Seminar on K-stability of Fano Varieties

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In this series of talks, we will discuss recent advances on the existence of Kähler-Einstein metrics on Fano varieties. More precisely, the works of Chen-Donaldson-Sun and Tian imply that a Fano variety X (i.e., a complex smooth projective variety such that $\mathscr{O}_X(-K_X) = \det(T_X)$ is ample) possesses a Kähler-Einstein metric if and only if it is K-polystable. This latter condition, introduced by Tian (1997) and Donaldson (2002), allows us to reduce a problem in Partial Differential Equations and Differential Geometry to the language of Algebraic Geometry. The objective of the seminar will be to introduce the definition of stability and the properties of such varieties, also aiming to learn how to use this language in explicit examples.

Contents

1 The Calabi problem for Fano varieties

All varieties will be defined over the field of complex numbers $\mathbf{C}$.

To study projective (smooth) varieties X from the point of view of Differential Geometry, we consider Hermitian metrics h (instead of Riemannian metrics), which in turn are associated with

$$\omega \underset{\text{loc}}{=} \frac{i}{2\pi}\sum_{ij} h_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j \text{ a } (1,1)\text{-form with } h=(h_{ij}) \text{ Hermitian matrix with } h_{ij}: X \to \mathbf{C}\ \mathscr{C}^\infty \text{ function.}$$

We say that (X,ω) is a **Kähler variety** if $\mathrm{d}\omega = 0$.

Example 1.1. In $\mathbf{P}^n$, the **Fubini-Study metric** associated to[1]

$$\omega_{\mathrm{FS}} := \frac{i}{2}\partial\overline{\partial}\log\|\mathbf{z}\|^2$$

is Kähler. Thus, for $X \subseteq \mathbf{P}^n$ smooth projective subvariety, we have that $\omega := \omega_{\mathrm{FS}}|_X$ is a Kähler metric on X, i.e., *every smooth projective variety is a Kähler variety.*

Remark 1.2. *If $X \cong \mathbf{P}^1$, $\mathbf{C}/\Lambda$ or $\mathbf{D}/\Gamma$ is a Riemann surface, there are classical metrics (Fubini-Study, Euclidean, and Poincaré, respectively) on X with constant curvature ($+1$, 0, -1, respectively).*

In 1954, Eugenio Calabi proposed studying the existence of a Kähler metric ω on every smooth projective variety X such that

$$\mathrm{Ric}(\omega) = \lambda\omega \text{ for some } \lambda \in \{-1,0,1\} \text{ (Kähler-Einstein Equation)}$$

where $\mathrm{Ric}(\omega) \underset{\text{loc}}{=} -i\partial\overline{\partial}\log\det(h_{ij})$ is the Ricci curvature of ω.

Example 1.3. In the affine chart $U_0 = \{\mathbf{z} = [Z_0, Z_1, Z_2] \in \mathbf{P}^2,\ Z_0 \neq 0\} \cong \mathbf{A}^2$ of $\mathbf{P}^2$ with coordinates (z_1, z_2) we have $\omega_{\mathrm{FS}} \underset{\text{loc}}{=} \frac{i}{2}\partial\overline{\partial}\log(1+|z_1|^2+|z_2|^2) = \frac{i}{2\pi}\sum_{ij} h_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j$ where

$$h = (h_{ij}) = \frac{\pi}{(1+|z_1|^2+|z_2|^2)^2}\begin{pmatrix} 1+|z_2|^2 & -\overline{z}_1 z_2 \\ -z_1\overline{z}_2 & 1+|z_1|^2 \end{pmatrix}$$

with $\det(h_{ij}) = \pi^2(1+|z_1|^2+|z_2|^2)^{-3}$ and then $\mathrm{Ric}(\omega_{\mathrm{FS}}) \overset{\text{def}}{=} 6\pi\omega_{\mathrm{FS}}$. Since Ric is invariant under rescalings $\omega \mapsto \lambda_0^{-1}\omega$, we can normalize to obtain $\lambda = 1$. Similarly, $\mathrm{Ric}(\omega_{\mathrm{FS}}) = 2\pi(n+1)\omega_{\mathrm{FS}}$ in $\mathbf{P}^n$.

