mod p.$$

By the previous theorem, we know that $p = \Phi_{n+2}(1-d)$ and $n = 2$ or $n+2$ is prime.
We divide the proof into two cases.
Case $n+2$ is prime: In this case $p = 1 + (1-d) + ... + (1-d)^{n+1}$. If $\beta_n < d-1$, to prove $M = p$. For this we will prove that M is greater than 0 and less than $2p$, so first let us see that $M > 0$. Since $0 \le \beta_i \le \beta_j$, for all $i < j$, we have

$$\beta_{j-2}(1-d)^{j-2} + \beta_{j-1}(1-d)^{j-1} \ge 0, \quad \forall j \ge 3 \text{ odd},$$

and then

$$
\begin{aligned}
M &= \sum_{i=0}^{n} \beta_i (1-d)^i + (1-d)^{n+1} \\
&= \beta_0 + (\beta_1(1-d) + \beta_2(1-d)^2) + ... \\
&+ (\beta_{n-2}(1-d)^{n+2} + \beta_{n-1}(1-d)^{n-1}) + (\beta_n(1-d)^n + (1-d)^{n+1}) \\
&\geq \beta_n(1-d)^n + (1-d)^{n+1},
\end{aligned}
$$

and since $\beta_n < d-1$, we have $\beta_n(1-d)^n + (1-d)^{n+1} > 0$, and then $M > 0$.
Now let us see that $M < 2p$. If i is even, $(1-d)^i < -(1-d)^{i+1}$, then $\beta_i(1-d)^i \leq -\beta_{i+1}(1-d)^{i+1}$, so

$$M = (\beta_0+\beta_1(1-d))+(\beta_2(1-d)^2+\beta_3(1-d)^3)+...+(\beta_{n-1}(1-d)^{n-1}+\beta_n(1-d)^n)+(1-d)^{n+1},$$

and then, $M \leq (1-d)^{n+1}$. Since $d \geq 3$

$$M \leq (1-d)^{n+1} = (1-d)^{n+1}(2-1) \leq (1-d)^{n+1}(2+\frac{2}{1-d}) = 2((1-d)^n+(1-d)^{n+1}) < 2p.$$

Then $0 < M < 2p$, therefore $M = p$. So the previous lemma shows that $\beta_i = 1, \forall i$. This corresponds to $\mathbf{x}^\alpha = x_{n+1}^{d-1}x_0$.
If $\beta_n = d-1$, to prove $M = 0$. Since $\beta_n = (d-1)$, we have $\beta_n(1-d)^n + (1-d)^{n+1} = 0$ and then

$$
\begin{aligned}
M &= \beta_0 + \beta_1(1-d) + ... + \beta_{n-2}(1-d)^{n-2} + \beta_{n-1}(1-d)^{n-1} \\
&= (\beta_0 + \beta_1(1-d)) + (\beta_2(1-d)^2 + \beta_3(1-d)^3) + ... \\
&+ (\beta_{n-3}(1-d)^{n-3} + \beta_{n-2}(1-d)^{n-2}) + \beta_{n-1}(1-d)^{n-1} \\
&\leq \beta_{n-1}(1-d)^{n-1} \\
&\leq (d-1)^n \\
&< \Phi_{n+2}(1-d) \\
&= p,
\end{aligned}
$$

and also

$$M = \beta_0 + (\beta_1(1-d) + \beta_2(1-d)^2) + ... + (\beta_{n-2}(1-d)^{n-2} + \beta_{n-1}(1-d)^{n-1}) \;\geq\; \beta_0 \;\geq\; 0,$$

therefore $M = 0$. So previous lemma shows that $\beta_i = 0, \forall i < n$. This corresponds to $\mathbf{x}^\alpha = x_n^{d-1}x_{n+1}$.

Let's see that β_0 cannot be d. If $\beta_0 = d$, then $\beta_i = d$, for all i, and then

$$M = d(1+(1-d)+...+(1-d)^n)+(1-d)^{n+1} = dp-(d-1)(1-d)^{n+1} = dp+(1-d)^{n+2},$$

but $(1-d)^{n+2}-1$ is a multiple of p, so $(1-d)^{n+2}$ can not be a multiple of p, which provides a contradiction. If $\beta_j = d, \forall j \geq k+1$ and $\beta_k < d$, for some $k < n$ then

$$\begin{aligned} M &= \sum_{i=0}^{k} \beta_i(1-d)^i + d((1-d)^{k+1}+...+(1-d)^n)+(1-d)^{n+1} \\ &= \sum_{i=0}^{k} \beta_i(1-d)^i + d(1+...+(1-d)^{n+1}) - d(1+...+(1-d)^k)+(1-d)^{n+1} - d(1-d)^{n+1} \\ &= \sum_{i=0}^{k} \beta_i(1-d)^i + dp + ((1-d)^{k+1}-1)+(1-d)^{n+2} \\ &= \sum_{i=0}^{k} \beta_i(1-d)^i + dp + (1-d)^{k+1} + ((1-d)^{n+2}-1) \\ &= \sum_{i=0}^{k} \beta_i(1-d)^i + dp + (1-d)^{k+1} - dp \\ &= \sum_{i=0}^{k} \beta_i(1-d)^i + (1-d)^{k+1}. \end{aligned}$$

By arguments similar to the previous ones, it can be seen that $-p < M < p$, therefore $M = 0$, and then the previous lemma shows that $\beta_k = (d-1)$ and $\beta_i = 0$, for all $i < k$. This corresponds to $\mathbf{x}^\alpha = x_k^{d-1}x_{k+1}$. Hence, $\mathcal{E} = \langle x_{n+1}^{d-1}x_0, x_k^{d-1}x_{k+1}; 0 \leq k \leq n\rangle$ and

$$F = a_0 \cdot x_0^{d-1}x_1 + a_1 \cdot x_1^{d-1}x_2 + ... + a_n \cdot x_n^{d-1}x_{n+1} + a_{n+1} \cdot x_{n+1}^{d-1}x_0.$$

Since $X = V(F)$ is smooth, by Lemma 1.1.5, $a_i \neq 0, \forall i$ and applying a linear change of coordinates we can put

$$F = x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0.$$

CASE $n = 2$: In this case $p = (d-1)^2+1$ and so

$$M = \beta_0+\beta_1(1-d)+\beta_2(1-d)^2+(1-d)^3 = \beta_0+\beta_1(1-d)+\beta_2((1-d)^2+1)-\beta_2+(1-d)^3$$

$$\equiv \beta_0+\beta_1(1-d)-\beta_2+(1-d)^3 \mod \text{p},$$

so

$$M \equiv \beta_0+\beta_1(1-d)-\beta_2+(1-d)^3+(1-d)-(1-d)$$

$$\equiv \beta_0 + \beta_1(1-d) - \beta_2 - (1-d) = (\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) \mod \text{p}.$$

So $(\beta_0 - \beta_2) + (\beta_1 - 1)(1-d)$ is a multiple of p. Also, we have

$$p > (d-1) \geq (\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) \geq -d - (1-d)^2 > -2p,$$

then $(\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) = 0$ or $(\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) = -p$. If $(\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) = 0$, then $(1-d)|(\beta_0 - \beta_2)$, furthermore we have to $-d \leq \beta_0 - \beta_2 \leq 0$, then $\beta_0 - \beta_2 = 0$ or $\beta_0 - \beta_2 = 1 - d$. If $\beta_0 - \beta_2 = 0$, then $\beta_1 = 1$, so $\beta_0 = \beta_1 = \beta_2 = 1$, this corresponds to $\mathbf{x}^\alpha = x_3^{d-1}x_0$. If $\beta_0 - \beta_2 = 1 - d$, then $\beta_2 = d, \beta_0 = 1$ or $\beta_2 = d-1, \beta_0 = 0$, in both cases $\beta_1 = 0$, then $\beta_0 = 0$ and $\beta_2 = d - 1$, this corresponds to $\mathbf{x}^\alpha = x_2^{d-1}x_3$. If $(\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) = -p$, we have

$$(\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) = -1 - (d-1)^2 \Rightarrow 1 + (\beta_0 - \beta_2) + (\beta_1 - 1)(1-d) + (1-d)^2 = 0$$

$$\Rightarrow 1 + (\beta_0 - \beta_2) + \beta_1(1-d) - d(1-d) = 0 \Rightarrow 1 + (\beta_0 - \beta_2) + (\beta_1 - d)(1-d) = 0$$

$$\Rightarrow 1 + (\beta_0 - \beta_2) = (d-1)(\beta_1 - d),$$

then, $(d-1)|(1 + \beta_0 - \beta_2)$, and since $1 - d \leq 1 + \beta_0 - \beta_2 \leq 1$, we have

$$1 + \beta_0 - \beta_2 = 0 \quad \text{or} \quad 1 + \beta_0 - \beta_2 = 1 - d.$$

If $1 + \beta_0 - \beta_2 = 1 - d$, then $\beta_0 - \beta_2 = -d$, so $\beta_0 = 0, \beta_1 = d-1$ and $\beta_2 = d$, this corresponds to $\mathbf{x}^\alpha = x_1^{d-1}x_2$. If $1 + \beta_0 - \beta_2 = 0$, then $\beta_0 = d-1, \beta_1 = d$ and $\beta_2 = d$, this corresponds to $\mathbf{x}^\alpha = x_0^{d-1}x_1$.
Hence, $\mathcal{E} = \langle x_0^{d-1}x_1, x_1^{d-1}x_2, x_2^{d-1}x_3, x_3^{d-1}x_0 \rangle$ and

$$F = a_0 \cdot x_0^{d-1}x_1 + a_1 \cdot x_1^{d-1}x_2 + a_2 \cdot x_2^{d-1}x_3 + a_3 \cdot x_3^{d-1}x_0.$$

With the same argument as above, we can apply a linear change of coordinates to put

$$F = x_0^{d-1}x_1 + x_1^{d-1}x_2 + x_2^{d-1}x_3 + x_3^{d-1}x_0.$$

□

1.4 The differential method

Fix a vector space V over $\mathbb{C}$ of dimension $n+2$, this time with $n \geq 1$. For an automorphism $\widetilde{\varphi} : V \to V$ in $GL(V)$ we denote its image $\pi(\widetilde{\varphi})$ by φ. On the other hand, given an automorphism $\varphi \in PGL(V)$ we choose a preimage by π and we denote it by $\widetilde{\varphi} \in GL(V)$.

The automorphism $\widetilde{\varphi}$ induces an automorphism $\widetilde{\varphi}^* : V^* \to V^*$ on the dual space V^* given by $\widetilde{\varphi}^*(L) = L \circ \widetilde{\varphi}$. Consequently, it also induces a grade automorphism $\widetilde{\varphi}^* : S(V^*) \to S(V^*)$ on the symmetric algebra $S(V^*)$ of the vector space V^* given also by $\widetilde{\varphi}^*(F) = F \circ \widetilde{\varphi}$. This automorphism restricts to an autmorphism $\widetilde{\varphi}^* : S^d(V^*) \to S^d(V^*)$ of forms of degree d.
Let now X be a hypersurface of $P(V)$ given as the zero set of a homogeneous form $F \in S^d(V^*)$ of degree d. From previous results, we know that if $d \geq 2$ we have the following isomorphism

$$Lin(X) \cong \{\varphi \in PGL(V) \mid \varphi(X) = X\} = \{\varphi \in PGL(V) \mid \widetilde{\varphi}^*(F) = \lambda F \text{ for some } \lambda \in \mathbb{C}^*\}.$$

The so called differential method has been used to constrain, in some particular cases via ad-doc constructions, the shape of the matrices in $PGL(V)$ defining an automorphism of a hypersurface X in $\mathbb{P}(V)$ given by a form having, in a certain bases, few monomials. We will give a general treatment of it in this and the following section.
Let us fix some notations in order to do computations in coordinates. Let $\beta = \{e_0, ..., e_{n+1}\}$ be a basis of V and $\beta^* = \{x_0, ..., x_{n+1}\}$ be the corresponding dual basis of V^*. This choice also endows V^* with a dot product for wich β^* is an orthonormal basis, i.e., $x_i \cdot x_j = 1$ if $i = j$ and $x_i \cdot x_j = 0$ if $i \neq j$. Furthermore, the choice of a basis also induces a canonical isomorphism of the symmetric algebra $S(V^*)$ and the polynomial ring $\mathbb{C}[x_0, ..., x_{n+1}]$. Under this isomorphism, a form $F \in S(V^*)$ of degree d corresponds to a homogeneous polynomial of total degree d. Moreover, after this choice of basis, an element $\widetilde{\varphi}$ in $GL(V)$ is given by a matrix $(\widetilde{\varphi}_{ij})$.

Definition 1.4.1. *Let D be the directional derivate operator*

$$D : V^* \times S(V^*) \to S(V^*), \quad (x, F) \mapsto \frac{\partial F}{\partial x} = \nabla(F) \cdot x.$$

For a fixed homogeneus form $F \in S(V^*)$, define the specialization of D given by

$$D_F : V^* \to S(V^*), \quad x \mapsto \frac{\partial F}{\partial x} = \nabla(F) \cdot x.$$

*Recall that the rank of a linear operator is the dimension of its image. We define the **differential rank** of a form $F \in S(V^*)$ as* $\operatorname{diff.rank}(F) := \operatorname{rank} D_F$.

The key idea of the differential method is that the differential rank does not change if we replace F by $\widetilde{\varphi}^*(F)$, where $\widetilde{\varphi} \in GL(V)$.

Proposition 1.4.2. *([OY19b]). Let $F, G \in S(V^*)$ and let $\widetilde{\varphi} \in GL(V)$ be such that $\widetilde{\varphi}^*(G) = \lambda F$ for some $\lambda \in \mathbb{C}^*$. Then* $\operatorname{diff.rank}(F) = \operatorname{diff.rank}(G)$.

Proof. Using the bilinearity of D and the chain rule, we obtain that

$$\lambda\frac{\partial F}{\partial x} = \nabla(\lambda F)\cdot x = \nabla(G\circ\widetilde{\varphi}^*)\cdot x = ((\nabla G)\circ\widetilde{\varphi}^*)\cdot\widetilde{\varphi}^*(x) \tag{1.4.1}$$

$$= (\nabla(G)\cdot\widetilde{\varphi}^*(x))\circ\widetilde{\varphi}^* = \frac{\partial G}{\partial(\widetilde{\varphi}^* x)}\circ\widetilde{\varphi}^* = \widetilde{\varphi}^*\left(\frac{\partial G}{\partial(\widetilde{\varphi}^* x)}\right),\ \forall x\in V^* \tag{1.4.2}$$

Hence

$$(\lambda(\widetilde{\varphi}^*)^{-1}\circ D_F)(x) = (D_G\circ\widetilde{\varphi}^*)(x),\ \forall x\in V^*$$

and then, the following diagram

$$\begin{array}{ccc} V^* & \xrightarrow{D_F} & S(V^*) \\ {\scriptstyle\widetilde{\varphi}^*}\downarrow & & \downarrow{\scriptstyle\lambda(\widetilde{\varphi}^*)^{-1}} \\ V^* & \xrightarrow{D_G} & S(V^*) \end{array}$$

is commutative. Hence, the ranks of D_F and D_G agree since both vertical arrows are isomorphisms.

□

As a corollary we obtain the following lemma that we will use to constrain the shape of the automorphism groups of certain hypersurfaces including Klein hypersurfaces.

Corollary 1.4.3. *Let X be a hypersurface in $\mathbb{P}(V)$ given by a homogeneous form $F\in S(V^*)$ and let $\varphi\in PGL(V)$ be a linear automorphism of X. Then for every $x\in V^*$ we have*

$$\operatorname{diff.rank}\left(\frac{\partial F}{\partial x}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^* x)}\right).$$

Proof. Since φ is an automorphism of X, we have that $\widetilde{\varphi}^*(F) = F\circ\widetilde{\varphi} = \lambda F$ for some $\lambda\in\mathbb{C}^*$. Now, if we consider $G = F$ and use the first part of the proof of the previous proposition, we obtain that

$$\lambda\frac{\partial F}{\partial x} = \widetilde{\varphi}^*\left(\frac{\partial G}{\partial(\widetilde{\varphi}^* x)}\right) = \widetilde{\varphi}^*\left(\frac{\partial F}{\partial(\widetilde{\varphi}^* x)}\right).$$

Then, we have that $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial(\widetilde{\varphi}^* x)}\in S(V^*)$ be such that $\widetilde{\varphi}^*\left(\frac{\partial F}{\partial(\widetilde{\varphi}^* x)}\right) = \lambda\frac{\partial F}{\partial x}$, with $\widetilde{\varphi}\in GL(V)$ and $\lambda\in\mathbb{C}^*$, and then all the hypotheses of the previous proposition are fulfilled, therefore diff.rank$\left(\frac{\partial F}{\partial x}\right)$ = diff.rank$\left(\frac{\partial F}{\partial(\widetilde{\varphi}^* x)}\right)$.

□

Remark 1.4.4. *For a generic homogeneous form $F\in S(V^*)$ we have ker $D_F = \{0\}$ and so the differential rank of F is dim$(V) = n+2$. In particular, if X is smooth of degree greather than one then the corresponding form F has differential rank dim(V). Nevertheless, we apply the previous corollary in the cases where the differential rank of $\frac{\partial F}{\partial x}$ for a particular* *choice of $x \in V^*$ drops, this puts constrains on the shape of the matrices that appear as automorphisms of X since the differential rank of $\frac{\partial F}{\partial(\widetilde{\varphi}^* x)}$ must also drop.*

Definition 1.4.5.

1. *A matrix is a generalized permutation matrix if there is at most one coefficient different from 0 in each row and each column. We denote by $GP(V, \beta)$ the subgroup of all $\widetilde{\varphi} \in GL(V)$ such that $\widetilde{\varphi}$ corresponds to a generalized permutation matrix with respect to the basis β. We also denote by $PGP(V, \beta)$ the image of $GP(V.\beta)$ in $PGL(V)$. Note that every generalized permutation matrix is the product of a diagonal matrix and a permutation matrix.*

2. *A matrix is a generalized triangular matrix if it is of the form P_1TP_2 where P_1 and P_2 are permutation matrices, and T is upper triangular. We denote by $GT(V, \beta)$ the subset of all $\widetilde{\varphi} \in GL(V)$ such that $\widetilde{\varphi}$ corresponds to a generalized triangular matrix with respect to the basis β. We also denote by $PGT(V, \beta)$ the image of $GT(V, \beta)$ in $PGL(V)$.*

The groups $GP(V, \beta)$, $PGP(V, \beta)$, and the sets $GT(V, \beta)$ and $PGT(V, \beta)$ are indeed dependent on the choice of a basis as the notation suggest.
In order to show that the automorphism group of most simple hypersurface consist of generalized permutations, we need a notion of sparsity of the set of monomials appearing in a homogeneous form in a given basis.

Definition 1.4.6. *Let $F \in S^d(V^*)$, and β and β^* be dual bases of V and V^*, respectively.*

1. *We define the distance between two monomials $\prod x_i^{a_i}$ and $\prod x_i^{b_i}$ as the sum $\sum_i |a_i - b_i|$.*

2. *We define the **sparsity** of F with respect to β, denoted by $Spar(F)$, as the minimun of the distance between the monomials of F after identifying $S(V^*) \simeq \mathbb{C}[x_0, ..., x_{n+1}]$ via the basis β.*

3. *We define the variables of F, denoted by $Vars(F)$, as the set of variables appearing in F, i.e., $Vars(F)$ is the smallest subset of β^* such that F is contained in $S^d(W)$ with W the span of $Vars(F)$.*

Note that the sparsity of a homogeneous form is always an even non-negative integer. When the sparsity of a form is greater than 2, we can compute its differential rank directly as we show in the following lemma.

Lemma 1.4.7. *Let $F \in S^d(V^*)$ be homogeneous of degree $d \geq 2$. If $Spar(F) > 2$, then diff.rank(F) equals the cardinality of $Vars(F)$.*

Proof. To prove that $Ker\ D_F = < \beta^* \backslash Vars(F) >$.
Let $J \subseteq \{0, ..., n+1\}$ the set of indices of the variables appearing in F, i.e., $x_j \in Vars(F)$ if

and only if $j \in J$. Let $x \in <\beta^* \backslash Vars(F)>$, then exists $\lambda_i \in \mathbb{C}$, with $i \in I := \{0, ..., n+1\} \backslash J$, such that

$$x = \sum_{i \in I} \lambda_i x_i,$$

then

$$D_F(x) = D_F(\sum_{i \in I} \lambda_i x_i) = \sum_{i \in I} \lambda_i D_F(x_i) = \sum_{i \in I} \lambda_i \frac{\partial F}{\partial x_i} = 0$$

because $x_i \notin Vars(F)$, i.e., x_i is not a variable of F, therefore $\frac{\partial F}{\partial x_i} = 0$, and then $x \in Ker\ D_F$. Therefore $<\beta^* \backslash Vars(F)> \subseteq Ker\ D_F$.
If $x \in <Vars(F)> \backslash \{0\}$, then

$$x = \sum_{j \in J} \lambda_j x_j,$$

where $\lambda_j \in \mathbb{C}$. Since $d \geq 2$, we have that $\frac{\partial F}{\partial x_j} \neq 0$, for all $j \in J$. Suppose by contradiction that $x \in Ker\ D_F$. Since $x \neq 0$, we have that $\lambda_{j^*} \neq 0$, for some $j^* \in J$. Let $\mathbf{x}^\alpha$ be a monomial of $\frac{\partial F}{\partial x_{j^*}}$, i.e.,

$$\mathbf{x}^\alpha := \prod_{j \in J} x_j^{\alpha_j}, \ \sum_{j \in J} \alpha_j = d - 1, \text{ and } \alpha_j \geq 0,$$

then, since $x \in Ker\ D_F$, we have

$$0 = D_F(x) = D_F\left(\sum_{j \in J} \lambda_j x_j\right) = \sum_{j \in J} \lambda_j D_F(x_j) = \sum_{j \in J} \lambda_j \frac{\partial F}{\partial x_j} = \lambda_{j^*} \frac{\partial F}{\partial x_{j^*}} + \sum_{j \in J \backslash \{j^*\}} \lambda_j \frac{\partial F}{\partial x_j},$$

and since the monomials are l.i., there exists some $j' \in J \backslash \{j^*\}$, such that $\mathbf{x}^\alpha$ is also a monomial of $\frac{\partial F}{\partial x_{j'}}$, but, if $\mathbf{x}^\alpha$ is a monomial of $\frac{\partial F}{\partial x_{j^*}}$ and $\frac{\partial F}{\partial x_{j'}}$, then $\mathbf{x}^\alpha x_{j^*}$ and $\mathbf{x}^\alpha x_{j'}$ are monomials of F, but the distance between these two monomials is 2, which is a contradiction since $Spars(F) > 2$, therefore $D_F(x) \neq 0$, and then $x \notin Ker\ D_F$, and then $Ker\ D_F = <\beta^* \backslash Vars(F)>$.
So diff.rank(F) = rank(D_F) = dim V^* - dim $Ker\ D_F = n + 2 - \#I = \#J = \#Vars(F)$.

