Ayudantía 4 (MAT426)

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1. We say that an affine variety $X \subseteq \mathbb{A}^{n+1}$ is a **cone** if $0 \in X$ and if $x \in X$ then $\lambda x \in X$ for all $\lambda \in k$. Prove that there is a bijection between cones in $\mathbb{A}^{n+1}$ and closed subvarieties of $\mathbb{P}^n$.

   Proof. First we claim that cones correspond to affine varieties generated by a set S of non-constant homogeneous polynomials.

   Let $S \subseteq k[x_0, \dots, x_n]$ be such a set. We need to show that $V(S) = \{x \in \mathbb{A}^{n+1} : f(x) = 0 \quad \forall f \in S\}$ is a cone. Since f is homogeneous and non-constant, we have $f(0) = 0$ for all $f \in S$, thus $0 \in V(S)$. Further, let $x \in V(S)$ and $\lambda \in k$. Then for every $f \in S$ it holds $f(\lambda x) = \lambda^{deg_f} f(x) = \lambda^{deg_f} 0 = 0$, where the first equality holds since f is homogeneous and $f(x) = 0$ since $x \in V(S)$. Thus, $\lambda x \in V(S)$ and $V(S)$ is a cone.

   Now, let conversely $X \subseteq \mathbb{A}^{n+1}$ be a cone. We will show that $\mathcal{I}(X)$ is generated by a set of non-constant homogeneous polynomials S. Then we can conclude $X = V(\mathcal{I}(X)) = V(\langle S \rangle) = V(S)$.

   Since $k[x_0, \dots, x_n] = \bigoplus_{d \in \mathbb{N}} \{ \sum_{\substack{i_0, \dots, i_n \in \mathbb{N} \\ i_0 + \dots + i_n = d}} a_{i_0, \dots, i_n} x_0^{i_0} \cdots x_n^{i_n} : a_{i_0, \dots, i_n} \in k \quad \forall i_0, \dots, i_n \}$, every $f \in k[x_0, \dots, x_n]$ has a unique homogeneous decomposition, i.e. $f = \sum_{d \in \mathbb{N}} f_d$ with f_d homogeneous for all $d \in \mathbb{N}$ and only finitely many $f_d \neq 0$. So let $0 \neq f \in \mathcal{I}(X)$ with homogeneous decomposition $f = \sum_{d \in \mathbb{N}} f_d$. We need to show $f_d \in \mathcal{I}(X)$. Let $x \in X$. Since X is a cone, we have $\lambda x \in X$ for all $\lambda \in k$ and since $f \in \mathcal{I}(X)$ we get $0 = f(\lambda x) = \sum_{d \in \mathbb{N}} f_d(\lambda x) = \sum_{d \in \mathbb{N}} \lambda^d f_d(x)$, where the last equality follows since the f_d are homogeneous. Interpreting the last term as a polynomial in λ which is equal to the zero polynomial we get $f_d(x) = 0$ for all $d \in \mathbb{N}$. Since $x \in X$ was arbitrary, this shows $f_d \in \mathcal{I}(X)$ for all $d \in \mathbb{N}$. Thus, every polynomial in $\mathcal{I}(X)$ can be written as a finite sum of homogeneous polynomials. Further note, that since $0 \in X$ (X is a cone), $f(0) = 0$ for all $f \in \mathcal{I}(X)$, the generating polynomials are non-constant. Thus, $\mathcal{I}(X)$ is as desired generated by a set of non-constant homogeneous polynomials. This shows the proposed claim.

   With this observation we can now proof the main claim. To distinguish between affine varieties and closed subvarieties of $\mathbb{P}^n$, we introduce the notation of $V_a(S) = \{x \in \mathbb{A}^{n+1} : f(x) = 0 \quad \forall f \in S\}$ for affine varieties and $V_p(S) = \{[x_0, \dots, x_n] \in \mathbb{P}^n : f(x) = 0 \quad \forall f \in S\}$. Note that for $V_p(S)$ to be well defined, we require f to be homogeneous for all $f \in S$.