Recall 1.4 (Kodaira's Theorem)**.** The Ricci curvature $\mathrm{Ric}(\omega)$ defines a real $(1,1)$-form such that $[\mathrm{Ric}(\omega)] = 2\pi c_1(X) \overset{\text{def}}{=} 2\pi[-K_X] \in \mathrm{H}^{1,1}(X,\mathbf{R})$. Thus, in the case $\lambda = -1$ (resp. $\lambda = 1$) we have that $[K_X] = [\omega]$ (resp. $[-K_X] = [\omega]$) is cohomologous to a positive $(1,1)$-form. Kodaira's embedding theorem ensures that K_X (resp. $-K_X$) is **ample**. Thus, the Kähler-Einstein equation implies that:

1. X is **canonically polarized** (i.e., K_X ample) if $\lambda = -1$.
2. X is such that $K_X \sim_{\mathbf{Q}} 0$ if $\lambda = 0$.
3. X is **Fano** (i.e., $-K_X$ ample) if $\lambda = 1$.

The existence of Kähler-Einstein metrics on **all** canonically polarized and Calabi-Yau varieties are fundamental results in Geometric Analysis by Aubin and Yau, respectively. On the other hand, we will see that **not every** *Fano variety admits a Kähler-Einstein metric.*

Example 1.5. Let X be a smooth Fano variety. Then,

- $(\dim(X) = 1)$ $X \cong \mathbf{P}^1$ is Kähler-Einstein.
- $(\dim(X) = 2)$ $X \cong \mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{P}^2$ or $\mathrm{Bl}_{p_1,\dots,p_r}(\mathbf{P}^2)$ blow-up at $r \geq 8$ points in general position. We will see that all of them are Kähler-Einstein **except for** $\mathrm{Bl}_p(\mathbf{P}^2) \cong \mathbf{F}_1$ and $\mathrm{Bl}_{p_1,p_2}(\mathbf{P}^2)$.
- $(\dim(X) = 3)$ Iskovskikh, Mori, and Mukai classified the 3-folds of Fano into $17 + 88 = 105$ families. In 2023, Araujo-Castravet-Cheltsov-Fujita-Kaloghiros-Martínez–García-Shramov-Suess-Viswanathan proved that for exactly 78 families, the general member admits a Kähler-Einstein metric.

[1] Here, $\partial f = \sum \frac{\partial f}{\partial z^i}\,\mathrm{d}z^i$ and $\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}^j}\,\mathrm{d}\overline{z}^j$.

Theorem 1.6 (Chen–Donaldson–Sun 2014, Tian 2015)**.** *A smooth Fano variety X admits a Kähler-Einstein metric if and only if $(X, -K_X)$ is K-polystable.*

Surprisingly, the notion of K-polystability is a **purely algebro-geometric** concept, and it arises from the idea of using Geometric Invariant Theory (GIT) on the Hilbert scheme:

Let X be a projective variety and $L \in \mathrm{Pic}(X)$. A **test configuration** of (X, L) is an equivariant degeneration by $\mathbf{G}_m$ obtained as follows:

1. Given an embedding $\varphi_{|L^{\otimes k}|} : X \hookrightarrow \mathbf{P}^N$ for some $k \gg 0$, and
2. Given a 1-parameter subgroup given by a group morphism $\rho : \mathbf{G}_m \to \mathrm{PGL}_{N+1}(\mathbf{C})$,

we consider the induced action of $\mathbf{G}_m$ on $\mathbf{P}^N$ and the embedding

$$j : X \times \mathbf{G}_m \hookrightarrow \mathbf{P}^N \times \mathbf{G}_m,\ (x,t) \mapsto (\rho(t)x, t).$$

Thus, the Zariski closure $\mathscr{X} := \overline{j(X \times \mathbf{G}_m)} \subseteq \mathbf{P}^N \times \mathbf{A}^1$ is endowed with a projection $\pi : \mathscr{X} \to \mathbf{A}^1$ which is a flat morphism and such that $\mathscr{L} := \mathscr{O}_{\mathscr{X}}(1) \in \mathrm{Pic}(\mathscr{X})$ is relatively ample over π.