□

We now define a relation on the dual basis β^* of V^*.

Definition 1.4.8. *Let $F \in S(V^*)$ be homogeneous of degree $d \geq 2$, and β and β^* be dual bases of V and V^*, respectively. We endow the set $\beta^* = \{x_0, ..., x_{n+1}\}$ with the relation $\leq_F$ given by*

$$x_i \leq_F x_j \iff Vars\left(\frac{\partial F}{\partial x_i}\right) \subseteq Vars\left(\frac{\partial F}{\partial x_j}\right).$$

Remark 1.4.9. *The set β^* together with the relation $\leq_F$ may not be a partial order. We* *call a partially ordered set a poset for brevity. Indeed, let* $\beta^* = \{x_0, x_1, x_2, x_3\}$*, and* $d \geq 3$ *and consider first*

$$F = x_0^{d-1}x_1 + x_1^{d-1}x_0 + x_2^d + x_3^d.$$

Then $(\beta^*, \leq_F)$ *is not a poset since* $x_0 \leq_F x_1$ *and* $x_1 \leq_F x_0$*. On the other hand, if we consider instead*

$$F = x_0^{d-1}x_1 + x_1^{d-1}x_2 + x_2^{d-1}x_3 + x_3^{d-1}x_0,$$

we have that $(\beta^*, \leq_F)$ *is a poset since* x_i *and* x_j *are not comparable whenever* $i \neq j$*. We say that such a poset, where* $x_i \leq_F x_j$ *implies* $i = j$*, is trivial.*

We now come to the main theorem of this section wich provides strong restrictions on the shape of the automorphism group of a smooth hypersurface under certain sparsity conditions.

Theorem 1.4.10. *Let* $X = V(F) \subset \mathbb{P}(V)$ *be a smooth hypersurface of dimension* $n \geq 1$ *and degree* $d \geq 3$ *with* $(n, d) \neq (1, 3), (2, 4)$*. Let also* β *and* β^* *be dual bases of* V *and* V^**, respectively. If* $Spars(F) > 4$ *and* $(\beta^*, \leq_F)$ *is a poset then* $Aut(X) \subseteq PGT(V, \beta)$*.*

Proof. Let $\varphi \in Aut(X)$. By Theorem 1.1.2, φ is the restriction of an element of $PGL(V)$ in X, and we know from Proposition 1.1.6 that such element is unique, so we can consider $\varphi \in PGL(V)$, and then, we can pick a representative $\widetilde{\varphi} \in GL(V)$ such that $\widetilde{\varphi}^*(F) = \lambda F$. We represent $\widetilde{\varphi}$ in the basis β by a matrix $(\widetilde{\varphi}_{ji})$ so that $\widetilde{\varphi}^* : V^* \to V^*$ is given in the basis β^* by the matrix $(\widetilde{\varphi_{ij}})$. With this notation, we have

$$\frac{\partial F}{\partial(\widetilde{\varphi}^*(x_i))} = \nabla F \cdot \widetilde{\varphi}^*(x_i) = \nabla F \cdot \sum_{j=0}^{n+1} \widetilde{\varphi}_{ij} x_j = \sum_{j=0}^{n+1} \widetilde{\varphi}_{ij} \nabla F \cdot x_j = \sum_{j=0}^{n+1} \widetilde{\varphi}_{ij} \frac{\partial F}{\partial x_j}.$$

And since $Spar(F) > 4$, we have that $\text{Spar}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^*(x_i))}\right) > 2$. Hence, by the previous Lemma we have

$$\text{diff.rank}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^*(x_i))}\right) = \#\text{Vars}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^*(x_i))}\right) = \# \bigcup_{\{j | \widetilde{\varphi}_{ij} \neq 0\}} \text{Vars}\left(\frac{\partial F}{\partial x_j}\right) \tag{1.4.3}$$

Here, the last equality follow from the fact that $Spar(F) > 4$, and then $Spar\left(\frac{\partial F}{\partial x_k}\right) > 2$, in particular, $Spar\left(\frac{\partial F}{\partial x_j}\right) > 2$, for all j such that $\widetilde{\varphi}_{ij} \neq 0$, and for that $\frac{\partial^2 F}{\partial x_j \partial x_k}$ and $\frac{\partial^2 F}{\partial x_{j'} \partial x_k}$ never share a common monomial if $j \neq j'$, therefore $\frac{\partial^2 F}{\partial \widetilde{\varphi}^*(x_i) \partial x_k} = \sum_{\{j | \widetilde{\varphi}^*_{ij} \neq 0\}} \widetilde{\varphi}_{ij} \frac{\partial^2 F}{\partial x_j \partial x_k}$ is a sum of l.i. monomials, and for that variables are not canceled.
By the previous corollary, we have $\text{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = \text{diff.rank}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^*(x_i))}\right)$.
Let us prove by induction that for all $i = 0, ..., n+1$, we have

$$\widetilde{\varphi}^*(x_i) = \widetilde{\varphi}_{ij} x_j + \widetilde{\varphi}_{ij_1} x_{j_1} + ... + \widetilde{\varphi}_{ij_k} x_{j_k},$$

with $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial x_j}\right)$ and $x_{j_\ell} \leq_F x_j$ for all $\ell \in \{1,...,k\}$ and that the assignment $x_i \mapsto x_j$ gives a permutation of the variables for which $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right)$ is equal to a fixed value.

Let x_i be such that $\frac{\partial F}{\partial x_i}$ has a minimal possible differential rank, hence we claim that $\widetilde{\varphi}^*(x_i) = \widetilde{\varphi}_{ij}x_j$, where x_j is such that $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_j}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right)$. Suppose, by contradiction, that

$$\widetilde{\varphi}^*(x_i) = \widetilde{\varphi}_{ij}x_j + \widetilde{\varphi}_{ij_1}x_{j_1} + ... + \widetilde{\varphi}_{ij_k}x_{j_k},$$

where $x_j \neq x_{j_l}$ for some $l \in \{1,...,k\}$, and both coefficients $\widetilde{\varphi}_{ij}, \widetilde{\varphi}_{ij_l} \neq 0$. Then, we would have:

$$\begin{aligned}
\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) &= \operatorname{diff.rank}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^* x_i)}\right)\\
&= \#\operatorname{Vars}\left(\frac{\partial F}{\partial(\widetilde{\varphi}^* x_i)}\right)\\
&= \# \bigcup_{\{m\mid\widetilde{\varphi}_{im}\neq 0\}} \operatorname{Vars}\left(\frac{\partial F}{\partial x_m}\right)\\
&\geq \#\left(\operatorname{Vars}\left(\frac{\partial F}{\partial x_j}\right)\bigcup\operatorname{Vars}\left(\frac{\partial F}{\partial x_{j_l}}\right)\right),
\end{aligned}$$

but since $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_j}\right) = \#\operatorname{Vars}\left(\frac{\partial F}{\partial x_j}\right)$ and $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_{j_l}}\right) = \#\operatorname{Vars}\left(\frac{\partial F}{\partial x_{j_l}}\right)$, the only way to preserve the minimality of $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right)$ is if

$$\operatorname{Vars}\left(\frac{\partial F}{\partial x_j}\right) = \operatorname{Vars}\left(\frac{\partial F}{\partial x_{j_l}}\right),$$

however, in that case, since $(\beta^*, \leq_F)$ is a poset, it follows that $x_j = x_{j_l}$, contradicting our assumption, therefore $\widetilde{\varphi}^*(x_i) = \widetilde{\varphi}_{ij}x_j$ and $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial x_j}\right)$. Note that since $(\widetilde{\varphi}_{ij})$ is an invertible matrix, the assignment $x_i \mapsto x_j$ gives a permutation of the variables for which the $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right)$ attains the minimum.

Assume the claim holds for all variables with differential rank less than N and that the assignment $x_i \mapsto x_j$ is a permutation of the set of variables x_i such that $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = m < N$.

Let

$$S := \langle\{x_m \in \beta^* \mid \operatorname{diff.rank}\left(\frac{\partial F}{\partial x_m}\right) < N\}\rangle.$$

By the induction hypothesis, we have

$$\widetilde{\varphi}^*(S) \subset S,$$

but since $\widetilde{\varphi}^* \in \mathrm{GL}(V^*)$, it follows that

$$\dim(\widetilde{\varphi}^*(S)) = \dim(S),$$

and therefore $\widetilde{\varphi}^*(S) = S$. If $x_i \in \beta^*$ is such that $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = N$, then $x_i \notin S$, which implies that

$$\widetilde{\varphi}^* x_i = \widetilde{\varphi}_{ij} x_j + \widetilde{\varphi}_{ij_1} x_{j_1} + ... + \widetilde{\varphi}_{ij_k} x_{j_k},$$

with

$$\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_j}\right) = N \text{ and } x_{j_l} \leq_F x_j, \forall l \in \{1, ..., k\}.$$

And again, using the fact that $\widetilde{\varphi}^* \in \mathrm{GL}(V^*)$, we conclude that the assignment $x_i \mapsto x_j$ defines a permutation of the variables x_i for which $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = N$. This prove the claim.

To conclude the proof, let P_1 be the permutation matrix that is the inverse of the assignments $x_i \mapsto x_j$ and let P_2 be any permutation matrix reordering the variables so that the differential rank of x_i is smaller or equal than that of x_j whenever $i < j$. Now $P_1 \widetilde{\varphi} P_2$ is an upper triangular matrix, proving that $\widetilde{\varphi} \in \mathrm{GT}(V, \beta)$ and so $\varphi \in \mathrm{PGT}(V, \beta)$.

□

1.5 Automorphisms of classical hypersurfaces

Corollary 1.5.1. *Let $X = V(F) \subset \mathbb{P}(V)$ be a smooth hypersurface of dimension $n \geq 1$ and degree $d \geq 3$ with $(n, d) \neq (1, 3), (2, 4)$. Let also β and β^* be dual bases of V and V^*, respectively. If $\mathrm{Spar}(F) > 4$ and $(\beta^*, \leq_F)$ is the trivial poset then $\mathrm{Aut}(X) \subseteq \mathrm{PGP}(V, \beta)$.*

Proof. Since $(\beta^*, \leq_F)$ is the trivial poset, in particular is a poset, and then all the hypotheses of the previous theorem are satisfied, therefore $\mathrm{Aut}(X) \subseteq \mathrm{PGT}(V, \beta)$.

Let be $\varphi \in \mathrm{Aut}(X)$ and $x_i \in \beta^*$, if we suppose by contradiction that $\varphi \notin \mathrm{PGP}(V, \beta)$ then exist $x_j, x_{j_1}, ..., x_{j_k} \in \beta^*$ where $x_j \neq x_{j_l}$ for all $l \in \{1, ..., k\}$, and

$$\widetilde{\varphi}(x_i) = \widetilde{\varphi}_{ij} x_j + \widetilde{\varphi}_{ij_1} x_{j_1} + ... + \widetilde{\varphi}_{ij_k} x_{j_k},$$

with $\operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial x_j}\right)$ and $x_{j_l} \leq_F x_i$, for all $l \in \{1, ..., k\}$, but how $(\beta^*, \leq_F)$ is the trivial poset, then $x_{j_l} = x_i$, for all $l \in \{1, ..., k\}$, and then

$$\widetilde{\varphi}(x_i) = \widetilde{\varphi}_{ij} x_j + \widetilde{\varphi}_{ii} x_i,$$

but

$$\#\operatorname{Vars}\left(\frac{\partial F}{\partial x_i}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial x_i}\right) = \operatorname{diff.rank}\left(\frac{\partial F}{\partial \widetilde{\varphi}^* x_i}\right) = \#(\operatorname{Vars}\left(\frac{\partial F}{\partial x_j}\right) \cup \operatorname{Vars}\left(\frac{\partial F}{\partial x_i}\right)),$$

and then $\operatorname{Vars}\left(\frac{\partial F}{\partial x_j}\right) \subset \operatorname{Vars}\left(\frac{\partial F}{\partial x_i}\right)$, and then $x_j \leq_F x_i$, and since $(\beta^*, \leq_F)$ is a trivial poset, $x_j = x_i$ which is a contradiction since $x_j \neq x_{j_l} = x_i$, therefore $\varphi \in \operatorname{PGP}(V, \beta)$ □

As an application of the previous result we will show that for simple hypersurfaces, all automorphisms are generalized permutations. Our definition of simple hypersurfaces is inspired on the main result of [Zhe22], wich claims that every abelian group acting on any smooth hypersurface can be realized as a subgroup of the automorphism groups of these simple hypersurfaces.

Definition 1.5.2. *We say that a smooth hypersurface $X = V(F)$ given by a homogeneous form $F \in S^d(V^*)$ of degree d is simple if, after possibly renaming the elements of the basis β^* of V^*, the form F is given by*

$$F = K_{i_1} + K_{i_2} + ... + K_{i_l} + T_{j_1} + ... + T_{j_m},$$

where $K_i = x_1^{d-1}x_2 + x_2^{d-1}x_3 + ... + x_i^{d-1}x_1$ is the Klein form of degree d in i variables, $T_j = y_1^{d-1}y_2 + ... + y_{j-1}^{d-1}y_j + y_j^d$ is the Delsarte form of degree d in j variables, and all such Klein and Delsarte forms have linearly independent variables. In consequence $i_1 + ... + i_l + j_1 + ... + j_m = dim(V) = n + 2$.

Corollary 1.5.3. *Let $X = V(F) \subset \mathbb{P}(V)$ be a simple hypersurface of dimension $n \geq 1$ and degree $d \geq 3$ with $(n, d) \neq (1, 3), (2, 4)$. Let also β and β^* be dual bases of V and V^*, respectively. If Spar(F) > 4 then* $\operatorname{Aut}(X) \subseteq \operatorname{PGP}(V, \beta)$.

Proof. If x_i is a variable appearing in the Klein

$$K = x_{i_1}^{d-1}x_{i_2} + x_{i_2}^{d-1}x_{i_3} + ... + x_{i_m}^{d-1}x_{i_1},$$

then we have that

$$\frac{\partial F}{\partial x_i} = \frac{\partial K_j}{\partial x_i} = x_i^{d-1} + (d-1)x_i^{d-2}x_{i+1}$$

if $i \neq i_1, i_m$. In case it is the first variable of this Klein, then

$$\frac{\partial F}{\partial x_{i_1}} = (d-1)x_{i_1}^{d-2}x_{i_2} + x_{i_m}^{d-1},$$

and if $i = i_m$, then

$$\frac{\partial F}{\partial x_{i_m}} = x_{i_{m-1}}^{d-1} + (d-1)x_{i_m}^{d-2}x_{i_1}.$$

Therefore, we have that if x_i, x_j are two distinct variables appearing in the Klein, then

$$\mathrm{Vars}\left(\frac{\partial F}{\partial x_i}\right) \subsetneq \mathrm{Vars}\left(\frac{\partial F}{\partial x_j}\right).$$

In the same way, we can see that in the case of Delsarte forms

$$D = y_{j_1}^{d-1} y_{j_2} + y_{j_2}^{d-1} y_{j_3} + ... + y_{j_{l-1}}^{d-1} y_{j_l} + y_{j_l}^{d},$$

the only nontrivial relations are $y_{j_1} \leq_F y_{j_2}$ and $y_{j_l} \leq_F y_{j_{l-1}}$, Therefore, these two are the only nontrivial relations in all of F. In particular $(\beta^*, \leq_F)$ is a poset.
Therefore, all hypothesis of the Theorem 1.4.10 are verified, then we have $\mathrm{Aut}(X) \subseteq \mathrm{PGT}(V, \beta)$. Let $\varphi \in \mathrm{Aut}(X)$, then we have $\varphi \in \mathrm{PGT}(V, \beta)$. Suppose by contradiction that $\varphi \notin \mathrm{PGP}(V, \beta)$. then, there exists a variable z_i such that

$$\widetilde{\varphi}^*(z_i) = \widetilde{\varphi}_{ij} u_j + \widetilde{\varphi}_{ij_1} u_{j_1} + ... + \widetilde{\varphi}_{ij_l} u_{j_l},$$

where $\mathrm{diff.rank}\left(\frac{\partial F}{\partial z_i}\right) = \mathrm{diff.rank}\left(\frac{\partial F}{\partial u_j}\right)$, and $u_{j_k} \leq_F u_j$ for all $k \in \{j_1, ..., j_l\}$,and the u_{j_k} are variables distinct from u_i.
Since the only nontrivial relations are $y_{j_1} \leq_F y_{j_2}$ and $y_{j_l} \leq_F y_{j_{l-1}}$, we have only two cases

$$\widetilde{\varphi}^*(z_i) = \widetilde{\varphi}_{ij_2} y_{j_2} + \widetilde{\varphi}_{ij_1} y_{j_1},$$

or

$$\widetilde{\varphi}^*(z_i) = \widetilde{\varphi}_{ij_l} y_{j_l} + \widetilde{\varphi}_{ij_{l-1}} y_{j_{l-1}}.$$

Without loss of generality, we can suposse that

$$\widetilde{\varphi}^*(z_i) = \widetilde{\varphi}_{ij_2} y_{j_2} + \widetilde{\varphi}_{ij_1} y_{j_1}.$$

We have that $y_{j_1} \leq_F y_{j_2}$, therefore $\mathrm{diff.rank}\left(\frac{\partial F}{\partial z_i}\right) = \mathrm{diff.rank}\left(\frac{\partial F}{\partial y_{j_2}}\right) = 3$, and then we have that z_i appears in F in two terms $z_{i-1}^{d-1} z_i + z_i^{d-1} z_{i+1}$.
Let u_i be a variable different from z_i. If

$$\mathrm{diff.rank}\left(\frac{\partial F}{\partial u_i}\right) = 2,$$

then $\widetilde{\varphi}^*(u_i) = c y_{j_1}$ or $\widetilde{\varphi}^*(u_i) = c y_{j_l}$. On the other hand, if

$$\mathrm{diff.rank}\left(\frac{\partial F}{\partial u_i}\right) = 3,$$

then $\widetilde{\varphi}^*(u_i)$ cannot contain the variable y_{j_2} as one of its terms. This is because, from the

previous theorem, we know that $\widetilde{\varphi}^*$ permutes variables with the same differential rank (up to adding multiples of other variables), and we already have $z_i \mapsto y_{j_2}$.
Therefore, the only way y_{j_2} could appear as one of the terms of $\widetilde{\varphi}^*(u_i)$ is if there exists another variable h_i such that

$$\text{diff. rank}\left(\frac{\partial F}{\partial h_i}\right) = 3$$

and $y_{j_2} \leq_F h_i$. However, this does not happen if $h_i \neq y_{j_2}$.
For all these reasons, the only monomials in $\widetilde{\varphi}^* F$ in which the variable y_{j_2} appears are those coming from

$$\widetilde{\varphi}^*(z_{i-1})^{d-1}\widetilde{\varphi}^*(z_i) + \widetilde{\varphi}^*(z_i)^{d-1}\widetilde{\varphi}^*(z_{i+1}),$$

these include the monomials

$$\widetilde{\varphi}^*(z_{i-1})^{d-1}y_{j_2}, y_{j_1}^{d-2}y_{j_2}\widetilde{\varphi}^*(z_{i+1}) + ... + y_{j_2}^{d-1}\widetilde{\varphi}^*(z_{i+1}).$$

On the other hand, the form F contains the monomial $y_{j_1}^{d-1}y_{j_2}$, which leaves us with only two possibilities, either $\widetilde{\varphi}^*(z_{i-1}) = \alpha y_{j_1}$ or $\widetilde{\varphi}^*(z_{i+1}) = \alpha y_{j_1}$.
If $\widetilde{\varphi}^*(z_{i-1}) = \alpha y_{j_1}$, then $\widetilde{\varphi}^*(z_{i+1})$ is composed of variables different from y_{j_1} and y_{j_2}. Let u_i be one of the variables appearing in $\widetilde{\varphi}^*(z_{i+1})$. Then one of the monomials in $y_{j_1}^{d-2}y_{j_2}\widetilde{\varphi}^*(z_{i+1})$ would be

$$y_{j_1}^{d-1}y_{j_2}u_i,$$

which does not appear elsewhere in $\widetilde{\varphi}^* F$, and thus will not cancel out. However, this is not a monomial of F, leading to a contradiction.
Similarly, if $\widetilde{\varphi}^*(z_{i+1}) = \alpha y_{j_1}$, then $\widetilde{\varphi}^*(z_{i-1})^{d-1}y_{j_2}$ contains a monomial that does not appear anywhere else in $\widetilde{\varphi}^* F$, and therefore does not cancel. However, this monomial is not part of F, which leads to a contradiction.
Then we have, $\widetilde{\varphi}^*(z_i) = \widetilde{\varphi}_{ij_2}y_{j_2}$ and therefore

$$\text{Aut}(X) \subseteq \text{PGP}(\text{V}, \beta).$$

□

Definition 1.5.4. ***Fermat hypersurfaces*** *The Fermat hypersuface $X = V(F)$ of dimension n and degree d is given in the basis β^* by the form*

$$F = x_0^d + x_1^d + x_2^d + x_3^d + ... + x_n^d + x_{n+1}^d \in S^d(V^*).$$

The following theorem is well known [Shi88,Kon02]. Nevertheless, we prove it as a straightforward application of the previous corollary. We remark that the Fermat hypersurface is simple since the form F is given as the sum of $n+1$ Delsarte forms of degree d in 1 variable.