   Let $X \subseteq \mathbb{A}^{n+1}$ be a cone. By our observation above, there exists a set $S \subseteq k[x_0, \dots, x_n]$ of non-constant homogeneous polynomials such that $X = V_a(S)$. Now we consider the „projectivization“ of X, which is defined as $\mathbb{P}(X) = \{[x_0, \dots, x_n] : (x_0, \dots, x_n) \in X \setminus \{0\}\} \subseteq \mathbb{P}^n$. Thus we have $\mathbb{P}(X) = \mathbb{P}(V_a(S)) = V_p(S)$ which is a closed subvariety of $\mathbb{P}^n$.

   Conversely, let X be a closed subvariety of $\mathbb{P}^n$. Thus, there exists a set of non-constant homogeneous polynomials S such that $X = V_p(S)$. We define the cone over X as $C(X) = \{0\} \cup \{(x_0, \dots, x_n) : [x_0, \dots, x_n] \in X\} \subseteq \mathbb{A}^{n+1}$. Thus we get $C(X) = C(V_p(S)) = V_a(S)$, which is a cone as we have proven above.

   Thus, we obtain a bijection as desired. □

2. By means of the Segre embedding, prove that $\mathbb{P}^n$ is a separated algebraic variety.

   Proof. As we know already that $\mathbb{P}^n$ is a algebraic variety, it only remains to show that it is separated, i.e. $\Delta_{\mathbb{P}^n} = \{(x, x) : x \in \mathbb{P}^n\}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. Using the Segre embedding we will show, that $\Delta_{\mathbb{P}^n}$ is isomorph to a closed subset of $\mathbb{P}^N \cong \mathbb{P}^n \times \mathbb{P}^n$ where $N = (n+1)^2 - 1$.

   Clearly we have by definition of the homogeneous coordinates

$$
\begin{aligned}
([x_0, \dots, x_n], [y_0, \dots, y_n]) \in \Delta_{\mathbb{P}^n} &\iff [x_0, \dots, x_n] = [y_0, \dots, y_n] \\
&\iff \begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix} \text{ has rank } \leq 1 \\
&\iff \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} = 0 \quad \forall\, i, j \\
&\iff x_i y_j - x_j y_i = 0 \quad \forall\, 0 \leq i, j \leq n
\end{aligned}
$$

   Thus, using the Segre coordinates $z_{ij} := x_i y_j$, we get that

$$
\Delta_{\mathbb{P}^n} \cong \{[z_{00}, \dots, z_{nn}] \in \mathbb{P}^N : z_{ij} = z_{ji}\ \forall\, 0 \leq i, j \leq n\} = V(\{z_{ij} - z_{ji} : 0 \leq i, j \leq n\}),
$$

   which is a closed subvariety of $\mathbb{P}^N$ generated by the homogenous polynomials $z_{ij} - z_{ji}$. □

3. Let X be a projective variety and assume that there is a closed embedding $X \hookrightarrow Y$, where Y is an **affine** algebraic variety. Prove that X is a finite set of points.

Proof. Since Y is an affine algebraic variety, there exists an embbeding $g : Y \hookrightarrow \mathbb{A}^m$ for a certain $m \in \mathbb{N}$. Let $f : X \hookrightarrow Y$ be a closed embedding of the projective variety X into Y. For $i = 1, \dots, m$ we denote with p_i as usual the projection on the i-th coordinate. Thus, we have

$$X \overset{f}{\hookrightarrow} Y \overset{g}{\hookrightarrow} \mathbb{A}^m \xrightarrow{p_i} \mathbb{A}^1 \cong k.$$

Since all these functions are regular, we get that $f_i := p_i \circ g \circ f : X \to k$ is regular for all i. Thus, we can apply Corollary 2.7.12 on each f_i to deduce that $f_i(X)$ is finite for all i. This however implies, that $g(f(X))$ is finite. Since f and g are injective (as they are embeddings), $g \circ f$ is injective and hence we can conclude that X is finite. □