We will say that $(\mathscr{X}, \mathscr{L}) \xrightarrow{\pi} \mathbf{A}^1$ is a **test configuration** and that the special fiber $\mathscr{X}_0 \overset{\text{def}}{=} \pi^{-1}(0)$ is the **flat limit** of $\rho(t) \cdot X \subseteq \mathbf{P}^N$ as $t \to 0$.

Example 1.7 (Li–Xu)**.** Let $(X, L) = (\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(3))$ be the *twisted cubic* and consider the non-normal variety

$$\mathscr{X} := \{s^2(x+w)w - z^2 = sx(x+w) - yz = xz - syw = y^2w - x^2(x+w) = 0\} \subseteq \mathbf{P}^3_{[x,y,z,w]} \times \mathbf{A}^1_s.$$

with $\mathbf{G}_m$ action given by $t \cdot ([x,y,z,w], s) = ([x,y,tz,w], ts)$. Thus, $\mathscr{X}_0 \cong \mathrm{Proj}(\mathbf{C}[x,y,z,w]/I_0)$ nodal plane cubic given by $I_0 = \langle z^2, yz, xz, y^2w - x^2(x+w)\rangle$ and $p = (0,0,0,1)$ non-reduced point of $\mathscr{X}_0$.

Analogous to the Hilbert-Mumford criterion for stability in the context of GIT, Futaki (1983) and Donaldson (2002) proposed studying the $\mathbf{G}_m$ action on the special fiber $\mathscr{X}_0$. More precisely, we will see that the $\mathbf{G}_m$ action on the section spaces $\mathrm{H}^0(\mathscr{X}_0, \mathscr{L}_0^{\otimes m})$ allows us to define the **Donaldson-Futaki invariant** $\mathrm{DF}(\mathscr{X}, \mathscr{L}) \in \mathbf{Q}$ associated with a test configuration $(\mathscr{X}, \mathscr{L})$ with normal total space $\mathscr{X}$ and say that:

1. (X, L) is **K-semistable** if $\mathrm{DF}(\mathscr{X}, \mathscr{L}) \geq 0$ for all test configurations $(\mathscr{X}, \mathscr{L})$ of (X, L).
2. (X, L) is **K-polystable** if it is K-semistable and if $\mathrm{DF}(\mathscr{X}, \mathscr{L}) = 0$ implies $(\mathscr{X}, \mathscr{L}) \cong (X, L) \times \mathbf{A}^1$.

Remark 1.8 (MMP)**.** *Very recent works (Li, Fujita, Odaka, Okada, Xu, etc.) have proven that the moduli space of smooth Kähler-Einstein Fano varieties is compactified by considering K-polystable Fano varieties with* **klt singularities** *(instead of slc), as conjectured by Tian.*

We will also see that the calculation of $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ can be done explicitly because $\mathscr{L}$ is a **$\mathbf{G}_m$-linearized line bundle** (cf. Chapter 7 of *Lectures on Invariant Theory* by I. Dolgachev):

Let X be an algebraic variety with $\mathbf{G}_m$ action $\alpha : \mathbf{G}_m \times X \to X$ and let $L \in \mathrm{Pic}(X)$. A $\mathbf{G}_m$-linearization of L is an action of $\mathbf{G}_m$ on the total space $\mathbf{V}(L)$ of L that makes the projection $\mathbf{V}(L) \xrightarrow{\pi} X$ a $\mathbf{G}_m$-equivariant morphism and such that the action on the fibers is linear. More formally, it is an action $\sigma : \mathbf{G}_m \times \mathbf{V}(L) \to \mathbf{V}(L)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{G}_m \times \mathbf{V}(L) & \xrightarrow{\ \sigma\ } & \mathbf{V}(L) \\ {\scriptstyle \mathrm{Id}_{\mathbf{G}_m} \times \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbf{G}_m \times X & \xrightarrow{\ \alpha\ } & X \end{array}$$

and such that the zero section $\mathbf{0}_L \subseteq \mathbf{V}(L)$ is $\mathbf{G}_m$-invariant. In particular, for all $t \in \mathbf{G}_m$ and all $x \in X$

$$\sigma_x(t) : L_x \xrightarrow{\simeq} L_{t \cdot x} \text{ is a linear isomorphism.}$$