Proposition 1.5.5. *The automorphism group of the Fermat hypersurface $X = V(F)$ of dimension $n \geq 1$ and degree $d \geq 3$, with $(n, d) \neq (1, 3), (2, 4)$ is ismorphic to*

$$\mathrm{Aut}(X) = (\mathbb{Z}/d\mathbb{Z})^{n+1} \rtimes S_{n+2}$$

Proof. If $\widetilde{\sigma} \in \mathrm{GL}(V)$ correspond in the basis β to a matrix of permutation, then exists a unique $s_{\widetilde{\sigma}} \in S_{n+2}$ such that $\widetilde{\sigma}(e_i) = e_{s_{\widetilde{\sigma}}(i)}$, for all $i = 0, ..., n+1$, and how $s_{\widetilde{\sigma}} \in S_{n+2}$, then also $s_{\widetilde{\sigma}}^{-1} \in S_{n+2}$. And then, we have that $x_i \circ \widetilde{\sigma} = x_{s_{\widetilde{\sigma}}^{-1}(i)}$, and for that $x_i^d \circ \widetilde{\sigma} = x_{s_{\widetilde{\sigma}}^{-1}(i)}^d$, therefore, how $s_{\widetilde{\sigma}}^{-1}$ is a permutation of the set $\{0, ..., n+1\}$ and the sum is associative and commutative, we have

$$\widetilde{\sigma}^* F = F \circ \widetilde{\sigma} = (x_0^d + x_1^d + ... + x_n^d + x_{n+1}^d) \circ \widetilde{\sigma} = x_0^d \circ \widetilde{\sigma} + x_1^d \circ \widetilde{\sigma} + ... + x_n^d \circ \widetilde{\sigma} + x_{n+1}^d \circ \widetilde{\sigma}$$

$$= x_{s_{\widetilde{\sigma}}^{-1}(0)}^d + x_{s_{\widetilde{\sigma}}^{-1}(1)}^d + ... + x_{s_{\widetilde{\sigma}}^{-1}(n)}^d + x_{s_{\widetilde{\sigma}}^{-1}(n+1)}^d = x_0^d + x_1^d + ... + x_n^d + x_{n+1}^d = F,$$

and then $\sigma \in \mathrm{Aut}(X)$. Therefore the image in $\mathrm{PGL}(V)$ of every permutation matrix (with respect the basis β) is an automorphism of X.

Also, taking any two monomials of F, we have that their distance is $2d$, and for that $Spar(F) = 2d \geq 2 \cdot 3 = 6 > 4$, and then by the previous corollary we have that $\mathrm{Aut}(X) \subseteq \mathrm{PGP}(V, \beta)$.
We already know that the image in $\mathrm{PGL}(V)$ of any permutation matrix is an automorphism of X, so we only need to determine the automorphisms of X that are given by images of diagonal matrices in $\mathrm{PGL}(V)$.

Let $D < \mathrm{Aut}(X)$ be the subgroup of automorphism of X such that they are the image in $\mathrm{PGL}(V)$ of diagonal matrices. We want to prove that $D \cong (\mathbb{Z}/d\mathbb{Z})^{n+1}$.

Let ξ be a primitive d-th root of unity. Let us define the morphism $\gamma : (\mathbb{Z}/d\mathbb{Z})^{n+1} \to D$ given by

$$\gamma(\overline{k}) = \pi(\widetilde{\gamma}(\overline{k})), \textit{ where } \widetilde{\gamma}(\overline{k})(e_0) = e_0 \textit{ and } \widetilde{\gamma}(\overline{k})(e_i) = \xi^{k_i} e_i, \quad \forall i = 1, ..., n+1,$$

$$\overline{k} = (k_1, ..., k_{n+1}) \in (\mathbb{Z}/d\mathbb{Z})^{n+1}.$$

It is not hard to see that γ is well defined, i.e., if $\overline{k} \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$, we have that $\gamma(\overline{k})$ is indeed the image in $PGL(V, \beta)$ of a diagonal matrix, this is by the definition of $\widetilde{\gamma}$, we have $\widetilde{\gamma}(\overline{k})(e_i)$ is a multiple of e_i(and also ξ^{k_i} no depends of the representative in $\mathbb{Z}/d\mathbb{Z}$ of k_i, since ξ is a primitive d-th root of unity), and $\gamma(\overline{k}) \in Aut(X)$ since $\widetilde{\gamma}(\overline{k})^* F = F$(since $\xi^d = 1$), and $\widetilde{\gamma}(\overline{k})$

by definition is a representative of $\gamma(\overline{k})$, so $\gamma(\overline{k}) \in D$. And γ is a morphism since

$$\widetilde{\gamma}((k_1,...,k_n)+(l_1,...,l_n))(e_i) = \widetilde{\gamma}((k_1+l_1,...,k_n+l_n))(e_i) = \xi^{k_i+l_i}e_i = \xi^{k_i}(\xi^{l_i}e_i)$$

$$= \widetilde{\gamma}((k_1,...,k_n))(\widetilde{\gamma}((l_1,...,l_n))(e_i)) = [\widetilde{\gamma}((k_1,...,k_n)) \circ \widetilde{\gamma}((l_1,...,l_n))](e_i), \quad \forall i = 1,...,n+1,$$

$$\textit{and } \widetilde{\gamma}((k_1,...,k_n)+(l_1,...,l_n))(e_0) = [\widetilde{\gamma}((k_1,...,k_n)) \circ \widetilde{\gamma}((l_1,...,l_n))](e_0), \ \textit{since}$$

$$\widetilde{\gamma}(\overline{u})(e_0) = e_0, \forall \overline{u} \in (\mathbb{Z}/d\mathbb{Z})^{n+1},$$

therefore $\widetilde{\gamma} : (\mathbb{Z}/d\mathbb{Z})^{n+1} \to \mathrm{GL}(V)$ is a morphism of groups, and since $\pi(\widetilde{\varphi_1} \circ \widetilde{\varphi_2}) = \pi(\widetilde{\varphi_1}) \circ \pi(\widetilde{\varphi_2})$, we have that γ is morphism of groups.
Injectivity: Let be $\overline{k} \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$ such that $\gamma(\overline{k}) = Id \in \mathrm{Aut}(X)$, then $\pi(\widetilde{\gamma}(\overline{k})) = Id \in \mathrm{PGL}(V)$, and then, exists $\lambda \in \mathbb{C}^*$ such that $\widetilde{\gamma}(\overline{k}) = \lambda Id \in \mathrm{GL}(V)$, but since $\widetilde{\gamma}(\overline{k})(e_0) = e_0$, we have $\lambda = 1$, and then $\widetilde{\gamma}(\overline{k}) = Id$, and for that, $\widetilde{\gamma}(\overline{k})(e_i) = e_i$, for all $i = 0,...,n+1$, therefore $\xi^{k_i} = 1$, for all $i = 1,...,n+1$, and since ξ is a primitive d-th root of unity and $k_i \in \mathbb{Z}/d\mathbb{Z}$, we have that $k_i = 0$, for all $i = 1,...,n+1$, and then $\overline{k} = \overline{0} = (0,...,0) \in (\mathbb{Z}/d\mathbb{Z})^{n+1}$.
Surjectivity: Let be $\varphi \in \mathrm{Aut}(X)$, then $\widetilde{\varphi}^*(F) = \lambda F$ for some $\lambda \in \mathbb{C}^*$ without lost of generality we can considerer $\lambda = 1$ (redefining if necessary $\widetilde{\varphi}$ by choosing another representative of φ), and then

$$\widetilde{\varphi}_{00}^d x_0^d + \widetilde{\varphi}_{11}^d x_1^d + ... + \widetilde{\varphi}_{nn}^d x_n^d + \widetilde{\varphi}_{n+1,n+1}^d x_{n+1}^d = x_0^d + x_1^d + ... + x_{nn}^d + x_{n+1,n+1}^d$$

$$\Rightarrow \widetilde{\varphi}_{ii}^d = 1, \forall i = 0,...,n+1.$$

And let be $\widetilde{\psi} = \frac{1}{\widetilde{\varphi}_{00}}\widetilde{\varphi}$, then we have $\widetilde{\psi}_{00} = 1$, and $\widetilde{\psi}_{ii}^d = \left(\frac{\widetilde{\varphi}_{ii}}{\widetilde{\varphi}_{00}}\right)^d = \frac{\widetilde{\varphi}_{ii}^d}{\widetilde{\varphi}_{00}^d} = 1$, for all $i = 1,...,n+1$, i.e, $\widetilde{\psi}_{ii}$ is a d-th root of unity, so exists $k_i^* \in \mathbb{Z}/d\mathbb{Z}$ such that $\widetilde{\psi}_{ii} = \xi^{k_i^*}$, and then, let defined $\overline{k^*} = (k_1^*,...,k_{n+1}^*)$, so

$$\varphi = \pi(\widetilde{\varphi}) = \pi(\widetilde{\psi}) = \pi(\widetilde{\gamma}((k_1^*,...,k_{n+1}^*))) = \pi(\widetilde{\gamma}(\overline{k^*})) = \pi(\gamma(\overline{k^*})).$$

Therefore $D \simeq (\mathbb{Z}/d\mathbb{Z})^{n+1}$, and then

$$\mathrm{Aut}(X) = (\mathbb{Z}/d\mathbb{Z})^{n+1} \rtimes S_{n+2}.$$

□

Definition 1.5.6. ***Delsarte hypersurfaces*** *The Delsarte hypersurface $X = V(T)$ of dimension n and degree d is given in the basis β^* by the form*

$$T = x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^d,$$

Proposition 1.5.7. *The automorphism group of the Delsarte hypersurfaces X of dimension $n \geq 2$ and degree $d \geq 4$, with $(n, d) \neq (2, 4)$ is isomorphic to*

$$\mathrm{Aut}(X) = \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}$$

Proof. Since $\mathrm{Spar}(T) = 2d - 2 \geq 2 \cdot 4 - 2 = 8 - 2 = 6 > 4$, by the previous corollary, we have $\mathrm{Aut}(X) \subseteq \mathrm{PGP}(V, \beta)$.
Let be $\sigma \in \mathrm{Aut}(X)$ with representative $\widetilde{\sigma} \in \mathrm{GL}(V)$ a matrix of permutation with respect to the basis β, then exists a unique $s_{\widetilde{\sigma}} \in S_{n+2}$ such that $\widetilde{\sigma}(e_i) = e_{s_{\widetilde{\sigma}}(i)}$, for all $i = 0, ..., n+1$, and how $s_{\widetilde{\sigma}} \in S_{n+2}$, then also $s_{\widetilde{\sigma}}^{-1} \in S_{n+2}$, and $\widetilde{\sigma}^*(x_i) = x_i \circ \widetilde{\sigma} = x_{s_{\widetilde{\sigma}}^{-1}(i)}$, and since $s_{\widetilde{\sigma}}^{-1}$ is a permutation, we have that $x_{s_{\widetilde{\sigma}}^{-1}(i)} \neq x_{s_{\widetilde{\sigma}}^{-1}(j)}$ if $i \neq j$, so

$$Vars(\widetilde{\sigma}^*(x_i^{d-1}x_{i+1})) = Vars(x^d_{s_{\widetilde{\sigma}}^{-1}(i)}x_{s_{\widetilde{\sigma}}^{-1}(i+1)}) = 2,$$

therefore $\widetilde{\sigma}^*T$ corresponds to the sum of the monomial $x^d_{s_{\widetilde{\sigma}}^{-1}(n+1)}$ plus $n+1$ monomials where each of these $n+1$ monomials is formed by two variables, so there is no way for monomial $x^d_{s_{\widetilde{\sigma}}^{-1}(n+1)}$ to cancel out, and for that, the only way for $\widetilde{\sigma}^*T$ to be a multiple of T is that $x^d_{s_{\widetilde{\sigma}}^{-1}(n+1)} = x^d_{n+1}$, and then $s_{\widetilde{\sigma}}^{-1}(n+1) = n+1$, then $s_{\widetilde{\sigma}}(n+1) = n+1$, and then $\widetilde{\sigma}(e_{n+1}) = e_{n+1}$.
Then, using similar reasoning, analyzing the variables of each of the monomials $x^d_{s_{\widetilde{\sigma}}^{-1}(i)}x_{s_{\widetilde{\sigma}}^{-1}(i+1)}$, we have that there is no way for monomial $x^d_{s_{\widetilde{\sigma}}^{-1}(n)}x_{s_{\widetilde{\sigma}}^{-1}(n+1)} = x^d_{s_{\widetilde{\sigma}}^{-1}(n)}x_{n+1}$ to be cancelled, and in addition, it is the only monomial that is the product of a variable raised to d multiplied by the variable x_{n+1}, so for $\widetilde{\sigma}^*T$ to be a multiple of T, necessarily $x^d_{s_{\widetilde{\sigma}}^{-1}(n)}x_{n+1} = x_n^d x_{n+1}^d$, and then $s_{\widetilde{\sigma}}^{-1}(n) = n$, so $s_{\widetilde{\sigma}}(n) = n$, therefore $\widetilde{\sigma}(e_n) = e_n$. Then, repeating this procedure n more times, we will obtain

$$\widetilde{\sigma}(e_{n-1}) = e_{n-1}, \widetilde{\sigma}(e_{n-2}) = e_{n-2}, ..., \widetilde{\sigma}(e_1) = e_1, \widetilde{\sigma}(e_0) = e_0,$$

therefore $\widetilde{\sigma} = Id \in \mathrm{GL}(V)$.
Then, we have $\mathrm{Aut}(X) = D$, where $D < PGL(V)$ is formed by the images of diagonal matrices in $\mathrm{PGL}(V)$. we will prove $D \simeq \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}$.
Let be $\gamma : \mathbb{Z}/(d-1)^{n+1}\mathbb{Z} \to D$ a morphism of groups given by

$$\gamma(k) = \pi(\widetilde{\gamma}(k)), \ \textit{where } \widetilde{\gamma}(k) \in GL(V) \textit{ is such that}$$

$$\widetilde{\gamma}(k)(e_i) = \xi^{k(1-d)^i}e_i, \ \textit{for all } i = 0, ..., n+1.$$

It is not difficult to see that γ is well defined and that it is indeed a group morphism. To prove γ is injective.
If $\gamma(k) = Id \in \mathrm{PGL}(V)$, then $\pi(\widetilde{\gamma}(k)) = Id$, and then $\widetilde{\gamma}(k) = \lambda Id \in \mathrm{GL}(V)$ for some $\lambda \in \mathbb{C}^*$,

but since ξ is a primitive $(d-1)^{n+1}$-th root of unity, we have

$$\lambda e_{n+1} = \widetilde{\gamma}(k)(e_{n+1}) = \xi^{k(1-d)^{n+1}} e_{n+1} = (\xi^{(1-d)^{n+1}})^k e_{n+1} = 1^k e_{n+1} = e_{n+1},$$

so $\lambda = 1$, and then $\widetilde{\gamma}(k) = Id$, therefore $\widetilde{\gamma}(k)(e_i) = e_i$, for all $i = 0, ..., n+1$, in particular $\xi^k e_0 = \xi^{k(d-1)^0} e_0 = \widetilde{\gamma}(k)(e_0) = e_0$, so $\xi^k = 1$, and then $k = 0 \in \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}$. To prove γ is surjective.
Let $\varphi \in Aut(X)$, so $\widetilde{\varphi}^* T = \lambda T$, for some $\lambda \in \mathbb{C}^*$, without loss of generality, we can consider $\lambda = 1$, and then $\widetilde{\varphi}^* T = T$, so we have

$$\widetilde{\varphi}_{00}^{d-1}\widetilde{\varphi}_{11}x_0^{d-1}x_1 + \widetilde{\varphi}_{11}^{d-1}\widetilde{\varphi}_{22}x_1^{d-1}x_2 + ... + \widetilde{\varphi}_{nn}^{d-1}\widetilde{\varphi}_{n+1,n+1}x_n^{d-1}x_{n+1} + \widetilde{\varphi}_{n+1,n+1}^{d}x_{n+1}^{d}$$

$$= x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^d,$$

so we have $\widetilde{\varphi}_{n+1,n+1}^d = 1$, and then, is we defined $\widetilde{\psi} = \frac{1}{\widetilde{\varphi}_{n+1,n+1}}\widetilde{\varphi}$, since T is a form of degree d

$$\widetilde{\psi}^* T = T \circ \widetilde{\psi} = T \circ \left(\frac{1}{\widetilde{\varphi}_{n+1,n+1}}\widetilde{\varphi}\right) = \left(\frac{1}{\widetilde{\varphi}_{n+1,n+1}}\right)^d T \circ \widetilde{\varphi} = T \circ \widetilde{\varphi} = \widetilde{\varphi}^* T = T,$$

therefore we have

$$\pi(\widetilde{\psi}) = \varphi, \widetilde{\psi}_{ii}^{d-1}\widetilde{\psi}_{i+1,i+1} = 1, \ \textit{for all } i = 0, ..., n \textit{ and } \widetilde{\psi}_{n+1,n+1} = 1,$$

then we have, $\widetilde{\psi}_{00}^{d-1} = \frac{1}{\widetilde{\psi}_{11}}$ or equivalent $\widetilde{\psi}_{00}^{1-d} = \widetilde{\psi}_{11}$, and also, we have $\widetilde{\psi}_{11}^{1-d} = \widetilde{\psi}_{22}$, $\widetilde{\psi}_{22}^{1-d} = \widetilde{\psi}_{33}$,...,$\widetilde{\psi}_{nn}^{1-d} = \widetilde{\psi}_{n+1,n+1}$, and for this reason

$$\widetilde{\psi}_{00}^{(1-d)^{n+1}} = (\widetilde{\psi}_{00}^{1-d})^{(1-d)^n} = \widetilde{\psi}_{11}^{(1-d)^n} = (\widetilde{\psi}_{11}^{1-d})^{(1-d)^{n-1}} = \widetilde{\psi}_{22}^{(1-d)^{n-1}} = \widetilde{\psi}_{33}^{(1-d)^{n-2}} = ...$$

$$= \widetilde{\psi}_{nn}^{1-d} = \widetilde{\psi}_{n+1,n+1} = 1 \Rightarrow \widetilde{\psi}_{00}^{(d-1)^{n+1}} = 1,$$

therefore exists $k^* \in \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}$ such that, $\widetilde{\psi}_{00} = \xi^{k^*}$,and also, $\widetilde{\psi}_{ii} = \xi^{k^*(1-d)^i}$, since $\widetilde{\psi}_{00}^{(1-d)^i} = \widetilde{\psi}_{ii}$, so we have $\widetilde{\gamma}(k^*)(e_i) = \xi^{k^*(1-d)^i} e_i = \widetilde{\psi}_{ii} e_i$ for all $i = 0, ..., n+1$, therefore $\widetilde{\gamma}(k^*) = \widetilde{\psi}$, and then $\gamma(k^*) = \pi(\widetilde{\gamma}(k^*)) = \pi(\widetilde{\psi}) = \pi(\widetilde{\varphi}) = \varphi$.
Therefore $D \simeq \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}$, and then

$$\mathrm{Aut}(X) = \mathbb{Z}/(d-1)^{n+1}\mathbb{Z}.$$

□

Definition 1.5.8. ***Klein hypersurfaces.*** *The Klein hypersurface $X = V(K)$ of dimension n and degree d is given in the basis β^* by the form*

$$K = x_0^{d-1}x_1 + x_1^{d-1}x_2 + x_2^{d-1}x_3 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0,$$

Proposition 1.5.9. *The isomorphism group of the Klein hypersurface X of dimension $n \geq 2$ and degree $d \geq 4$, with $(n,d) \neq (2,4)$ is isomorphic to*

$$\mathrm{Aut}(X) = (\mathbb{Z}/m\mathbb{Z}) \rtimes \mathbb{Z}/(n+2)\mathbb{Z}$$

where $m = \frac{(d-1)^{n+2}-(-1)^{n+2}}{d}$.

Proof. Since $\mathrm{Spar}(K) = 2d-2 \geq 2\cdot 4 - 2 = 6 > 4$, by the previous corollary, we have $\mathrm{Aut}(X) \subseteq \mathrm{PGP}(V,\beta)$. Let be $P < \mathrm{Aut}(X)$ the subgroup of autmorphisms of X such that they are the image in $\mathrm{PGL}(V)$ of permutation matrices. Let us prove that $P \simeq \mathbb{Z}/(n+2)\mathbb{Z}$. Let be $\gamma : \mathbb{Z}/(n+2)\mathbb{Z} \to P$ the morphism of groups given by $\gamma(k) = \pi(\widetilde{\gamma}(k))$, where $\widetilde{\gamma}(k)(e_i) = e_{i+k}$, for all $i = 0,...,n+1$ (here $i+k$ denotes the remainder obtained by dividing it by $n+2$, for example, if $k=1$, then $\widetilde{\gamma}(1)(e_{n+1}) = e_0$).
γ is well defined, indeed if $k \in \mathbb{Z}/(n+2)\mathbb{Z}$, then

$$\begin{aligned}
\widetilde{\gamma}(k)^*K &= K \circ \widetilde{\gamma}(k) \\
&= x_0^{d-1}x_1 \circ \widetilde{\gamma}(k) + x_1^{d-1}x_2 \circ \widetilde{\gamma}(k) + \cdots + x_n^{d-1}x_{n+1} \circ \widetilde{\gamma}(k) + x_{n+1}^{d-1}x_0 \circ \widetilde{\gamma}(k) \\
&= x_{0-k}^{d-1}x_{1-k} + x_{1-k}^{d-1}x_{2-k} + \cdots + x_{n-k}^{d-1}x_{n+1-k} + x_{n+1-k}^{d-1}x_{0-k} \\
&= K,
\end{aligned}$$

therefore $\gamma(k) = \pi(\widetilde{\gamma}(k)) \in Aut(X)$, and is a morphism, since

$$\begin{aligned}
\widetilde{\gamma}(k_1+k_2)(e_i) &= \widetilde{\gamma}(k_2+k_1)(e_i) \\
&= e_{i+(k_2+k_1)} \\
&= e_{(i+k_2)+k_1} \\
&= \widetilde{\gamma}(k_1)(e_{i+k_2}) \\
&= \widetilde{\gamma}(k_1)(\widetilde{\gamma}(k_2)(e_i)) \\
&= [\widetilde{\gamma}(k_1) \circ \widetilde{\gamma}(k_2)](e_i), \quad \forall i = 0,\ldots,n+1 \\
\Rightarrow \quad \widetilde{\gamma}(k_1+k_2) &= \widetilde{\gamma}(k_1) \circ \widetilde{\gamma}(k_2) \\
\Rightarrow \quad \gamma(k_1+k_2) &= \gamma(k_1) \circ \gamma(k_2).
\end{aligned}$$

To prove γ is injective. If $\gamma(k) = Id \in Aut(X)$, then $\pi(\widetilde{\gamma}(k)) = Id \in PGL(V)$, then $\widetilde{\gamma}(k) = \lambda Id \in GL(V)$, for some $\lambda \in \mathbb{C}$, but since $\widetilde{\gamma}(k)$ is a permutation matrix, we have $\lambda = 1$, and then $\widetilde{\gamma}(k) = Id$, then $\widetilde{\gamma}(k)(e_0) = e_0$, and then $e_{0+k} = e_0$, therefore $k \equiv 0(modn+2)$, and then $k = 0 \in \mathbb{Z}/(n+2)\mathbb{Z}$. To prove γ is surjective. Let $\sigma \in P$ and $\widetilde{\sigma} \in GL(V)$ the permutation matrix representative of σ, then $\widetilde{\sigma}^*K = K$, and then, is not hard to see that

$\widetilde{\sigma}$ must satisfy that if $\widetilde{\sigma}(e_i) = e_j$, then $\widetilde{\sigma}(e_{i+1}) = e_{j+1}$. Let be $k^* = j - i$, then we have $\widetilde{\gamma}(k^*) = \widetilde{\sigma}$, and then $\gamma(k^*) = \sigma$. Therefore $P \simeq \mathbb{Z}/((n+2)\mathbb{Z}$.
Let be $D < \mathrm{Aut}(X)$ the subgroup of atumorphisms of X such that they are the image in $PGL(V)$ of diagonal matrices. To prove $D \simeq \mathbb{Z}/m\mathbb{Z}$, where $m = \frac{(d-1)^{n+2}-(-1)^{n+2}}{d}$.
Let be $\alpha : \mathbb{Z}/dm\mathbb{Z} \to D$, where $\alpha = \pi(\widetilde{\alpha}(k))$, and $\widetilde{\alpha}(k)(e_i) = \xi^{k(1-d)^i} e_i$, for all $i = 0, ..., n+1$. Is not hard to see that α is well defined and is a morphisms of groups. To prove α is surjective. Let be $\varphi \in D$ and $\widetilde{\varphi} \in \mathrm{GL}(V)$ a representative of φ such that $\widetilde{\varphi}^* K = K$, then we have $\widetilde{\varphi}_{ii}^{d-1}\widetilde{\varphi}_{i+1,i+1} = 1$, for all $i = 0, ..., n$, and $\widetilde{\varphi}_{n+1,n+1}^{d-1}\widetilde{\varphi}_{00} = 1$, so we have

$$\widetilde{\varphi}_{00} = \widetilde{\varphi}_{n+1,n+1}^{1-d} = \widetilde{\varphi}_{nn}^{(1-d)^2} = ... = \widetilde{\varphi}_{11}^{(1-d)^{n+1}} = \widetilde{\varphi}_{00}^{(1-d)^{n+2}}$$

$$\Rightarrow \widetilde{\varphi}_{00}^{(d-1)^{n+2}-(-1)^{n+2}} = 1,$$

so $\widetilde{\varphi}_{00}$ is a dm-th root of unity, therefore $\exists k^* \in \mathbb{Z}/dm\mathbb{Z} : \widetilde{\varphi}_{00} = \xi^{k^*}$, then, since $\widetilde{\varphi}_{ii}^{1-d} = \widetilde{\varphi}_{i+1,i+1}$, for all $i = 0, ..., n$, we have $\widetilde{\varphi}_{ii} = \widetilde{\varphi}_{00}^{(1-d)^i}$, for all $i = 0, ..., n+1$, and then we have $\xi^{k^*(1-d)^i} = \widetilde{\varphi} ii$, for all $i = 0, ..., n+1$, therefore $\widetilde{\varphi} = \widetilde{\gamma}(k^*)$, and then $\gamma(k^*) = \varphi$.
But γ is not a isomorphism, since if $k^{**} \in \mathbb{Z}/dm\mathbb{Z}$ is such that $(\xi^{k^{**}})^d = 1$, then we have

$$\xi^{k^{**}[(1-d)^i-(1-d)^j]} = [(\xi^{k^{**}})^d]^{\frac{(1-d)^i-(1-d)^j}{d}} = 1^{\frac{(1-d)^i-(1-d)^j}{d}} = 1,$$

so $\xi^{k^{**}(1-d)^i} = \xi^{k^{**}(1-d)^j}$, and the*) $= \widetilde{\gamma}(k^*) = \xi^{k^{**}} Id$, therefore $\gamma(k^{**}) = Id \in Aut(X)$, so $k^{**} \in Ker(\gamma)$. On the other hand, if $k^{**} \in Ker(\gamma)$, then $\gamma(k^{**}) = Id \in Aut(X)$, so $\widetilde{\gamma}(k^*) = \lambda Id \in GL(V)$, but since $\widetilde{\gamma}(k^*)(e_0) = \xi^{k^{**}} e_0$, we have $\lambda = \xi^{k^{**}}$, and then $\xi^{k^{**}} = \xi^{k^{**}(1-d)^i}$, for all $i = 0, ..., n+1$, so

$$\xi^{k^{**}} = \xi^{k^{**}(1-d)} \Rightarrow \xi^{k^{**}d} = 1 \Rightarrow (\xi^{k^{**}})^d = 1,$$

so k^{**} is such that $\xi^{k^{**}}$ is a d-th root of unity. Therefore

$$Ker(\gamma) = \{k^{**} \in \mathbb{Z}/dm\mathbb{Z} : \xi^{k^{**}} = 1\},$$

so $D \simeq \mathbb{Z}/m\mathbb{Z}$. Therefore

$$\mathrm{Aut}(X) = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/(n+2)\mathbb{Z}.$$

□

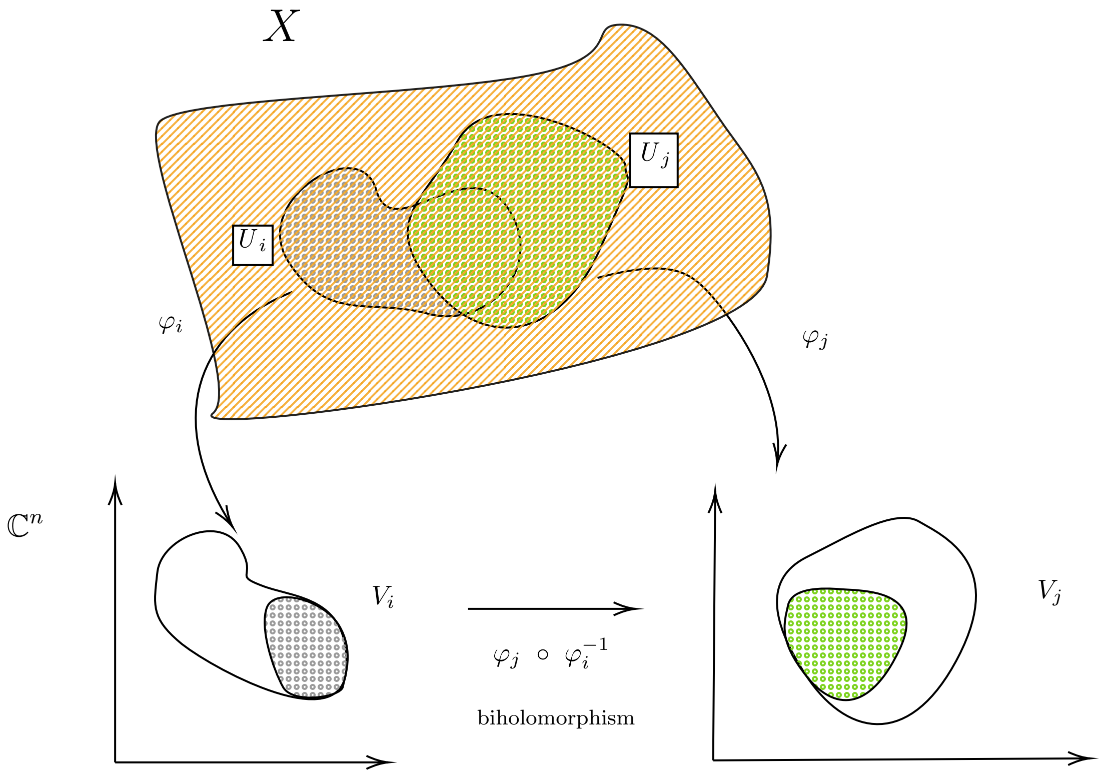
Chapter 2

HODGE STRUCTURES

In this chapter, we will study the concept of Hodge structures, both in their abstract form and as they arise from the cohomology groups of a complex algebraic variety X.
To this end, we will first review the notions of complex manifolds, tangent and cotangent spaces, and differential forms, in order to introduce de Rham and Dolbeault cohomology.
We will then proceed with the definition of a Hermitian metric on a complex manifold, the Hodge star operator, and the notions of harmonic forms on Kahler manifolds.
We will also present foundational results, including the Universal Coefficient Theorem for homology—a fundamental tool in algebraic topology—and de Rham's theorem, which connects differential forms with singular cohomology and thus links differential geometry with algebraic topology. Subsequently, we introduce the notions of Hodge structures and polarized Hodge structures, and conclude by showing that the automorphism group of a polarized Hodge structure is finite.

2.1 Differential forms on complex manifolds

Fix a complex manifold X of dimension n, then there exists an open cover of X, $\{U_i\}_{i\in I}$ (i.e, $X = \cup_{i\in I} U_i$, where $U_i \subseteq X$ are open) and homeomorphisms $\varphi_i : U_i \to V_i$, where V_i are open sets of $\mathbb{C}^n$, such that whenever $U_i \cap U_j \neq \emptyset$, $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a biholomorphism. Then $\{(U_i, \varphi_i)\}_{i\in I}$ is an holomorphic atlas of X.



$(z_1, z_2, ..., z_n)$ *local coordinates via* φ_i.

So, X has a structure of a differentiable manifold of dimension $2n$, this follow from the diffeomorphism $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, the biyection $(z_1, ..., z_n) \mapsto (x_1, y_1, ..., x_n, y_n)$, where $z_j = x_j + iy_j$.

For $p \in X$, exists $U \subseteq X$ an open neighborhood of p, and a homeomorphism $\varphi : U \to W$, where W is an open subset of $\mathbb{R}^{2n}$. We have the identification via φ

$$T_p^{\mathbb{R}} X = T_{\varphi(p)}^{\mathbb{R}} \mathbb{C}^n = \bigoplus_{j=1}^{n} (\mathbb{R}\frac{\partial}{\partial x_j} \oplus \mathbb{R}\frac{\partial}{\partial y_j}),$$

where $T_p^{\mathbb{R}} X$ corresponds to the space of $\mathbb{R}$-linear derivations of $\mathcal{C}_p^{\infty}(X)$, and $\mathcal{C}_p^{\infty}(X)$ denotes the algebra of germs of smooth real-valued functions at p.

The complexified tangent space is

$$T_p^{\mathbb{C}}X := T_p^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{j=1}^{n}(\mathbb{C}\frac{\partial}{\partial x_j} \oplus \mathbb{C}\frac{\partial}{\partial y_j}).$$

In this setting, derivatives are extended not only in terms of their range but also in terms of their domain, so that they now apply to functions taking values in $\mathbb{C}$, i.e., exists a morphism $()_{\mathbb{C}} : T_p^{\mathbb{R}}X \to T_p^{\mathbb{C}}X$, $D_{\mathbb{R}} \mapsto D_{\mathbb{C}}$, where

$$D_{\mathbb{C}}(f + ig) := D_{\mathbb{R}}(f) + iD_{\mathbb{R}}(g),\ for\ all\ D_{\mathbb{R}} \in T_p^{\mathbb{R}}X, f, g \in \mathcal{C}_p^{\infty}(X).$$

So, we have $T_p^{\mathbb{C}}X = Im(()_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C}$.
This induces the canonical decomposition

$$T_p^{\mathbb{C}}X = T_p^{1,0}X \oplus T_p^{0,1}X := (\bigoplus_{j=1}^{n}\mathbb{C}\frac{\partial}{\partial z_j}) \oplus (\bigoplus_{j=1}^{n}\mathbb{C}\frac{\partial}{\partial \overline{z_j}}),$$

where $\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$, $\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$.
And we have

$$T_p^{1,0}X \simeq T_pX,\ T_p^{0,1}X \simeq \overline{T_pX},$$

where T_pX is the holomorphic tangent space in X at p, this is the space of $\mathbb{C}$-linear derivations of the local ring of germs of holomorphic functions at p, $\mathcal{O}_{X,p}$. So we have that $T_p^{1,0}X$ consist of derivations that vanish on antiholomorphic functions, and $T_p^{0,1}X$ consist of derivations that vanish on holomorphic functions. Here, we have the morphism $()|_{\mathcal{O}_{X,p}} : T_p^{\mathbb{C}}X \to T_pX$, wich sends a derivation $D \in T_p^{\mathbb{C}}X$ to the derivation $D|_{\mathcal{O}_{X,p}} \in T_pX$. This is a surjective morphism, with kernel $T_p^{0,1}X$.

Using the holomorphic charts of X, we can equip

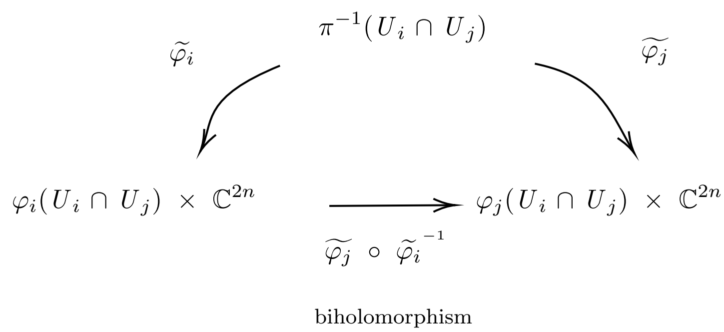
$$T^{\mathbb{C}}X := \bigsqcup_{p \in X} T_p^{\mathbb{C}}X \xrightarrow{\pi} X$$

of a structure of a holomorphic complex vector bundle of range $2n$

$$\{\pi^{-1}(U_i)\}_{i \in I} \subseteq T^{\mathbb{C}}X,$$

where $\{U_i\}_{i \in I} \subseteq X$ are the opens of the holomorphic atlas of X, and we have

$$\pi^{-1}(U_i) = \bigsqcup_{p \in U_i} T_p^{\mathbb{C}}X$$



and $[\widetilde{\varphi}_j \circ \widetilde{\varphi}_i^{-1}](x,.) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is an $\mathbb{C}$-lineal isomorphism, for all $x \in \varphi_i(U_i \cap U_j)$.

The complexified tangent bundle $T^{\mathbb{C}}X$ decomposes has the direct sum of the holomorphic and antiholomorphic tangent bundles:

$$T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

were both are complex vector bundles of range n.
Here we have $T^{1,0}X = \bigsqcup_{p\in X} T_p^{1,0}X$ and $T^{0,1}X = \bigsqcup_{p\in X} T_p^{0,1}X$, and $T^{1,0}X \simeq TX = \bigsqcup_{p\in X} T_pX$.
We may now consider the complexified cotangent bundle $(T^{\mathbb{C}}X)^*$ given by

$$(T^{\mathbb{C}}X)^* = \bigsqcup_{p\in X} (T_p^{\mathbb{C}}X)^*,$$

here the superscript * denotes the dual vector space. In the same way, we have a decomposition of the complexified cotangent bundle

$$(T^{\mathbb{C}}X)^* = (T^{1,0}X)^* \oplus (T^{0,1}X)^*,$$

when $(T^{1,0}X)^* = \bigsqcup_{p\in X}(T_p^{1,0}X)^*$, and $(T^{0,1}X)^* = \bigsqcup_{p\in X}(T_p^{0,1}X)^*$, so

$$(T_p^{\mathbb{C}}X)^* = (T_p^{1,0}X)^* \oplus (T_p^{0,1}X)^* = (\bigoplus_{j=1}^{n} \mathbb{C}dz_j) \oplus (\bigoplus_{j=1}^{n} \mathbb{C}d\overline{z_j}),$$

when

$$d\overline{z_j}\left(\frac{\partial}{\partial \overline{z_k}}\right) = dz_j\left(\frac{\partial}{\partial z_k}\right) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

and $dz_j\left(\frac{\partial}{\partial\overline{z_k}}\right) = d\overline{z_j}\left(\frac{\partial}{\partial z_k}\right) = 0$. These element satisfy

$$dz_j = dx_j + idy_j, d\overline{z_j} = dx_j + idy_j.$$

With all the above, we now give the following definition

Definition 2.1.1. *We define the sheaf of complex $\mathcal{C}^\infty$ 1-forms on X denoted $\Omega^1_{X^\infty}$, as the sheaf of $\mathcal{C}^\infty$ sections of the complexified cotangent bundle $(T^{\mathbb{C}}X)^* \xrightarrow{\pi} X$, i.e.*

$$\Omega^1_{X^\infty}(U) = \{s : U \xrightarrow{\mathcal{C}^\infty} (T^{\mathbb{C}}X)^* : \pi \circ s = id_U\}, \forall U \subseteq X \text{ open.}$$

The decomposition $(T^{\mathbb{C}}X)^* = (T^{1,0}X)^* \oplus (T^{0,1}X)^*$ induces the decomposition

$$\Omega^1_{X^\infty} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

where

$$\Omega^{1,0}_X = \{ \text{ sections } \mathcal{C}^\infty \text{ of } (T^{1,0}X)^* \to X\},$$

$$\Omega^{0,1}_X = \{ \text{ sections } \mathcal{C}^\infty \text{ of } (T^{0,1}X)^* \to X\}.$$

Locally in coordinate charts, a form $\omega \in \Omega^{1,0}_X(U \cap U_i)$ can be written as

$$(\varphi_i)_*\omega = \sum_{j=1}^n a_j dz_j,$$

where $a_j \in \mathcal{C}^\infty(\varphi(U \cap U_i), \mathbb{C})$, and a form $\eta \in \Omega^{0,1}_X(U \cap U_i)$ can be written as

$$(\varphi_i)_*\eta = \sum_{j=1}^n b_j d\overline{z_j},$$

where $b_j \in \mathcal{C}^\infty(\varphi(U \cap U_i), \mathbb{C})$.
We say that $\omega \in \Omega^{1,0}_X$ is holomorphic if it is a holomorphic section of $(TX)^* \simeq (T^{1,0}X)^*$, then, we define

$$\Omega^1_X = \{ \text{ sheaf of the holomorphic sections of } (TX)^* \simeq (T^{1,0}X)^*\},$$

Locally in coordinate charts, a form $\omega \in \Omega^1_X(U)$ can be written as

$$(\varphi_i)_*\omega = \sum_{j=1}^n a_j dz_j,$$

where $a_j \in \mathcal{C}^\infty(\varphi(U \cap U_i), \mathbb{C})$ is a holomorphic function.

In general we consider

$$\wedge^k (T^{\mathbb{C}}X)^* = \bigoplus_{p+q=k} \wedge^p (T^{1,0}X)^* \otimes \wedge^q (T^{0,1}X)^*,$$

$$\Omega^k_{X^\infty} = \{\mathcal{C}^\infty \textit{ sections of } \wedge^k (T^{\mathbb{C}}X)^*\},$$

$$\Omega^{p,q}_X = \{\mathcal{C}^\infty \textit{ sections of } \wedge^p (T^{1,0}X)^* \otimes \wedge^q (T^{0,1}X)^*\}.$$

Then we have

$$\Omega^k_{X^\infty} = \bigoplus_{p+q=k} \Omega^{p,q}_X,$$

and locally in coordinate charts, a form $\omega \in \Omega^{p,q}_X(U)$ can be written as

$$(\varphi_i)_*\omega = \sum_{\substack{I=\{i_1<\ldots<i_p\},\\ J=\{j_1<\ldots<j_q\}}} a_{I,J} dz_I \wedge d\overline{z_J},$$

where $a_{I,J} \in \mathcal{C}^\infty(\varphi(U \cap U_i), \mathbb{C})$ and $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\overline{z_J} = d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}$.

$$\wedge^p (T^{1,0}_p X)^* \otimes \wedge^q (T^{0,1}_p X)^* = \bigoplus_{\substack{i_1<\ldots<i_p\\ j_1<\ldots<j_q}} \mathbb{C} dz_I \wedge d\overline{z_J},$$

and $\wedge^k (TX)^* \simeq \wedge (T^{1,0}X)^*$. The holomorphic k-forms are

$$\Omega^k_X = \{ \textit{ holomorphic sections of } \wedge^k (TX)^*\},$$

and locally in coordinate charts, a form $\omega \in \Omega^k_X(U)$ can be written as

$$(\varphi_i)_*\omega = \sum_{i_1<\ldots<i_k} a_I dz_I,$$

where $a_I \in \mathcal{C}^\infty(\varphi(U \cap U_i), \mathbb{C})$ is holomorphic.
Then the exterior derivate defines a morphism of sheaves

$$d : \Omega^k_{X^\infty} \to \Omega^{k+1}_{X^\infty}$$

defined on sections $\omega \in \Omega^k_{X^\infty}(U)$, locally in coordinates by

$$\begin{aligned}(\varphi_i)_*(d\omega) :&= d((\varphi_i)_*\omega)\\ &= d\left(\sum_{\substack{I,J\\|I|+|J|=k}} a_{I,J}dz_I \wedge d\overline{z_J}\right)\\ &= \sum_{\substack{I,J\\|I|+|J|=k}} da_{I,J}\wedge dz_I \wedge d\overline{z_J} \in \Omega^{k+1}_{X^\infty}(U\cap U_i),\end{aligned}$$

where $df := \sum_{j=1}^n (\frac{\partial f}{\partial z_j}dz_j + \frac{\partial f}{\partial \overline{z_j}}d\overline{z_j}) = \sum_{j=1}^n(\frac{\partial f}{\partial x_j}dx_j + \frac{\partial f}{\partial y_j}dy_j) \in \Omega^1_{X^\infty}(U),\ f\in\mathcal{C}^\infty(U)$.

The exterior derivate satisfies $d\circ d = 0$, since in local coordinates we have

$$\begin{aligned}d(d\omega) &= d\left(\sum_{\substack{I,J\\|I|+|J|=k}} da_{I,J}\wedge dz_I\wedge d\overline{z_J}\right)\\ &= d\left(\sum_{\substack{I,J\\|I|+|J|=k}}\sum_{j=1}^n(\frac{\partial a_{I,J}}{\partial z_j}dz_j + \frac{\partial a_{I,J}}{\partial \overline{z_j}}d\overline{z_j})\wedge dz_I\wedge d\overline{z_J}\right)\\ &= \sum_{\substack{I,J\\|I|+|J|=k}} d\left(\sum_{j=1}^n \frac{\partial a_{I,J}}{\partial z_j}dz_j\wedge dz_I\wedge d\overline{z_J}\right) + d\left(\sum_{j=1}^n\frac{\partial a_{I,J}}{\partial\overline{z_j}}d\overline{z_j}\wedge dz_I\wedge d\overline{z_J}\right)\\ &= \sum_{\substack{I,J\\|I|+|J|=k}}\sum_{j=1}^n d\left(\frac{\partial a_{I,J}}{\partial z_j}\right)dz_j\wedge dz_I\wedge d\overline{z_J} + d\left(\frac{\partial a_{I,J}}{\partial\overline{z_j}}\right)d\overline{z_j}\wedge dz_I\wedge d\overline{z_J}\\ &= \sum_{\substack{I,J\\|I|+|J|=k}}\sum_{j=1}^n\sum_{k=1}^n(\frac{\partial^2 a_{I,J}}{\partial z_j\partial z_k}dz_k\wedge dz_j\wedge dz_I\wedge d\overline{z_J} + \frac{\partial^2 a_{I,J}}{\partial z_j\partial\overline{z_k}}d\overline{z_k}\wedge dz_j\wedge dz_I\wedge d\overline{z_J}\\ &+ \frac{\partial^2 a_{I,J}}{\partial\overline{z_j}\partial z_k}dz_k\wedge d\overline{z_j}\wedge dz_I\wedge d\overline{z_J} + \frac{\partial^2 a_{I,J}}{\partial\overline{z_j}\partial\overline{z_k}}d\overline{z_k}\wedge d\overline{z_j}\wedge dz_I\wedge d\overline{z_J}),\end{aligned}$$

and since $\frac{\partial^2 a_{I,J}}{\partial\overline{z_j}\partial\overline{z_k}} = \frac{\partial^2 a_{I,J}}{\partial\overline{z_k}\partial\overline{z_j}}$, $\frac{\partial^2 a_{I,J}}{\partial z_j\partial\overline{z_k}} = \frac{\partial^2 a_{I,J}}{\partial\overline{z_k}\partial z_j}$, $\frac{\partial^2 a_{I,J}}{\partial z_j\partial z_k} = \frac{\partial^2 a_{I,J}}{\partial z_k\partial z_j}$, and $d\alpha\wedge d\beta = -d\beta\wedge d\alpha$, we have

$$[d\circ d](\omega) = 0.$$

Then we have a sheaf complex (the $\mathcal{C}^\infty$ De Rham complex of sheaves)

$$\mathcal{C}_X^\infty = \Omega^0_{X^\infty} \xrightarrow{d} \Omega^1_{X^\infty} \xrightarrow{d} \Omega^2_{X^\infty} \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{2n}_{X^\infty} \xrightarrow{d} 0,$$

and for any open subset $U \subseteq X$, we have an induced De Rham complex $(\Omega^\bullet_{X^\infty}(U), d)$, in particular for $U = X$, we have $(\Omega^\bullet_{X^\infty}(X), d)$.
The cohomology groups of this complex are the so-called **De Rham cohomology groups**

$$H^k_{dR}(X) = H^k_{dR}(X, \mathbb{C}) := \frac{Ker(\Omega^k_{X^\infty}(X) \xrightarrow{d} \Omega^{k+1}_{X^\infty}(X))}{Im(\Omega^{k-1}_{X^\infty}(X) \xrightarrow{d} \Omega^k_{X^\infty}(X))}.$$

We have that $H^k_{dR}(X, \mathbb{C}) = H^k_{dR}(X, \mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$.
If we restrict the exterior differential to

$$\Omega^{p,q}_X \xrightarrow{d=\partial+\overline{\partial}} \Omega^{p+1,q}_X \oplus \Omega^{p,q+1}_X,$$

we have for $\omega \in \Omega^{p,q}_X$

$$d\omega = d(\sum a_{I,J} dz_I \wedge d\overline{z_J}) = \sum \sum_{i=1}^n \frac{a_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge d\overline{z_J} + \sum \sum_{i=1}^n \frac{a_{I,J}}{\partial \overline{z_i}} d\overline{z_i} \wedge dz_I \wedge d\overline{z_J} = \partial\omega + \overline{\partial}\omega,$$

i.e., $d = \partial + \overline{\partial}$, $\partial : \Omega^{p,q}_X \to \Omega^{p+1,q}_X$, and $\overline{\partial} : \Omega^{p,q}_X \to \Omega^{p,q+1}_X$.
How $d \circ d = 0 \Rightarrow 0 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial}) = \partial \circ \partial + \partial \circ \overline{\partial} + \overline{\partial} \circ \partial + \overline{\partial} \circ \overline{\partial}$, and if $\omega \in \Omega^{p,q}_X \Rightarrow$ $(\partial \circ \partial)(\omega) \in \Omega^{p+2,q}_X$, $(\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)(\omega) \in \Omega^{p+1,q+1}_X$, $(\overline{\partial} \circ \overline{\partial})(\omega) \in \Omega^{p,q+2}_X$, therefore

$$\partial \circ \partial = 0, \ \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0, \ \overline{\partial} \circ \overline{\partial} = 0.$$

The above relations give us the Dolbeault complex of sheaves

$$\Omega^{p,0}_X \xrightarrow{\overline{\partial}} \Omega^{p,1}_X \xrightarrow{\overline{\partial}} \Omega^{p,2}_X \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{p,n}_X \xrightarrow{\overline{\partial}} 0,$$

where $ker(\overline{\partial} : \Omega^{p,0}_X \to \Omega^{p,1}_X) = \Omega^p_X$.
The Dolbeault complex is $(\Omega^{p,\bullet}_X, \overline{\partial})$, and it gives rise to the Dolbeault cohomology groups

$$H^{p,q}_{\overline{\partial}}(X) := \frac{Ker(\Omega^{p,q}_X(X) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}_X(X))}{Im(\Omega^{p,q-1}_X(X) \xrightarrow{\overline{\partial}} \Omega^{p,q}_X(X))}.$$

2.2 Harmonic forms on compact Hermitian manifolds

Definition 2.2.1. *Let $\pi : E \to X$ be a complex vector bundle. A Hermitian metric on E is a global $\mathcal{C}^\infty$ section of $(E \otimes \overline{E})^*$ (i.e., a sesquilinear form on each fiber) such that*

$$h_p : E_p \times E_p \to \mathbb{C},$$

$$h_p(u,v) = \overline{h_p(v,u)},$$

$$h_p(u,u) > 0, \forall u \neq 0.$$

Definition 2.2.2. *A Hermitian manifold is a complex manifold X together with a Hermitian metric on the holomorphic tangent bundle $TX \to X$.*

In local holomorphic coordinates $(z_1, ..., z_n)$, the Hermitian metric h can be expressed as

$$h = \sum_{1 \leq i,j \leq n} h_{i,j} dz_i \otimes d\overline{z_j},$$

where the matrix (h_{ij}) is hermitian and positive definite, and the coefficients h_{ij} are $\mathcal{C}^\infty$ since h is a global $\mathcal{C}^\infty$ section of the bundle $(TX \otimes \overline{TX})^*$. To the metric h we associate the form $\omega \in \Omega^{1,1}_{X^\infty}(X)$ given by

$$\omega := -Im(h) = \frac{i}{2}(h - \overline{h}) = \frac{i}{2} \sum_{i,j} h_{ij}(dz_i \otimes d\overline{zj} - d\overline{zj} \otimes dz_i) = \frac{i}{2} \sum_{i,j} h_{ij} dz_i \wedge d\overline{z_j},$$

this, in turn, induces a canonical volume form

$$\Phi := \frac{\omega^n}{n!} = \frac{\omega \wedge ... \wedge \omega}{n!} \in \Omega^{n,n}_{X^\infty}(X),$$

which in local coordinates is

$$\Phi = \left(\frac{i}{2}\right)^n det((h_{ij})) dz_1 \wedge d\overline{z_1} \wedge ... \wedge dz_n d\overline{z_n}.$$

With all this, we have the following definition:

Definition 2.2.3. *Let X be a compact Hermitian manifold. We define the L^2 inner product on $\Omega^k_{X^\infty}(X)$ by*

$$(\psi, \nu)_{L^2} := \int_X (\psi(x), \nu(x)) \Phi(x),$$

where ψ, ν are global sections of $\wedge^k (T^{\mathbb{C}} X)^*$.

To induce the metric $(.,.)$ on $\wedge^k (T^{\mathbb{C}} X)^*$, we proceed as follows: Let us choose an orthonormal frame $\mathcal{C}^\infty$, $\varphi_1, ..., \varphi_n$ of $TX \to X$ with respect to the metric h. Then, let $\phi_1, ..., \phi_n$ be the dual frame, that is,

$$\phi_i(x)(\varphi_j(x)) = \delta_{ij}.$$

Next, we impose that

$$\{\phi_{i_1} \wedge ... \wedge \phi_{i_p} \wedge \overline{\phi_{j_1}} \wedge ... \wedge \overline{\phi_{j_p}}\}$$

is an orthonormal frame of $\wedge^p(T^{1,0}X)^* \otimes \wedge^q(T^{0,1}X)^* \to X$. Then, $\{\phi_1, ..., \phi_n, \overline{\phi_1}, ..., \overline{\phi_n}\}$ is a frame for $T^{\mathbb{C}}X^* \to X$. That is, we impose that

$$(\phi_i(x), \phi_j(x)) = \delta_{ij}.$$

Since $\overline{\omega} = -\frac{i}{2}\sum \overline{h_{ij}} d\overline{z_i} \wedge dz_j = \frac{i}{2}\sum h_{ji} dz_j \wedge d\overline{z_i} = \omega$, we have

- $\overline{(\psi, \nu)}_{L^2} = (\nu, \psi)_{L^2}$

- $(\psi, \psi)_{L^2} \geq 0$

- $(\psi, \psi)_{L^2} = 0 \Rightarrow \psi = 0.$

Definition 2.2.4. *The Hodge star operator associated to the Hermitian metric h is the unique $\mathbb{C}$-linear isomorphism*

$$* : \Omega_X^{p,q}(X) \to \Omega_X^{n-p,n-q}(X)$$

such that $\forall \psi, \nu \in \Omega_X^{p,q}(X)$:

$$(\psi(x), \nu(x))\Phi(x) = \psi(x) \wedge *\nu(x).$$

Proposition 2.2.5. *The Hodge star operator satisfies $* * \nu = (-1)^{p+q}\nu$.*

For the proof, see Voisin (2002) [Voi02b]

Definition 2.2.6. *We define $\overline{\partial}^* : \Omega_X^{p,q}(X) \to \Omega_X^{p,q-1}(X)$ given by $\overline{\partial}^* := - * \overline{\partial} *$.*

Proposition 2.2.7. $(\overline{\partial}\psi, \nu)_{L^2} = (\psi, \overline{\partial}^*\nu)_{L^2}$, $\forall \psi \in \Omega_X^{p,q-1}(X), \nu \in \Omega_X^{p,q}(X)$.

Proof. We have

$$
\begin{aligned}
(\overline{\partial}\psi,\nu)_{L^2} &= \int_X (\overline{\partial}\psi(x),\nu(x))\Phi(x) \\
&= \int_X \overline{\partial}\psi(x)\wedge *\nu(x) \\
&= \int_X \overline{\partial}(\psi(x)\wedge *\nu(x)) + (-1)^{p+q}\int_X \psi(x)\wedge \overline{\partial} * \nu(x) \\
&= \int_X d(\psi(x)\wedge *\nu(x)) + (-1)^{p+q}\int_X \psi(x)\wedge \overline{\partial} * \nu(x) \\
&= \int_{\partial X} \psi(x)\wedge *\nu(x) + (-1)^{p+q}\int_X \psi(x)\wedge \overline{\partial} * \nu(x) \\
&= \int_{\emptyset} \psi(x)\wedge *\nu(x) + (-1)^{p+q}\int_X \psi(x)\wedge \overline{\partial} * \nu(x) \\
&= (-1)^{p+q}\int_X \psi(x)\wedge \overline{\partial} * \nu(x) \\
&= -\int_X \psi(x)\wedge * * \overline{\partial} * \nu(x) \\
&= \int_X \psi(x)\wedge * \overline{\partial}^* \nu(x) \\
&= \int_X (\psi(x),\overline{\partial}^*\nu(x))\Phi(x) \\
&= (\psi,\overline{\partial}^*\nu)_{L^2},\ \forall \psi\in\Omega_X^{p,q-1}(X), \nu\in\Omega_X^{p,q}(X).
\end{aligned}
$$

□

Proposition 2.2.8. *Let $\psi \in \Omega_X^{p,q}(X)$, such that $\overline{\partial}\psi = 0$, then ψ has the minimal norm within its cohomology class $[\psi] = \psi + \overline{\partial}\Omega_X^{p,q-1}(X) \in H_{\overline{\partial}}^{p,q}(X)$ if and only if $\overline{\partial}^*\psi = 0$.*

Proof. $(\Rightarrow)$ $\forall \nu \in \Omega_X^{p,q-1}(X)$, we have

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}||\psi + t\overline{\partial}\nu||_{L^2}^2|_{t=0} \\
&= \frac{\partial}{\partial t}(||\psi||_{L^2}^2 + t(\psi,\overline{\partial}\nu)_{L^2} + t(\overline{\partial}\nu,\psi)_{L^2} + t^2||\overline{\partial}\nu||_{L^2}^2)|_{t=0} \\
&= ((\psi,\overline{\partial}\nu)_{L^2} + (\overline{\partial}\nu,\psi)_{L^2} + 2t||\overline{\partial}\nu||_{L^2}^2)|_{t=0} \\
&= (\psi,\overline{\partial}\nu)_{L^2} + (\overline{\partial}\nu,\psi)_{L^2} \\
&= (\psi,\overline{\partial}\nu)_{L^2} + \overline{(\psi,\overline{\partial}\nu)}_{L^2} \\
&= 2Re((\psi,\overline{\partial}\nu)_{L^2}),
\end{aligned}
$$

therefore $Re((\psi,\overline{\partial}\nu)_{L^2}) = 0$, and similarly consider $\frac{\partial}{\partial t}||\psi + it\overline{\partial}\nu||_{L^2}|_{t=0} = 0$, we also have $Im((\psi,\overline{\partial}\nu)_{L^2}) = 0$. Therefore $(\psi,\overline{\partial}\nu)_{L^2} = (\overline{\partial}\nu,\psi)_{L^2} = 0$, then we have $(\nu,\overline{\partial}^*\psi)_{L^2} = 0$, for all $\nu \in \Omega_X^{p,q-1}(X)$, hence, letting $\nu = \overline{\partial}^*\psi$, it follows that $||\overline{\partial}^*\psi||_{L^2}^2 = 0$, and then $\overline{\partial}^*\psi = 0$.

($\Leftarrow$) Suppose that $\overline{\partial}^* \psi = 0$. Let $\nu \in \Omega_X^{p,q-1}(X)$ with $\overline{\partial}\nu \neq 0$, then we have

$$\begin{aligned}
||\psi + \overline{\partial}\nu||_{L^2}^2 &= ||\psi||_{L^2}^2 + ||\overline{\partial}\nu||_{L^2}^2 + 2Re((\psi, \overline{\partial}\nu)_{L^2}) \\
&= ||\psi||_{L^2}^2 + ||\overline{\partial}\nu||_{L^2}^2 + 2Re((\overline{\partial}\nu, \psi)_{L^2}) \\
&= ||\psi||_{L^2}^2 + ||\overline{\partial}\nu||_{L^2}^2 + 2Re((\nu, \overline{\partial}^*\psi)_{L^2}) \\
&= ||\psi||_{L^2}^2 + ||\overline{\partial}\nu||_{L^2}^2 + 2Re((\nu, 0)_{L^2}) \\
&= ||\psi||_{L^2}^2 + ||\overline{\partial}\nu||_{L^2}^2 \\
&> ||\psi||_{L^2}^2.
\end{aligned}$$

□

Definition 2.2.9. *We define the Dolbeault Laplacian, denoted by $\Delta_{\overline{\partial}}$, as the elliptic differential operator given by*

$$\Delta_{\overline{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \,:\Omega_X^{p,q}(X) \to \Omega_X^{p,q}(X).$$

Proposition 2.2.10.

$$\Delta_{\overline{\partial}}\psi = 0 \Leftrightarrow \bar{\partial}\psi = 0 \text{ and } \bar{\partial}^*\psi = 0.$$

Proof.

$$\begin{aligned}
\Delta_{\overline{\partial}}\psi = 0 \quad \Rightarrow \quad 0 &= (\Delta_{\overline{\partial}}\psi, \psi)_{L^2} \\
&= (\bar{\partial}\bar{\partial}^*\psi + \bar{\partial}^*\bar{\partial}\psi, \psi)_{L^2} \\
&= (\bar{\partial}\bar{\partial}^*\psi, \psi)_{L^2} + (\bar{\partial}^*\bar{\partial}\psi, \psi)_{L^2} \\
&= (\bar{\partial}^*\psi, \bar{\partial}^*\psi)_{L^2} + (\bar{\partial}\psi, \bar{\partial}\psi)_{L^2} \\
&= \|\bar{\partial}^*\psi\|_{L^2}^2 + \|\bar{\partial}\psi\|_{L^2}^2 \\
&\Leftrightarrow \quad \|\bar{\partial}^*\psi\|_{L^2}^2 = 0 \quad \text{and} \quad \|\bar{\partial}\psi\|_{L^2}^2 = 0 \\
&\Leftrightarrow \quad \bar{\partial}^*\psi = 0 \quad \text{and} \quad \bar{\partial}\psi = 0 \\
&\Rightarrow \Delta_{\overline{\partial}}\psi = 0.
\end{aligned}$$

□

A form $\psi \in \Omega_X^{p,q}(X)$ satisfying $\Delta_{\overline{\partial}}\psi = 0$ is called a harmonic $(p,q)-$form.

Definition 2.2.11. *We denote by $\mathcal{H}_{\overline{\partial}}^{p,q}(X)$ the subset of $\Omega_X^{p,q}(X)$ consisting of harmonic $(p,q)-$forms.*

Proposition 2.2.12. *The morphism*

$$\begin{aligned}
\mathcal{H}_{\overline{\partial}}^{p,q}(X) &\to H_{\overline{\partial}}^{p,q}(X) \\
\psi &\mapsto [\psi]
\end{aligned}$$

is injective.

Proof. Let be $\psi_1, \psi_2 \in \mathcal{H}^{p,q}_{\bar\partial}(X)$ such that $[\psi_1] = [\psi_2]$. Since $\psi_1, \psi_2 \in \mathcal{H}^{p,q}_{\bar\partial}(X)$, we have $\Delta_{\bar\partial}\psi_1 = \Delta_{\bar\partial}\psi_2 = 0$, and then $\bar\partial\psi_1 = \bar\partial^*\psi_1 = \bar\partial\psi_2 = \bar\partial^*\psi_2 = 0$. And since $[\psi_1] = [\psi_2]$, exists $\nu \in \Omega^{p,q-1}_X(X)$ such that $\psi_1 = \psi_2 + \bar\partial\nu$.
Suppose, for the sake of contradiction, that ψ_1 and ψ_2 are distinct, and then we have $\bar\partial\nu \neq 0$. Then by the previous proposition we have $||\psi_1||_{L^2} = ||\psi_2 + \bar\partial\nu||_{L^2} > ||\psi_2||_{L^2}$, but also we have $\bar\partial^*\psi_1 = 0$, and $\bar\partial\psi_1 = 0$, and then also by the previous proposition we have that ψ_1 has the minimal norm within its cohomology class $[\psi_1] = [\psi_2]$, therefore $||\psi_1||_{L^2} \leq ||\psi_2||_{L^2}$ which is a contradiction. Therefore $\bar\partial\nu = 0$ and then $\psi_1 = \psi_2$. □

Theorem 2.2.13 (Hodge Harmonic Forms)**.** *Let X be a compact Hermitian manifold. Then for every pair of integers p, q, there is a natural isomorphism:*

$$\mathcal{H}^{p,q}_{\bar\partial}(X) \cong H^{p,q}_{\bar\partial}(X).$$

Remark 2.2.14. *One can define the operator $d^* = - * d* = \partial^* + \bar\partial^*$, which is the formal adjoint of*

$$d : \Omega^k_{X^\infty}(X) \to \Omega^{k+1}_{X^\infty}(X)$$

with respect to the L^2-inner product $\langle ., . \rangle_{L^2}$.
We can then define the Hodge Laplacian

$$\Delta_d = dd^* + d^*d,$$

and we obtain an analogous theorem at the level of $k-$forms:

$$\mathcal{H}^k_d(X) \cong H^k_{dR}(X),$$

where $\mathcal{H}^k_d(X)$ denotes the space of harmonic $k-$forms with respect to Δ_d.

The proof relies on elliptic PDE theory. In fact, since X is compact, we have

$$dim\mathcal{H}^k_d < \infty \ \text{ and } \ dim\mathcal{H}^{p,q}_{\bar\partial} < \infty.$$

See [Voi02a, Chapter 6] for a complete proof.

We have the following corollary.

Corollary 2.2.15 (Poincaré Duality)**.** *Let X be a compact Hermitian manifold of complex dimension n, and consider de Rham cohomology with complex coefficients. Then, for each*

k, the space of harmonic k-forms $\mathcal{H}_d^k(X)$ admits a non-degenerate Hermitian pairing

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\bar{\beta},$$

which induces an isomorphism

$$\mathcal{H}_d^k(X) \cong \left(\mathcal{H}_d^{2n-k}(X)\right)^*.$$

Via the Hodge isomorphism $\mathcal{H}_d^k(X) \cong H_{dR}^k(X;\mathbb{C})$, this yields the complex Poincaré duality:

$$H_{dR}^k(X;\mathbb{C}) \cong \left(H_{dR}^{2n-k}(X;\mathbb{C})\right)^*.$$

Proof. It suffices to show that the wedge product pairing is non-degenerate on the space of harmonic forms. Indeed, suppose $\omega \in \mathcal{H}_d^k(X)$ satisfies

$$\int_X \omega \wedge \mu = 0 \quad \text{for all } \mu \in \mathcal{H}_d^{2n-k}(X).$$

Then, in particular, taking $\mu = *\omega$, we have:

$$0 = \int_X \omega \wedge *\omega = \langle \omega, \omega \rangle_{L^2} = \|\omega\|_{L^2}^2.$$

Therefore $\omega = 0$. □

2.3 Hodge Decomposition

Corollary 2.3.1 (Kodaira–Serre Duality)**.** *Let X be a compact Hermitian manifold of complex dimension n. Then, for each p, q, the wedge product followed by integration induces a non-degenerate bilinear pairing:*

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X) \times \mathcal{H}_{\bar{\partial}}^{n-p,n-q}(X) \xrightarrow{\wedge} \mathcal{H}_{\bar{\partial}}^{n,n}(X) \xrightarrow{\int_X} \mathbb{C}.$$

In particular, there is a canonical isomorphism:

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X) \cong \left(\mathcal{H}_{\bar{\partial}}^{n-p,n-q}(X)\right)^*.$$

Via the Hodge theorem, this induces the corresponding duality on Dolbeault cohomology:

$$H_{\bar{\partial}}^{p,q}(X) \cong \left(H_{\bar{\partial}}^{n-p,n-q}(X)\right)^*.$$

Definition 2.3.2. *Let X be a complex manifold equipped with a Hermitian metric h. We say that X is a Kähler manifold if the associated $(1,1)-$form $\omega = -Im(h)$ is closed, i.e.,*

$$d\omega = 0.$$

Theorem 2.3.3. *If X is a Kähler manifold then we have*

$$\Delta_d = 2\Delta_{\bar{\partial}} : \Omega^{p,q}_{X^\infty} \to \Omega^{p,q}_{X^\infty}.$$

Therefore $\mathcal{H}^{p,q}_d = \mathcal{H}^{p,q}_{\bar{\partial}}$. And how $\mathcal{H}^k_d = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_d$, we have

$$\mathcal{H}^k_d(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(X).$$

For a detailed proof of the theorem, see [Voi02b].
In the following theorem we state the Universal Coefficient Theorem for Homology.
See [Hat02, Chapter 3] for the following theorem.

Theorem 2.3.4 (Universal Coefficient Theorem for Homology)**.** *Let X be a topological space. Then for any abelian group G, there exists a natural short exact sequence:*

$$0 \to H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G \to H_k(X;G) \to Tor^{\mathbb{Z}}_1(H_{k-1}(X;\mathbb{Z}), G) \to 0.$$

In particular, if $G = \mathbb{C}$, which is a flat $\mathbb{Z}-$module, the Tor term vanishes and we have a natural isomorphism:

$$H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H_k(X;\mathbb{C}).$$

The following classical result relates de Rham cohomology with singular cohomology and homology via duality:

Theorem 2.3.5 (de Rham Theorem and Duality)**.** *Let X be a smooth manifold. Then, for each integer k, there are natural isomorphisms:*

$$H^k(X;\mathbb{C}) \cong H^k_{dR}(X) \quad \textit{and} \quad H^k(X;\mathbb{C}) \cong H_k(X;\mathbb{C})^*.$$

Hence,

$$H^k_{dR}(X) \cong H_k(X;\mathbb{C})^*.$$

See [BT82, Chapter 4] for a complete discussion.

Definition 2.3.6 (Hodge Structure)**.** *A* Hodge structure of weight n *consists of a free $\mathbb{Z}$-module H of finite rank (i.e., $H \cong \mathbb{Z}^r$) together with a decomposition of its complexification*

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^r$$

into a direct sum of complex subspaces:

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q},$$

such that the decomposition satisfies the symmetry condition

$$\overline{H^{p,q}} = H^{q,p}.$$

This is equivalent to giving a decreasing filtration, called the *Hodge filtration*,

$$\{0\} = F^{n+1}H_{\mathbb{C}} \subseteq F^n H_{\mathbb{C}} \subseteq \cdots \subseteq F^1 H_{\mathbb{C}} \subseteq F^0 H_{\mathbb{C}} = H_{\mathbb{C}},$$

by complex subspaces, satisfying the condition

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{q+1} H_{\mathbb{C}}}, \quad \text{for all } p+q=n.$$

In this case, the original Hodge decomposition is recovered via

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}.$$

Conversely, given such a Hodge decomposition, the filtration is reconstructed as

$$F^p H_{\mathbb{C}} := \bigoplus_{r \geq p} H^{r,n-r}.$$

Definition 2.3.7 (Lefschetz Operator)**.** *Let X be a compact Kähler manifold of complex dimension n, with associated $(1,1)$-form ω corresponding to the Kähler metric. We define the Lefschetz operator $L : H^k(X,\mathbb{C}) \to H^{k+2}(X,\mathbb{C})$ by*

$$L(\alpha) = \alpha \wedge \omega = \omega \wedge \alpha, \forall \alpha \in H^k(X,\mathbb{C}).$$

Theorem 2.3.8 (Hard Lefschetz Theorem)**.** *Let X be a compact Kähler manifold of complex dimension n. Then for every integer $k \geq 0$, the map*

$$L^k : H^{n-k}(X,\mathbb{C}) \to H^{n+k}(X,\mathbb{C})$$

is an isomorphism.

Definition 2.3.9. *We define the the primitive cohomology group $H^{n-k}(X,\mathbb{C})_{prim}$ as*

$$H^{n-k}(X,\mathbb{C})_{prim} := Ker(L^{k+1} : H^{n-k}(X,\mathbb{C}) \to H^{n+k+2}(X,\mathbb{C})).$$

Theorem 2.3.10 (Gysin Exact Sequence and Primitive Cohomology)**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree d and dimension n, and let $U = \mathbb{P}^{n+1} \setminus X$ be the open complement of X in $\mathbb{P}^{n+1}$. Consider the inclusions $i : X \to \mathbb{P}^{n+1}$ and $j : U \to \mathbb{P}^{n+1}$. Then, we have the following exact sequence of cohomology groups:*

$$\cdots \to H^n(X, \mathbb{C}) \xrightarrow{i_*} H^{n+2}(\mathbb{P}^{n+1}, \mathbb{C}) \xrightarrow{j^*} H^{n+2}(U, \mathbb{C}) \to \cdots$$

The map i_* corresponds to cupping with the Kähler class $\omega \in H^2(\mathbb{P}^{n+1}, \mathbb{C})$, i.e.,

$$i_*(\alpha) = \omega \wedge \alpha \quad \textit{for} \quad \alpha \in H^n(X, \mathbb{C}).$$

Thus, the primitive cohomology is the kernel of this map:

$$H^n(X, \mathbb{C})_{\text{prim}} = \ker\left(\omega \wedge (\cdot) \colon H^n(X, \mathbb{C}) \to H^{n+2}(\mathbb{P}^{n+1}, \mathbb{C})\right).$$

For a detailed treatment of this result see Hartshorne's *Algebraic Geometry* [Har77].

Corollary 2.3.11. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree d. Then the primitive cohomology in middle degree is given by:*

- *If n is odd, then*
$$H^n(X, \mathbb{C})_{\text{prim}} = H^n(X, \mathbb{C}).$$
- *If n is even, then*
$$H^n(X, \mathbb{C})_{\text{prim}} = \left(\mathbb{C} \cdot \omega^{n/2}\right)^{\perp} \subset H^n(X, \mathbb{C}),$$
where $\omega \in H^2(X, \mathbb{C})$ is the Kähler class induced from the ambient projective space, and the orthogonal is taken with respect to the Hodge–Riemann bilinear form.

In particular, the primitive cohomology has codimension

$$\operatorname{codim}\left(H^n(X, \mathbb{C})_{\text{prim}} \subset H^n(X, \mathbb{C})\right) = \begin{cases} 0, & \textit{if } n \textit{ is odd}, \\ 1, & \textit{if } n \textit{ is even}. \end{cases}$$

Proof. Consider the Gysin long exact sequence associated to the inclusion $i \colon X \hookrightarrow \mathbb{P}^{n+1}$ and its complement $j \colon U := \mathbb{P}^{n+1} \setminus X \hookrightarrow \mathbb{P}^{n+1}$. In the relevant degrees, we have:

$$\cdots \to H^n(X, \mathbb{C}) \xrightarrow{i_*} H^{n+2}(\mathbb{P}^{n+1}, \mathbb{C}) \xrightarrow{j^*} H^{n+2}(U, \mathbb{C}) \to \cdots$$

The map i_* corresponds to cupping with the Kähler class $\omega \in H^2(\mathbb{P}^{n+1}, \mathbb{C})$, i.e.,

$$i_*(\alpha) = \omega \wedge \alpha.$$

Thus, the primitive cohomology is the kernel of this map:

$$H^n(X,\mathbb{C})_{\text{prim}} = \ker\left(\omega \wedge (\cdot)\colon H^n(X,\mathbb{C}) \to H^{n+2}(\mathbb{P}^{n+1},\mathbb{C})\right).$$

Now, observe that:

- If n is odd, then $n+2$ is odd, so $H^{n+2}(\mathbb{P}^{n+1},\mathbb{C}) = 0$, since cohomology of projective space is concentrated in even degrees. Hence the map is identically zero, and the kernel is the whole space:

$$H^n(X,\mathbb{C})_{\text{prim}} = H^n(X,\mathbb{C}).$$

- If n is even, then $H^{n+2}(\mathbb{P}^{n+1},\mathbb{C}) \cong \mathbb{C}$, and the image of the map is generated by $\omega^{(n+2)/2}$. Hence, the kernel is the orthogonal complement of the one-dimensional subspace generated by $\omega^{n/2}$, and has codimension 1.

□

Theorem 2.3.12 (Griffiths)**.** *Let $X = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ a smooth hypersurface of degree d, then we have*

$$R^F_{d(q+1)-n-2} \simeq F^pH^n(X,\mathbb{C})_{prim}/F^{p+1}H^n(X,\mathbb{C})_{prim} \simeq H^{p,q}(X)_{prim},$$

where

$$R^F = \frac{\mathbb{C}[x_0,...,x_{n+1}]}{J^F}$$

is the Jacobian ring of F and $J^F = \langle \frac{\partial F}{\partial x_0},...,\frac{\partial F}{\partial x_{n+1}}\rangle$ is the Jacobian ideal of F.

For a detailed treatment of Griffiths
base theorem and the relationship with primitive cohomology, see [Cle00] and [Gri70].
Explicitly, to any $P \in R^F_{d(q+1)-n-2}$ we associate the form

$$\omega_P := res\left(\frac{P\Omega}{F^{q+1}}\right) \in \frac{F^pH^n(X,\mathbb{C})_{prim}}{F^{p+1}H^n(X,\mathbb{C})_{prim}},$$

where

$$\Omega = \sum_{i=0}^{n+1}(-1)^i x_i dx_0 \wedge ... \wedge \hat{dx_i} \wedge ... \wedge dx_{n+1}$$

is the standard top-degree form on $\mathbb{P}^{n+1}$.
A basis of $H^{p,q}(X)_{prim}$ is given by

$$\omega_\beta = res\left(\frac{x^\beta\Omega}{F^{q+1}}\right),$$

where $degx^\beta = \beta_0 + \beta_1 + ... + \beta_{n+1} = d(q+1) - n - 2$, with $\beta = (\beta_0, ..., \beta_{n+1})$, and $x^\beta = x_0^{\beta_0} \cdot ... \cdot x_{n+1}^{\beta_{n+1}}$.
The monomials x^β form a basis of the graded component $R^F_{d(q+1)-n-2}$.
In the case of the Klein hypersurface, one has:

$$R^F_{d(q+1)-n-2} = \frac{\mathbb{C}[x_0, ..., x_{n+1}]}{\langle (d-1)x_0^{d-2}x_1 + x_{n+1}^{d-1}, ..., x_n^{d-1} + (d-1)x_{n+1}^{d-2}x_0 \rangle}.$$

2.4 Polarized Hodge structures

Definition 2.4.1 (Polarized Hodge Structure)**.** *A* polarized Hodge structure *is a Hodge structure H with a bilinear form $Q : H \times H \to \mathbb{Z}$, called the* polarization*,*
*such that the $\mathbb{C}$-linear extension $Q : H_\mathbb{C} \times H_\mathbb{C} \to \mathbb{C}$ satisfies the following **Hodge–Riemann bilinear relations**:*

1. ***Orthogonality:***

$$Q(H^{p,q}, H^{p',q'}) = 0 \quad \textit{unless } p' = n - p \textit{ and } q' = n - q.$$

2. ***Symmetry (or skew-symmetry):*** *Q is $(-1)^n$-symmetric, i.e.,*

$$Q(x, y) = (-1)^n Q(y, x) \quad \textit{for all } x, y \in H.$$

3. ***Positivity (Riemann–Hodge bilinear relations):*** *For every nonzero $v \in H^{p,q}$, we have*

$$i^{p-q} Q(v, \bar{v}) > 0.$$

We can define the hermitian form $H : H_\mathbb{C} \times H_\mathbb{C} \to \mathbb{C}$ given by

$$H(\varphi, \psi) := i^n Q(\varphi, \bar{\psi}),$$

indeed, since Q is the $\mathbb{C}$-bilinear extension of a bilinear form with values in $\mathbb{Z}$, we have

$\overline{Q(\varphi,\psi)} = Q(\bar{\varphi},\bar{\psi})$. It follows that

$$\begin{aligned}\overline{H(\varphi,\psi)} &= \overline{i^n Q(\varphi,\bar{\psi})}\\ &= \overline{i^n}\,\overline{Q(\varphi,\bar{\psi})}\\ &= \overline{i^n}\,Q(\bar{\varphi},\bar{\bar{\psi}})\\ &= \overline{i^n}\,Q(\bar{\varphi},\psi)\\ &= (-1)^n\,\overline{i^n}\,Q(\psi,\bar{\varphi})\\ &= i^n\,Q(\psi,\bar{\varphi})\\ &= H(\psi,\varphi), \end{aligned} \tag{2.4.1}$$

(if n is even, we have $(-1)^n = 1$ and $i^n \in \mathbb{R}$ and then $(-1)^n(\bar{i^n}) = (\bar{i^n}) = i^n$, if n is odd, we have $(-1)^n = -1$ and $i^n \in \mathbb{C}\setminus\mathbb{R}$ and then $(-1)^n\bar{i^n} = -\bar{i^n} = -(-i^n) = i^n$.)

Proposition 2.4.2. *The Hermitian form H, previously defined, satisfies the following properties:*

1. ***Orthogonality of Hodge components:***

$$H\big(H^{p,q}, H^{p',q'}\big) = 0 \quad \text{if } p \neq p'.$$

2. ***Signature on each $H^{p,q}$:***

$$H(\varphi,\varphi)\begin{cases} > 0 & \text{if } q \text{ is even},\\ < 0 & \text{if } q \text{ is odd},\end{cases} \quad \text{for all } \varphi \in H^{p,q}\setminus\{0\}.$$

Proof. 1. Let be $\varphi \in H^{p,q}$ and $\psi \in H^{p',q'}$ with $p \neq p'$, then we have $\bar{\psi} \in \overline{H^{p',q'}} = H^{q',p'}$, and $p' \neq p = n-q$. Then, using the orthogonality property of Q, we obtain

$$H(\varphi,\psi) = i^n Q(\varphi,\bar{\psi}) = 0.$$

2. Let $\varphi \in H^{p,q}\setminus\{0\}$, then we have

$$\begin{aligned} H(\varphi,\varphi) &= i^n Q(\varphi,\bar{\varphi})\\ &= i^{p-q} i^{n+q-p} Q(\varphi,\bar{\varphi})\\ &= i^{p-q} Q(\varphi,\bar{\varphi}) i^{2q}\\ &= (-1)^q i^{p-q} Q(\varphi,\bar{\varphi}), \end{aligned} \tag{2.4.2}$$

and since $i^{p-q}Q(\varphi,\bar{\varphi}) > 0$, if q is even, we have $H(\varphi,\varphi) > 0$, and if q is odd we have $H(\varphi,\varphi) < 0$.

□

Theorem 2.4.3 (The Hodge-Riemann Bilinear Relations with the Hodge Index Theorem). *Let X be a compact Kähler manifold of complex dimension n. Then for every integer $k \geq 0$, we have a polarized Hodge structure (H, Q), where $H := H^k(X, \mathbb{Z})_{prim}$ and the polarization Q is given by $Q(\alpha, \beta) = Tr(\alpha \cup \beta \cup \omega^{n-k}) = \int_X \alpha \wedge \beta \wedge \omega^{n-k}$.*

For a detailed discussion of the Hodge-Riemann bilinear relations and the Hodge index theorem, see Griffiths and Harris' *Principles of Algebraic Geometry* [GH78], Hartshorne's *Algebraic Geometry* [Har77], and Demailly's *Complex Geometry and Lie Groups* [Dem12].

Definition 2.4.4. *A* morphism of Hodge structures *is a group homomorphism $f\colon H \to H'$ such that the complexified map $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ satisfies*

$$f_{\mathbb{C}}(H^{p,q}) \subseteq H'^{p,q} \quad \textit{for all } p, q.$$

If (H, Q) and (H', Q') are polarized Hodge structures, then f is a morphism of polarized Hodge structures *if, in addition, it satisfies:*

$$Q'\big(f_{\mathbb{C}}(\alpha), f_{\mathbb{C}}(\beta)\big) = Q(\alpha, \beta) \quad \textit{for all } \alpha, \beta \in H_{\mathbb{C}}.$$

Example 2.4.5. *Let X, Y be kahler compact manifolds and $f : X \to Y$ be an holomorphic function, then we have*

$$f^* : \Omega^k_{Y^\infty}(Y) \to \Omega^k_{X^\infty}(X),$$

further more

$$f^* : \mathcal{H}^k_d(Y) \to \mathcal{H}^k_d(X),$$

Since the composition of a harmonic function with a holomorphic function is harmonic. Then we have

$$f^* : \mathcal{H}^{p,q}_d(Y) \to \mathcal{H}^{p,q}_d(X),$$

$$f^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is a morphism of Hodge structures.

Remark 2.4.6. *Every morphism $f : X \to Y$ between smooth projective varieties induces morphisms*

$$f^* : H^k(Y, \mathbb{C})_{prim} \to H^k(X, \mathbb{C})_{prim}$$

of polarized Hodge structures.

Definition 2.4.7. *Given a polarized Hodge structure (H, Q), we define* Aut(H, Q) *to be the* *set of all automorphisms of polarized Hodge structures*

$$\varphi : (H, Q) \to (H, Q).$$

Proposition 2.4.8. *If* (H, Q) *is a polarized Hodge structure, then we have that* $\mathrm{Aut}(H, Q)$ *is finite.*

Proof. Let (H, Q) be a polarized Hodge structure. Then $\mathrm{Aut}(H, Q)$ is a subgroup of the group of automorphisms of H as a Hodge structure, and the latter is in turn a subgroup of the group of automorphisms of H as an abstract abelian group, that is, $H \simeq \mathbb{Z}^r$. Therefore,

$$\mathrm{Aut}(H, Q) < \mathrm{GL}(\mathbb{Z}^r).$$

Let $\sigma \in \mathrm{Aut}(H, Q)$. Then we have $\sigma_{\mathbb{C}}(H^{p,q}) = H^{p,q}$, since σ is an automorphism of Hodge structures. Moreover, $Q(\sigma_{\mathbb{C}}(u), \sigma_{\mathbb{C}}(v)) = Q(u, v)$, for all $u, v \in H_{\mathbb{C}}$, because σ is an automorphism of polarized Hodge structures.
Let $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$ be the complex dimension of $H^{p,q}$.
Let $\sigma_p : H^{p,q} \to H^{p,q}$ be such that $\sigma_p = \sigma_{\mathbb{C}}|_{H^{p,q}}$.
Then we have

$$\sigma_{\mathbb{C}} = \bigoplus_{p \leq n} \sigma_p.$$

Therefore, we have

$$\mathrm{Aut}(H, Q) = \prod_{p \leq \frac{n}{2}} \{\sigma_p \oplus \sigma_q : H^{p,q} \oplus H^{q,p} \to H^{p,q} \oplus H^{q,p}, \overline{\sigma_p(\alpha)} = \sigma_q(\bar{\alpha}), \sigma_p, \sigma_q \textit{ are isometries}\},$$

here we say that σ_p and σ_q are isometries because, for each $p \leq n$, we have

$$Q(\sigma_p(u), \sigma_p(v)) = Q(u, v), \forall u, v \in H_{\mathbb{C}},$$

and in particular,

$$Q(\sigma_p(u), \overline{\sigma_p(v)}) = Q(\sigma_{\mathbb{C}}(u), \overline{\sigma_{\mathbb{C}}(v)}) = Q(\sigma_{\mathbb{C}}(u), \sigma_{\mathbb{C}}(\bar{v})) = Q(u, \bar{v}).$$

Let us note that if $\alpha \in H^{p,q}$, then

$$\sigma_{\mathbb{C}}(\alpha + \bar{\alpha}) = \sigma_p(\alpha) + \sigma_q(\bar{\alpha}) = \overline{\sigma_p(\alpha) + \sigma_q(\bar{\alpha})}.$$

Thus, if $\alpha + \bar{\alpha} \in (H^{p,q} \oplus H^{q,p}) \cap H_{\mathbb{R}}$, then $\sigma_{\mathbb{C}}(\alpha + \bar{\alpha}) \in H_{\mathbb{R}}$, which implies that there exists a real-linear map

$$\sigma_{\mathbb{R}} : (H^{p,q} \oplus H^{q,p}) \cap H_{\mathbb{R}} \to (H^{p,q} \oplus H^{q,p}) \cap H_{\mathbb{R}}$$

such that

$$\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

Moreover, since $\overline{H^{p,q} \oplus H^{q,p}} = H^{p,q} \oplus H^{q,p}$, we have

$$H^{p,q} \oplus H^{q,p} = ((H^{p,q} \oplus H^{q,p}) \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Therefore

$$Aut(H,Q) < \prod_{p \leq \frac{n}{2}} \{\sigma_{\mathbb{R}} : (H^{p,q} \oplus H^{q,p}) \cap H_{\mathbb{R}} \to (H^{p,q} \oplus H^{q,p}) \cap H_{\mathbb{R}}, \sigma_{\mathbb{R}} \textit{ is an isometry}\} = \prod_{p \leq \frac{n}{2}} O(2h^{p,q}),$$

where the last product is a real compact group.
Since $\mathrm{Aut}(H,Q) < \mathrm{GL}(\mathbb{Z}^r)$, we have

$$\mathrm{Aut}(H,Q) < \prod_{p \leq \frac{n}{2}} O(2h^{p,q}) \cap \mathrm{GL}(\mathbb{Z}^r) < \prod_{p \leq \frac{n}{2}} O(2h^{p,q})$$

and $\prod_{p \leq \frac{n}{2}} O(2h^{p,q}) \cap \mathrm{GL}(\mathbb{Z}^r)$ is a discrete subset of the compact group $\prod_{p \leq \frac{n}{2}} O(2h^{p,q})$, therefore $\prod_{p \leq \frac{n}{2}} O(2h^{p,q}) \cap \mathrm{GL}(\mathbb{Z}^r)$ is finite and then $\mathrm{Aut}(H,Q)$ is finite.

□

Chapter 3

TORELLI PRINCIPLE

In this chapter, we will first provide, in the opening section, a motivation for the punctual Torelli principle by discussing the moduli space $\mathcal{H}_d^n$ and recalling the classical Torelli theorem. We will then state the strong Torelli principle, and finally conclude with the statement of the punctual Torelli principle. In the second section of this chapter, we will restrict our study to the Klein hypersurface. We begin by examining the polarized Hodge structure on $H^n(X,\mathbb{Z})_{prim}$, and we present a theorem giving a decomposition of $H^n(X,\mathbb{C})_{prim}$, into eigenspaces for the pullback action of the automorphism σ. Using this decomposition, we construct a homomorphism from the normalizer of σ^* in $\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}$ to a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$, which we denote by Δ. Finally, we show that if $C(\sigma^*)=\langle\sigma^*\rangle$, $N(\langle\sigma^*\rangle)=\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}$ and $\mu\simeq\langle\tau^*\rangle$, then

$$\mathrm{Aut}(X)\simeq\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}.$$

Finally, in the last section we present the groups Δ computed for several values of (n,d). In all these cases we find that $|\Delta|=n+2$, and hence

$$\Delta\simeq\mathbb{Z}/(n+2)\mathbb{Z}=\langle\tau\rangle.$$

This provides evidence in favor of the punctual Torelli principle for Klein hypersurfaces.

3.1 Torelli Principle

Let $\mathcal{H}_d^n$ be the moduli space of smooth degree d hypersurface in $\mathbb{P}^{n+1}$. Classically, this moduli space is constructed as the **GIT** quotient

$$\mathcal{H}_d^n = \mathscr{U}//\mathrm{PGL}_{n+2}(\mathbb{C})$$

of the open subset $\mathscr{U}$, corresponding to the complement of the discriminant divisor $\Delta \subseteq \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$, by the natural action of the reductive group $\mathrm{PGL}_{n+2}(\mathbb{C})$.
It follows from the classical results of [MM63, Cha78](Theorem 1.1.2) that, whenever $n \geq 1, d \geq 3$, and $(n,d) \notin \{(1,3),(2,4)\}$, the automorphism group of any hypersurface in $\mathcal{H}_d^n$ is a finite subgroup of $\mathrm{PGL}_{n+2}(\mathbb{C})$. For a general hypersurface $X \in \mathcal{H}_d^n$, the automorphism group is trivial; consequently, such points are smooth in $\mathcal{H}_d^n$. Conversely, any finite group G determines a (possibly empty) sublocus $\mathcal{H}_d^n(G)$, consisting of those smooth hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ for which $G \subseteq \mathrm{Aut}(X)$. In the case $G \subseteq \mathrm{PGL}_{n+2}(\mathbb{C})$ is not generated by pseudo-reflections we have $\mathcal{H}_d^n(G) \subseteq \mathrm{Sing}(\mathcal{H}_d^n)$, and thus in order to study the components of $\mathrm{Sing}(\mathcal{H}_d^n)$ it is natural to analyze which finite groups G are admissible (i.e., for which $\mathcal{H}_d^n(G) \neq \emptyset$). Since the correspondence $G \mapsto \mathcal{H}_d^n(G)$ is inclusion reversing, one may expect that the minimal admissible groups give information about the irreducible components of the singular locus of the moduli space $\mathcal{H}_d^n$. In the hierarchy of finite groups the simplest ones are cyclic groups $G \simeq \mathbb{Z}/m\mathbb{Z}$ and their admissibility in $\mathcal{H}_d^n$ was studied in [GAL11, VGA13, Zhe22]. Moreover, in [VGA13](Theorem 1.3.10) it was shown that if $n \geq 2$ and $d \geq 3$ with $(n,d) \neq (2,4)$, and if $\Phi_{n+2}(1-d)$ is prime, then the Klein hypersurface is the unique smooth projective hypersurface in $\mathcal{H}_d^n$ admitting an automorphism of prime order $p > (d-1)^n$

$$X = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + ... + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0 = 0\} \subseteq \mathbb{P}^{n+1},$$

(i.e. $\mathcal{H}_d^n(\mathbb{Z}/p\mathbb{Z}) = \{X\}$ is 0-dimensional) and in that case

$$p = \Phi_{n+2}(1-d).$$

In order to understand which irreducible components of $\mathrm{Sing}(\mathcal{H}_d^n)$ contain the Klein hypersurface one would like to know which other groups act faithfully on it, which leads us to analyze its full automorphism group.
The study of automorphisms of Klein hypersurfaces is a classical subject and begins with the paper by Felix Klein [Kle79b] studying the automorphism group of the quartic Klein curve $X \subseteq \mathbb{P}^2$, where $d = 4$ and $n = 1$, and proving that $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$ in that case. The case of Klein cubic curve $X \subseteq \mathbb{P}^2$ is also classical.

In [GALMVL24], the authors proved the following theorem, which provides evidence in favor of the so-called Strong Torelli Principle for hypersurfaces.

Theorem 3.1.1. *The middle primitive cohomology group of the Klein hypersurface of Wagstaff type X of dimension $n \geq 3$ and degree $d \geq 3$ is an extremal polarized Hodge structure. Moreover,*

$$\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim})/\{\pm 1\} \simeq \mathrm{Aut}(X)$$

in the following cases:
(a) $d \mid n+3$,
(b) $d = 3$ and $n \geq 5$.

Recall that the classical Torelli Theorem [And58, ACGH85] states that for any pair of curves X and X' of genus g, every isomorphism between their poralized Jacobian varieties $J(X) \simeq J(X')$ is induced, possible up to an involution, by a unique isomorphism of curves $X \simeq X'$. Since the Jacobian of a curve X is determined by its Hodge structure $J(X) = (H^{1,0}(X))^*/H_1(X,\mathbb{Z})$, an isomorphism between polarized Jacobian varieties $J(X) \simeq J(X')$ is represented by an isomorphism between the corresponding polarized Hodge structures. The Torelli Theorem motivated the following conjecture commonly called the Strong Torelli Principle:

Conjecture 3.1.2 (Strong Torelli Principle)**.** *Let X and X' be two smooth hypersurfaces of degree d and dimension n on $\mathbb{P}^{n+1}$. Then every isomorphism of polarized Hodge structures $\varphi : H^n(X,\mathbb{Z})_{prim} \to H^n(X',\mathbb{Z})_{prim}$ preserving polarizations is induced, possible up to an involution, by a unique isomorphism $\psi : X \to X'$.*

The weaker statement where we only ask for the existence of the isomorphisms φ and ψ without requering the former being induced y the latter is called the Global Torelli Principle.

If we take $X = X'$ in the Strong Torelli Principle, we obtain what we call the Punctual Torelli Principle

Conjecture 3.1.3 (Punctual Torelli Principle)**.** *Let X be a smooth hypersurface of degree d and and dimension n on $\mathbb{P}^{n+1}$. Then every automorphism of polarized Hodge structures $H^n(X,\mathbb{Z})_{prim}$ preserving polarizations is induced, possible up to an involution, by unique automorphism of X.*

Thus, the Strong Torelli Principle tells us that if $X = \{F = 0\}, Y = \{G = 0\} \subseteq \mathbb{P}^{n+1}$ are smooth hypersurfaces of degree d and dimension n, then, up to sign, every morphism

$$\varphi : H^n(Y,\mathbb{Z})_{prim} \to H^n(X,\mathbb{Z})_{prim}$$

of polarized Hodge structures arises from a unique morphism of varieties

$$f : X \to Y$$

such that

$$f^* = \pm\varphi.$$

In particular, if $X = Y$, the Punctual Torelli Principle tells us that

$$Aut(X) \simeq Aut(H^n(X,\mathbb{Z})_{prim}, \langle .,. \rangle)/\{\pm 1\}.$$

3.2 Automorphisms of polarized Hodge structures

In the case where $X = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + ...x_{n+1}^{d-1}x_0 = 0\}$ is the Klein hypersurface of degree d and dimension n, we previously (1.5.9) proved that

$$Aut(X) = \langle \sigma \rangle \rtimes \langle \tau \rangle = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/(n+2)\mathbb{Z},$$

where

$$m = \frac{(d-1)^{n+2} - (-1)^{n+2}}{d},$$

in the following cases:

- $n = 2$ and $d \geq 5$,
- $n = 3$ and $d \geq 4$,
- $n \geq 4$ and $d \geq 3$.

Recall that in the case where X is the Klein hypersurface, we have:

$$Aut(X) = \langle \sigma \rangle \rtimes \langle \tau \rangle = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/(n+2)\mathbb{Z},$$

where $\sigma : X \to X$ is given by

$$\sigma([x_0 : ... : x_{n+1}]) = [\xi_{md}x_0 : \xi_{md}^{1-d} : ... : \xi_{md}^{(1-d)^{n+1}} x_{n+1}],$$

and

$$m = \frac{(d-1)^{n+2} - (-1)^{n+2}}{d},$$

and ξ_{md} denotes a primitive md-th root of unity.
And in this case, since F is the Klein form (of degree d in $n+2$ variables), a basis for

$R^F_{d(q+1)-n-2}$ is given by the monomials x^β, where $\beta = (\beta_0, ..., \beta_{n+1})$ satisfies

$$0 \leq \beta_i \leq d-2 \quad \text{for all } i, \quad \text{and} \quad \sum_{i=0}^{n+1} \beta_i = d(q+1) - n - 2.$$

The automorphism σ induces an automorphism of polarized Hodge structures, $\sigma^* : H^n(X, \mathbb{Z})_{prim} \to H^n(X, \mathbb{Z})_{prim}$, then, the complexified map $\sigma^*_{\mathbb{C}} : H^n(X, \mathbb{C})_{prim} \to H^n(X, \mathbb{C})_{prim}$ satisfies

$$\sigma^*_{\mathbb{C}}(H^{p,q}(X)_{prim}) \subseteq H^{p,q}(X)_{prim},$$

where

$$\sigma^*_{\mathbb{C}}(\omega_P) = res\left(\frac{\sigma^* P \sigma^* \Omega}{(\sigma^* F)^{q+1}}\right),$$

$\sigma^* F = F$, since $\sigma \in Lin(X)$ and $H^n(X, \mathbb{C})_{prim} = \bigoplus_{p+q=n} H^{p,q}(X)_{prim}$, and for a fix q, a basis for $H^{p,q}(X)_{prim}$ is given by the forms $\omega_\beta := \omega_{x^\beta}$, where $\beta = (\beta_0, ..., \beta_{n+1})$ satisfies

$$0 \leq \beta_i \leq d-2 \quad \text{for all } i, \quad \text{and} \quad \sum_{i=0}^{n+1} \beta_i = d(q+1) - n - 2.$$

We have

$$\begin{aligned}
\sigma^*_{\mathbb{C}}(x^\beta) &= \xi_{md}^{\beta_0 + (1-d)\beta_1 + \ldots + (1-d)^{n+1}\beta_{n+1}} x^\beta, \\
\sigma^*_{\mathbb{C}}(\Omega) &= \sum_{i=0}^{n+1} (-1)^i (\xi_{md}^{(1-d)^i} x_i) d(\widehat{\xi_{md}^{(1-d)^i} x_i}) \\
&= \xi_{md}^{1 + (1-d) + \ldots + (1-d)^{n+1}} \Omega
\end{aligned}$$

Therefore

$$\begin{aligned}
\sigma^*_{\mathbb{C}}(\omega_\beta) &= \xi_{md}^{\sum_{i=0}^{n+1} (\beta_i + 1)(1-d)^i} \omega_\beta \\
&= \xi_m^{\sum_{i=0}^{n+1} \frac{(\beta_i+1)(1-d)^i}{d}} \omega_\beta,
\end{aligned}$$

we claim that $\sum_{i=0}^{n+1} \frac{(\beta_i+1)(1-d)^i}{d} \in \mathbb{Z}$. Since

$$\sum_{i=0}^{n+1} \beta_i = d(q+1) - n - 2,$$

we have that $\sum_{i=0}^{n+1} (\beta_i + 1)$ is a multiple of d. And for $i \in \mathbb{N}$, we have

$$(1-d)^i = 1 + M_i d,$$

where $M_i \in \mathbb{Z}$ (this follows directly from the binomial theorem). Therefore, we have

$$\begin{aligned}\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i &= \sum_{i=0}^{n+1}(\beta_i+1)(1+M_i d)\\ &= \sum_{i=0}^{n+1}(\beta_i+1)+\sum_{i=0}^{n+1}(\beta_i+1)M_i d\\ &= d(q+1)+d\sum_{i=0}^{n+1}(\beta_i+1)M_i\\ &= d[(q+1)+\sum_{i=0}^{n+1}(\beta_i+1)M_i],\end{aligned}$$

Therefore, $\sum_{i=0}^{n+1}\frac{(\beta_i+1)(1-d)^i}{d} \in \mathbb{Z}$ and then $\xi_m^{\sum_{i=0}^{n+1}\frac{(\beta_i+1)(1-d)^i}{d}} \in \mathbb{Q}(\xi_m)$.
Therefore $\omega_\beta \in V(\xi_m^j)$, where $j \equiv \sum_{i=0}^{n+1}\frac{(\beta_i+1)(1-d)^i}{d} \pmod{m}$.

Proposition 3.2.1. *Let n be an even number, and $d \geq 3$. Let $\beta = (\beta_0, ..., \beta_{n+1}), \beta' = (\beta'_0, ..., \beta'_{n+1})$ be such that $0 \leq \beta_i, \beta'_i \leq d-2$. If $\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i \equiv \sum_{i=0}^{n+1}(\beta'_i+1)(1-d)^i$ $(\operatorname{mod} md)$, then $\beta = \beta'$, or $\beta, \beta' \in \{(0, d-2, 0, d-2, ..., 0, d-2), (d-2, 0, d-2, 0, ..., d-2, 0)\}$.*

Proof. Let n be an even number, and $d \geq 3$. If $\beta, \beta' \in \{0, 1, ..., d-2\}^{n+2}$, we have

$$|\beta_i - \beta'_i| \leq (d-2), \forall i = 0, ..., n+1,$$

therefore

$$\begin{aligned}|\sum_i(\beta_i-\beta'_i)(1-d)^i| &\leq \sum_i |(\beta_i-\beta'_i)|(d-1)^i\\ &\leq \sum_i (d-2)(d-1)^i\\ &= \sum_i (d-1)^{i+1}-(d-1)^i\\ &= (d-1)^{n+2}-1\\ &= (d-1)^{n+2}-(-1)^{n+2}\\ &= md.\end{aligned}$$

If we have

$$|\sum_i(\beta_i-\beta'_i)(1-d)^i| = md,$$

then

$$\sum_i (\beta_i - \beta_i')(1-d)^i = md \ or \ \sum_i (\beta_i - \beta_i')(1-d)^i = -md.$$

If

$$\sum_i (\beta_i - \beta_i')(1-d)^i = md,$$

then

$$\sum_{i=0}^{n+1} (\beta_i - \beta_i')(1-d)^i = (d-1)^{n+2} - 1.$$

Since

$$\begin{aligned}\sum_{i=0}^{n} (\beta_i - \beta_i')(1-d)^i &\leq |\sum_{i=0}^{n} (\beta_i - \beta_i')(1-d)^i| \\ &\leq \sum_{i=0}^{n} |\beta_i - \beta_i'|(d-1)^i \leq \sum_{i=0}^{n} (d-2)(d-1)^i \\ &= \sum_{i=0}^{n} (d-1)^{i+1} - (d-1)^i \\ &= (d-1)^{n+1} - 1,\end{aligned}$$

we have

$$\begin{aligned}(\beta_{n+1} - \beta_{n+1}')(1-d)^{n+1} &= (d-1)^{n+2} - 1 - \sum_{i=0}^{n} (\beta_i - \beta_i')(1-d)^i \\ &\geq (d-1)^{n+2} - 1 - ((d-1)^{n+1} - 1) \\ &= (d-1)^{n+2} - (d-1)^{n+1} \\ &= (d-2)(d-1)^{n+1},\end{aligned}$$

therefore

$$\beta_{n+1} - \beta_{n+1}' \leq 2 - d$$

therefore, we have $\beta_{n+1} - \beta_{n+1}' = 2 - d$, then, the only option is $\beta_{n+1} = 0$ and $\beta_{n+1}' = d-2$. So, we have

$$\sum_{i=0}^{n} (\beta_i - \beta_i')(1-d)^i + (2-d)(1-d)^{n+1} = (d-1)^{n+2} - 1$$

$$\Rightarrow \sum_{i=0}^{n} (\beta_i - \beta_i')(1-d)^i = (d-1)^{n+1} - 1,$$

again, we can see that

$$\sum_{i=0}^{n-1}(\beta_i-\beta'_i)(1-d)^i \leq (d-1)^n-1,$$

therefore

$$\begin{aligned}(\beta_n-\beta'_n)(1-d)^n &= (d-1)^{n+1}-1-\sum_{i=0}^{n-1}(\beta_i-\beta'_i)(1-d)^i\\ &\geq (d-1)^{n+1}-1-((d-1)^n-1)\\ &= (d-2)(d-1)^n,\end{aligned}$$

therefore $\beta_n-\beta'_n=d-2$, and then $\beta_n=d-2$ and $\beta'_n=0$.
By repeating the same procedure, we obtain that $\beta_{n-1}=0$ and $\beta'_{n-1}=d-2$. Proceeding inductively, we arrive at

$$\beta=(d-2,0,d-2,0,...,d-2,0,d-2,0) \text{ and } \beta'=(0,d-2,0,d-2,...,0,d-2,0,d-2).$$

If

$$\sum_i(\beta_i-\beta'_i)(1-d)^i=-md,$$

then

$$\sum_i(\beta'_i-\beta_i)(1-d)^i=md,$$

therefore $\beta=(0,d-2,0,d-2,...,0,d-2,0,d-2)$ and $\beta'=(d-2,0,d-2,0,...,d-2,0,d-2,0)$.
Let $\beta,\beta'\in\{0,...,d-2\}^{n+2}$. If

$$\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i \equiv \sum_{i=0}^{n+1}(\beta'_i+1)(1-d)^i \pmod{md},$$

we have

$$\sum_{i=0}^{n+1}(\beta_i-\beta'_i)(1-d)^i\equiv 0 \pmod{md}.$$

Since

$$|\sum_{i=0}^{n+1}(\beta_i-\beta'_i)(1-d)^i|\leq md,$$

we have

$$\sum_{i=0}^{n+1}(\beta_i-\beta'_i)(1-d)^i=0 \text{ or } |\sum_{i=0}^{n+1}(\beta_i-\beta'_i)(1-d)^i|=md.$$

If

$$\sum_{i=0}^{n+1}(\beta_i - \beta'_i)(1-d)^i = 0,$$

since $\beta \in \{0, ..., d-2\}^{n+2}$, we have $|\beta_i - \beta'_i| \in \{0, ..., d-2\}$, therefore, by the lemma 1.3.9, we have $\beta_i - \beta'_i = 0$, for all $i = 0, ..., n+1$, therefore

$$\beta = \beta'.$$

If

$$|\sum_{i=0}^{n+1}(\beta_i - \beta'_i)(1-d)^i| = md,$$

we have $\beta, \beta' \in \{(0, d-2, 0, d-2, ..., 0, d-2), (d-2, 0, d-2, 0, ..., d-2, 0)\}$. □

By the previous proposition, we have that if n is an even number, and $d \geq 3$, if $\beta = (\beta_0, ..., \beta_{n+1}), \beta' = (\beta'_0, ..., \beta'_{n+1})$ be such that $0 \leq \beta_i, \beta'_i \leq d-2$ and

$$\sum_{i=0}^{n+1}\beta_i = \sum_{i=0}^{n+1}\beta'_i = d(q+1) - n - 2,$$

then we have that if $\sum_{i=0}^{n+1} \frac{(\beta_i+1)(1-d)^i}{d} \equiv \sum_{i=0}^{n+1} \frac{(\beta'_i+1)(1-d)^i}{d} \pmod{m}$, then $\beta = \beta'$ or $\beta, \beta' \in \{(0, d-2, 0, d-2, ..., 0, d-2), (d-2, 0, d-2, 0, ..., d-2, 0)\}$.
By Griffiths' base theorem, we have that

$$\dim H^n(X, \mathbb{C})_{prim} = \begin{cases} m-1 & \text{if } n \text{ is odd} \\ m+1 & \text{if } n \text{ is even} \end{cases}$$

In the case where n is even, we have

$$\dim H^n(X, \mathbb{C})_{\text{prim}} = m+1,$$

therefore

$$\bigcup_{p+q=n}\left\{\omega_\beta \;\middle|\; 0 \leq \beta \leq d-2,\ \sum_i \beta_i = d(q+1) - n - 2\right\},$$

which is a basis of $H^n(X, \mathbb{C})_{prim}$, has $m+1$ elements.
That is, we have a set of $m+1$ vectors

$$\beta^0 = (\beta^0_0, ..., \beta^0_{n+1}), \beta^1 = (\beta^1_0, ..., \beta^1_{n+1}), ..., \beta^m = (\beta^m_0, ..., \beta^m_{n+1}),$$

such that the elements

$$\omega_{\beta^0}, ..., \omega_{\beta^m}$$

form a basis of $H^n(X,\mathbb{C})_{prim}$. We have that the vectors

$$\beta' = (0, d-2, 0, d-2, ..., 0, d-2, 0, d-2), \beta'' = (d-2, 0, d-2, 0, ..., d-2, 0, d-2, 0)$$

are such that $\omega_{\beta'}, \omega_{\beta''} \in H^{\frac{n}{2},\frac{n}{2}}(X)_{prim}$, since

$$\sum_i \beta'_i = \sum_i \beta''_i = (d-2)\frac{n+2}{2} = d(\frac{n}{2}+1) - n - 2.$$

And we have $\omega_{\beta'}, \omega_{\beta''} \in V(\xi_m^0) = V(1)$, since

$$\sum_i (\beta'_i + 1)(1-d)^i = -md \ and \ \sum_i (\beta''_i + 1)(1-d)^i = 0.$$

Therefore, each of the remaining $m-1$ vectors β^k has an associated eigenvalue $\xi_m^{j(k)}$, with the $j(k)$ being pairwise distinct.
That is, we have an injective function

$$j : I \to \{1, ..., m-1\},$$

where I is the set indexing the remaining $m-1$ vectors.
Since both I and the codomain have $m-1$ elements, j is a biyection.
Therefore

$$H^n(X,\mathbb{C})_{prim} = \bigoplus_{j=0}^{m-1} V(\xi_m^j),$$

where

$$\dim V(\xi_m^j) = \begin{cases} 1 & \text{if } m > 0 \\ 2 & \text{if } m = 0. \end{cases}$$

If n is odd, since

$$(\beta_k + 1)(1-d)^k + (\beta_{k+1} + 1)(1-d)^{k+1} \leq (d-1)^{k+1}(1 - (\beta_{k+1} + 1)) \leq 0, \forall k \ even$$

we have

$$\begin{aligned}
\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i &= (\beta_{n+1}+1)(1-d)^{n+1} \\
&+ \sum_{k\in\{0,2,4,\dots,n-1\}} (\beta_k+1)(1-d)^k + (\beta_{k+1}+1)(1-d)^{k+1} \\
&\leq (\beta_{n+1}+1)(1-d)^{n+1} \\
&\leq (d-1)^{n+2} \\
&< (d-1)^{n+2}+1 \\
&= (d-1)^{n+2}-(-1)^{n+2} \\
&= dm,
\end{aligned}$$

therefore

$$\sum_{i=0}^{n+1}\frac{(\beta_i+1)(1-d)^i}{d} < m.$$

And since

$$(\beta_k+1)(1-d)^k + (\beta_{k+1}+1)(1-d)^{k+1} \geq \beta_{k+1}(d-1)^{k+1} \geq 0, \forall k \ odd,$$

we have

$$\begin{aligned}
\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i &= \beta_0+1 \\
&+ \sum_{k\in\{1,3,5,\dots,n\}} (\beta_k+1)(1-d)^k + (\beta_{k+1}+1)(1-d)^{k+1} \\
&\geq \beta_0+1 \\
&\geq 1 \\
&> 0,
\end{aligned}$$

therefore

$$\sum_{i=0}^{n+1}\frac{(\beta_i+1)(1-d)^i}{d} > 0.$$

Therefore $V(1)=V(\xi_m^0)=\emptyset$.

And we have

$$\begin{aligned}
|\sum_{i=0}^{n+1}(\beta_i-\beta_i')(1-d)^i| &\leq \sum_{i=0}^{n+1}|(\beta_i-\beta_i')||(1-d)^i| \\
&\leq \sum_{i=0}^{n+1}(d-2)(d-1)^i \\
&= \sum_{i=0}^{n+1}(d-1)^{i+1}-(d-1)^i \\
&= (d-1)^{n+2}-1 \\
&< (d-1)^{n+2}+1 \\
&= (d-1)^{n+2}-(-1)^{n+2} \\
&= md,
\end{aligned}$$

therefore, if we have

$$\sum_{i=0}^{n+1}\frac{(\beta_i+1)(1-d)^i}{d} \equiv \sum_{i=0}^{n+1}\frac{(\beta_i'+1)(1-d)^i}{d} \pmod m,$$

then we have

$$\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i \equiv \sum_{i=0}^{n+1}(\beta_i'+1)(1-d)^i \pmod{md},$$

and then

$$\sum_{i=0}^{n+1}(\beta_i+1)(1-d)^i = \sum_{i=0}^{n+1}(\beta_i'+1)(1-d)^i,$$

and by the lemma 1.3.9 we have

$$\beta_i-\beta_i'=0, \forall i=0,...,n+1,$$

Therefore $\beta=\beta'$.
Since n is odd, we have

$$\dim H^n(X,\mathbb{C})_{\text{prim}} = m-1,$$

therefore

$$\bigcup_{p+q=n}\left\{\omega_\beta \,\middle|\, 0\leq\beta\leq d-2,\ \sum_i\beta_i=d(q+1)-n-2\right\},$$

which is a basis of $H^n(X,\mathbb{C})_{prim}$, has $m-1$ elements.
That is, we have a set of $m-1$ vectors

$$\beta^0=(\beta_0^0,...,\beta_{n+1}^0),\beta^1=(\beta_0^1,...,\beta_{n+1}^1),...,\beta^{m-2}=(\beta_0^{m-2},...,\beta_{n+1}^{m-2}),$$

such that the elements

$$\omega_{\beta^0}, ..., \omega_{\beta^{m-2}}$$

form a basis of $H^n(X,\mathbb{C})_{prim}$.
Therefore, each of the $m-1$ vectors β^k has an associated eigenvalue $\xi_m^{j(k)}$, with the $j(k)$ being pairwise distinct. That is, we have an injective function

$$j : I \to \{1, ..., m-1\},$$

where I is the set indexing the $m-1$ vectors.
Since both I and the codomain have $m-1$ elements, j is a biyection.
Therefore

$$H^n(X,\mathbb{C})_{prim} = \bigoplus_{j=1}^{m-1} V(\xi_m^j),$$

where

$$\dim V(\xi_m^j) = 1, \forall j = 1, ..., m-1.$$

Definition 3.2.2. *For every $p+q=n$, we define the set*

$$C^{p,q} := \{j = \sum_{i=0}^{n+1} \frac{(\beta_i+1)(1-d)^i}{d} \mid 0 \le \beta_i \le d-2, \sum_{i=0}^{n+1} \beta_i = d(q+1)-n-2\}.$$

With all this, we have the following theorem.

Theorem 3.2.3 (Villaflor et.al.)**.** *Let $X = \{x_0^{d-1}x_1 + ... + x_{n+1}^{d-1}x_0 = 0\}$ be the Klein hypersuperface of dimension n and degree $d \ge 3$. And let $V(\xi_m^j) = \{\omega \in H^n(X,\mathbb{C})_{prim} | \sigma_{\mathbb{C}}^*(\omega) = \xi_m^j \omega\}$ be the eigenspace of $\sigma_{\mathbb{C}}^*$ associated to the eigenvalue ξ_m^j.*
Then, for all $p+q=n$, we have

$$H^{p,q}(X)_{prim} = \bigoplus_{j \in C^{p,q}} V(\xi_m^j).$$

If n is an even number, we have

$$H^n(X,\mathbb{C})_{prim} = \bigoplus_{j=0}^{m-1} V(\xi_m^j),$$

where $\dim_{\mathbb{C}} V(\xi_m^j) = 1$ *if* $j = 1, ..., m-1$ *and* $\dim_{\mathbb{C}} V(\xi_m^j) = 2$ *if* $j = 0$.
If n is an odd number, we have

$$H^n(X,\mathbb{C})_{prim} = \bigoplus_{j=1}^{m-1} V(\xi_m^j),$$

where $\dim_{\mathbb{C}} V(\xi_m^j) = 1$ *for all* $j = 1, ..., m-1$.
And for $j \in C^{p,q}$, *we have*

$$V(\xi_m^j) = \langle \{\omega_\beta \,|\, 0 \leq \beta_i \leq d-2, \sum_i \beta_i = d(q+1) - n - 2, \sum_i \frac{(\beta_i+1)(1-d)^i}{d} = j\} \rangle,$$

where

$$\omega_\beta = res\left(\frac{x^\beta \Omega}{F^{q+1}}\right).$$

Definition 3.2.4. *Let define the set*

$$\Delta := \{t \in (\mathbb{Z}/m\mathbb{Z})^\times : t \cdot C^{p,q} = C^{p,q}, \forall p+q=n\},$$

i.e.

$$t \in \Delta \Leftrightarrow t \in (\mathbb{Z}/m\mathbb{Z})^\times \text{ and } \forall p+q=n, \text{ we have } t \cdot j \in C^{p,q}, \forall j \in C^{p,q}.$$

Definition 3.2.5. *Let we define the morphism* $\varphi : N(\langle \sigma^* \rangle) < \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim}, \langle .,. \rangle)/\{\pm 1\} \to \Delta$ *given by* $g \mapsto i_g$, *where* $g\sigma^* g^{-1} = \sigma^{*i_g}$.

Proposition 3.2.6. φ *is well defined*

Proof. Let $g \in N(\langle \sigma^* \rangle)$, then we have $g \in \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim}, \langle .,. \rangle)/\{\pm 1\}$, therefore it must preserve the Hodge decomposition

$$g_{\mathbb{C}}(H^{p,q}(X)_{prim}) \subseteq H^{p,q}(X)_{prim},$$

let $i_g = \varphi(g)$ and $\omega \in H^n(X,\mathbb{C})_{prim}$. If $\omega \in H^{p,q}(X)_{prim}$, then $g^{-1}\omega \in H^{p,q}(X)_{prim}$, therefore $g_{\mathbb{C}}^{-1}\omega \in V(\xi_m^j)$, for some $j \in C^{p,q}$. Then we have

$$g_{\mathbb{C}} \sigma_{\mathbb{C}}^* g_{\mathbb{C}}^{-1} \omega = g_{\mathbb{C}} \xi_m^j g_{\mathbb{C}}^{-1} \omega = \xi_m^j \omega.$$

On the other hand, we have

$$g\sigma^* g^{-1} = \sigma^{*i_g},$$

then, if $\omega \in V(\xi_m^k)$, we have

$$g_{\mathbb{C}} \sigma_{\mathbb{C}}^* g_{\mathbb{C}}^{-1} \omega = \sigma^{*i_g} \omega = (\xi_m^k)^{i_g} \omega = \xi_m^{k \cdot i_g} \omega.$$

Therefore

$$j \equiv k \cdot i_g \pmod m,$$

therefore $i_g \cdot C^{p,q} \subseteq C^{p,q}$ and since g is an automorphism, we have $i_g \in (\mathbb{Z}/m\mathbb{Z})^\times$.

□

Let $\mu := \mathrm{Im}\,\varphi$. It is easy to see that $\mathrm{Ker}\,\varphi = C(\sigma^*)$, the centralizer of σ^*. Therefore, by the

First Isomorphism Theorem, we obtain

$$N(\langle\sigma^*\rangle)/C(\sigma^*) \simeq \mu.$$

Recall that

$$\mathrm{Aut}(X) = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/(n+2)\mathbb{Z} = \langle\sigma\rangle \rtimes \langle\tau\rangle.$$

$\tau : X \to X$ is given by

$$\tau([x_0 : \,...\, : x_{n+1}]) = [x_1 : \,...\, : x_0],$$

then we have that the automorphism τ induces an automorphism of polarized Hodge structures, $\tau^* : H^n(X,\mathbb{Z})_{prim} \to H^n(X,\mathbb{Z})_{prim}$.
Let us show that τ^* lies in $N(\langle\sigma^*\rangle)$. For $[x_0 : \,...\, : x_{n+1}] \in X$, we have

$$\begin{aligned}\tau\sigma\tau^{-1}(x_0 : \,...\, : x_{n+1}) &= \tau\sigma(x_{n+1} : \,...\, : x_n)\\ &= \tau(\xi_{md}x_{n+1} : \,...\, : \xi_{md}^{(1-d)^{n+1}}x_n)\\ &= [\xi_{md}^{1-d}x_0 : \,...\, : \xi_{md}x_{n+1}]\\ &= \sigma^{1-d}(x_0 : \,...\, : x_{n+1}),\end{aligned}$$

therefore $\tau\sigma\tau^{-1} = \sigma^{1-d}$. Passing to cohomology, we obtain

$$\tau^*\sigma^*\tau^{*-1} = \sigma^{*1-d}$$

hence $\langle\tau^*\rangle < N(\langle\sigma^*\rangle)$.
Let us show that $\langle tau^*\rangle \cap C(\sigma^*) = \{e\}$.
Take $j \in \mathbb{Z}/(n+2)\mathbb{Z}$ such that

$$\tau^j\sigma\tau^{-j} = \sigma.$$

Using the relation $\tau^j\sigma\tau^{-j} = \sigma^{(1-d)^j}$, we obtain

$$(1-d)^j \equiv 1 \,(\mathrm{mod}\ m).$$

Since the order of $(1-d)$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ is $n+2$, it follows that j is a multiple of $n+2$. Therefore

$$\langle\tau\rangle \cap C(\sigma) = \{e\}.$$

Applying pullback, we conclude that

$$\langle\tau^*\rangle \cap C(\sigma^*) = \{e\}.$$

in $\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}$.
Therefore, via the homomorphism φ, we obtain a commutative diagram of short exact sequences, with exact rows and injective vertical maps.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C(\sigma^*) & \longrightarrow & N(\langle\sigma^*\rangle) & \longrightarrow & \mu & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \langle\sigma\rangle & \longrightarrow & \langle\sigma\rangle \rtimes \langle\tau\rangle & \longrightarrow & \langle\tau\rangle & \longrightarrow & 1 \end{array}$$

We have

$$\langle\sigma^*\rangle < C(\sigma^*),\ \langle\tau^*\rangle < \mu < \Delta,\ \mathrm{Aut}(X) \simeq \langle\sigma^*\rangle \rtimes \langle\tau^*\rangle < N(\langle\sigma^*\rangle) < \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\},$$

therefore is we can prove that

$$\langle\sigma^*\rangle = C(\sigma^*) \ \textit{and}\ \mu \simeq \langle\tau^*\rangle,$$

then it follows that

$$\mathrm{Aut}(X) \simeq N(\langle\sigma^*\rangle).$$

Moreover, if we further show that $\langle\sigma^*\rangle$ is normal in $\mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}$, then we obtain

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\},$$

which is exactly what we want.

3.3 Evidence supporting the punctual Torelli principle on Klein varieties

At the end of the previous section, we showed that if we could prove $\langle\sigma^*\rangle = C(\sigma^*)$, $\mu \simeq \langle\tau^*\rangle$, and $N(\langle\sigma^*\rangle) = \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}$, then we would obtain $\mathrm{Aut}(X) \simeq \mathrm{Aut}(H^n(X,\mathbb{Z})_{prim},\langle .,.\rangle)/\{\pm 1\}$.
We conjecture that $|\Delta| = n+2$. Since

$$\mathbb{Z}/(n+2)\mathbb{Z} = \langle\tau^*\rangle < \mu < \Delta$$

if $|\Delta| = n+2$, it follows that

$$\Delta = \mathbb{Z}/(n+2)\mathbb{Z},$$

and hence

$$\mu = \mathbb{Z}/(n+2)\mathbb{Z} = \langle\tau^*\rangle.$$

For fixed (n, d), one can compute the group Δ explicitly by first determining the sets $C^{p,q}$ (Definition 3.2.2). However, when (n, d) are not small, the computation of these sets becomes prohibitively expensive, even with computer assistance.
We developed a Python code in which we implemented a function to compute the groups Δ for given ordered pairs (n, d).

We now present the tables of the groups Δ for $2 \le n \le 8$. In each table we record the corresponding value of m, the elements of Δ, the order $|\Delta|$, and a generator of this group. Thus, in these cases we verify that Δ is a cyclic group and that its order is $n + 2$.

Table 3.1. Values of $|\Delta|$ for $n = 2$ and $3 \le d \le 20$.

d	n	m	τ_1	τ_2	τ_3	τ_4	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	2	5	1	2	3	4	4	4	2	Yes
4	2	20	1	13	9	17	4	4	13	Yes
5	2	51	16	1	38	47	4	4	38	Yes
6	2	104	1	83	99	25	4	4	83	Yes
7	2	185	1	154	179	36	4	4	154	Yes
8	2	300	1	293	257	49	4	4	293	Yes
9	2	455	64	1	398	447	4	4	398	Yes
10	2	656	81	583	1	647	4	4	583	Yes
11	2	909	1	818	899	100	4	4	818	Yes
12	2	1220	1	1209	1109	121	4	4	1209	Yes
13	2	1595	144	1	1462	1583	4	4	1462	Yes
14	2	2040	169	1	1883	2027	4	4	1883	Yes
15	2	2561	1	2378	2547	196	4	4	2378	Yes
16	2	3164	1	2953	225	3149	4	4	2953	Yes
17	2	3855	256	1	3614	3839	4	4	3614	Yes
18	2	4640	1	4623	289	4367	4	4	4623	Yes
19	2	5525	1	5218	5507	324	4	4	5218	Yes
20	2	6516	1	6497	6173	361	4	4	6497	Yes

For example, taking $d = 20$, we can check that the group

$$\Delta = \{1, 6497, 363, 6173\}$$

is indeed a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$, and that it is generated by 6497. o see this, note first that for these values of n and d we obtain

$$m = \frac{(19)^4 - (-1)^4}{20} = 6516$$

which agrees with the value of m reported in Table 3.1. And if we consider the generator 6497, we have

$$6497 \cdot 6497 = 42211009,$$

and

$$42211009 \equiv 361 (\bmod 6516),$$

$$361 \cdot 6497 = 2345417 \equiv 6173 (\bmod 6516),$$

$$6173 \cdot 6497 = 40105981 \equiv 1 (\bmod 6516).$$

Table 3.2. Values of $|\Delta|$ for $n = 2$ and $21 \leq d \leq 40$.

d	n	m	τ_1	τ_2	τ_3	τ_4	$n+2$	$\|\Delta\|$	Generator	Cyclic
21	2	7619	400	1	7238	7599	4	4	7238	Yes
22	2	8840	441	1	8819	8419	4	4	8819	Yes
23	2	10185	1	9722	10163	484	4	4	9722	Yes
24	2	11660	1	11153	11637	529	4	4	11153	Yes
25	2	13271	576	1	12718	13247	4	4	12718	Yes
26	2	15024	1	14423	625	14999	4	4	14423	Yes
27	2	16925	1	16274	16899	676	4	4	16274	Yes
28	2	18980	1	729	18277	18953	4	4	18277	Yes
29	2	21195	784	1	20438	21167	4	4	20438	Yes
30	2	23576	841	22763	1	23547	4	4	22763	Yes
31	2	26129	1	25258	26099	900	4	4	25258	Yes
32	2	28860	1	961	28829	27929	4	4	28829	Yes
33	2	31775	1024	1	30782	31743	4	4	30782	Yes
34	2	34880	1089	33823	1	34847	4	4	33823	Yes
35	2	38181	1	37058	38147	1156	4	4	37058	Yes
36	2	41684	1	41649	1225	40493	4	4	41649	Yes
37	2	45395	1296	1	44134	45359	4	4	44134	Yes
38	2	49320	1369	47987	49283	1	4	4	47987	Yes
39	2	53465	1	52058	53427	1444	4	4	52058	Yes
40	2	57836	1	1521	57797	56353	4	4	57797	Yes

Table 3.3. Values of $|\Delta|$ for $n = 3$ and $3 \leq d \leq 17$.

d	n	m	τ_1	τ_2	τ_3	τ_4	τ_5	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	3	11	1	3	4	5	9	5	5	3	Yes
4	3	61	1	34	9	20	58	5	5	34	Yes
5	3	205	1	201	141	16	51	5	5	201	Yes
6	3	521	1	516	104	396	25	5	5	516	Yes
7	3	1111	1	36	1105	185	895	5	5	36	Yes
8	3	2101	1	300	2094	49	1758	5	5	300	Yes
9	3	3641	64	1	455	3633	3129	5	5	64	Yes
10	3	5905	1	5896	656	81	5176	5	5	5896	Yes
11	3	9091	1	100	909	9081	8091	5	5	100	Yes
12	3	13421	1	13410	1220	121	12090	5	5	13410	Yes
13	3	19141	1	17413	144	19129	1595	5	5	17413	Yes
14	3	26521	1	24324	169	26508	2040	5	5	24324	Yes
15	3	35855	1	35841	2561	196	33111	5	5	35841	Yes
16	3	47461	1	225	47446	44086	3164	5	5	225	Yes
17	3	61681	256	1	61665	3855	57585	5	5	256	Yes

Table 3.4. Values of $|\Delta|$ for $n = 4$ and $3 \leq d \leq 11$.

d	n	m	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	4	21	1	4	10	13	16	19	6	6	10	Yes
4	4	182	1	9	81	179	121	155	6	6	179	Yes
5	4	819	256	1	614	815	16	755	6	6	614	Yes
6	4	2604	1	2083	2599	2479	625	25	6	6	2083	Yes
7	4	6665	1	6659	36	1296	6449	5554	6	6	6659	Yes
8	4	14706	2401	1	14699	49	14363	12605	6	6	14699	Yes
9	4	29127	64	1	4096	28615	25486	29119	6	6	25486	Yes
10	4	53144	1	6561	47239	53135	81	52415	6	6	47239	Yes
11	4	90909	1	100	10000	90899	89909	81818	6	6	90899	Yes

Table 3.5. Values of $|\Delta|$ for $n = 5$ and $3 \leq d \leq 8$.

d	n	m	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	5	43	1	35	4	41	11	16	21	7	7	35	Yes
4	5	547	544	1	520	9	304	81	182	7	7	544	Yes
5	5	3277	256	1	3273	2253	3213	16	819	7	7	256	Yes
6	5	13021	12896	1	9896	2604	625	13016	25	7	7	12896	Yes
7	5	39991	1	36	6665	1296	39985	32215	39775	7	7	36	Yes
8	5	102943	2401	1	102600	49	14706	102936	86136	7	7	2401	Yes

Table 3.6. Values of $|\Delta|$ for $n = 6$ and $3 \leq d \leq 6$.

d	n	m	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	6	85	64	1	4	42	77	16	83	53	8	8	42	Yes
4	6	1640	1	1637	1093	9	1613	81	1397	729	8	8	1637	Yes
5	6	13107	4096	1	256	9830	13103	16	12083	13043	8	8	9830	Yes
6	6	65104	1	15625	65099	625	52083	64979	25	61979	8	8	65099	Yes

Table 3.7. Values of $|\Delta|$ for $n = 7$ and $3 \leq d \leq 5$.

d	n	m	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	7	171	64	1	163	4	169	43	139	16	85	9	9	4	Yes
4	7	4921	1	4678	1640	9	2734	81	4918	729	4894	9	9	4678	Yes
5	7	52429	4096	1	256	52425	36045	51405	52365	16	13107	9	9	256	Yes

Table 3.8. Values of $|\Delta|$ for $n = 8$ and $3 \leq d \leq 4$.

d	n	m	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_10	$n+2$	$\|\Delta\|$	Generator	Cyclic
3	8	341	64	1	256	4	170	333	16	339	213	309	10	10	170	Yes
4	8	14762	1	6561	14759	9	14735	81	9841	14519	729	12575	10	10	6561	Yes

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