Moreover, since $\mathrm{Pic}(\mathbf{G}_m) \cong \{1\}$, every $L \in \mathrm{Pic}(X)$ admits a linearization and the possible classes of linearizations are parametrized by the character group $\mathfrak{X}(\mathbf{G}_m) \overset{\text{def}}{=} \mathrm{Hom}_{\mathrm{gr}}(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbf{Z}$. For example, if $X = \mathbf{P}^1$ and $L = \mathscr{O}_{\mathbf{P}^1}(-1)$ then $\mathbf{V}(L) \overset{\text{def}}{=} \{([x_0, x_1], \lambda(x_0, x_1)),\ [x_0, x_1] \in \mathbf{P}^1,\ \lambda \in \mathbf{C}\}$ and the $\mathbf{G}_m$ action given by $t \cdot ([x_0, x_1], \lambda(x_0, x_1)) := ([x_0, tx_1], \lambda(x_0, tx_1))$ determines a $\mathbf{G}_m$-linearization of L.

2 Test Configurations and Rees Algebras

Let X be a (separated, finite type over $k = \mathbf{C}$) projective scheme and $L \in \mathrm{Pic}(X)$ an ample line bundle.

Definition 2.1. A **test configuration** of (X, L) is a pair $(\mathscr{X}, \mathscr{L})$ along with

1. a proper and flat morphism $\pi : \mathscr{X} \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[t])$,
2. an action of $\mathbf{G}_m$ on $\mathscr{X}$ such that π is equivariant for the standard action $(a, t) \mapsto at$ of $\mathbf{G}_m$ on $\mathbf{A}^1$,
3. a $\mathbf{Q}$-line bundle $\mathscr{L} \in \mathrm{Pic}(\mathscr{X}) \otimes_{\mathbf{Z}} \mathbf{Q}$ that is π-ample and $\mathbf{G}_m$ linearized on $\mathscr{X}$,
4. an isomorphism $(\mathscr{X}_1, \mathscr{L}_1) \cong (X, L)$ between the general fiber and the original polarized variety.

Example 2.2. Let (X, L) be as in the previous definition.

1. The **trivial test configuration** $(X_{\mathbf{A}^1}, L_{\mathbf{A}^1}) := (X, L) \times \mathbf{A}^1$ is the one where the action of $\mathbf{G}_m$ on $X_{\mathbf{A}^1}$ is the product action, with the action on X being **trivial** and the action on $\mathbf{A}^1$ being the standard action.
2. A **product test configuration** is $(X_{\mathbf{A}^1}, L_{\mathbf{A}^1})$ as before, except that the action of $\mathbf{G}_m$ on X is not necessarily trivial. If $\mathrm{Aut}^\circ(X) \cong \{1\}$, then every product configuration is trivial.
3. Let $Z \subseteq X$ be a closed subscheme and let $\sigma : \mathscr{X} := \mathrm{Bl}_{Z \times 0}(X \times \mathbf{A}^1) \to X \times \mathbf{A}^1$. Then $p := \mathrm{pr}_{\mathbf{A}^1} \circ \sigma : \mathscr{X} \to \mathbf{A}^1$ is a proper and flat morphism[2]. Also, $\mathscr{X}_0 = E + F$, where $E = \sigma^{-1}(Z \times 0)$ is the exceptional divisor and where $F \cong \mathrm{Bl}_Z(X)$ is the strict transform of $X \times 0$. If $Z \subseteq X$ is $\mathbf{G}_m$-invariant (e.g., by the trivial action on X), then there is an induced action of $\mathbf{G}_m$ on $\mathscr{X}$, and since $-E$ is σ-ample, we have that $\mathscr{L} := \sigma^* L_{\mathbf{A}^1} \otimes \mathscr{O}_{\mathscr{X}}(-tE)$ is π-ample for $0 < t \ll 1$. Thus, $(\mathscr{X}, \mathscr{L})$ is a test configuration.
4. Let $r \in \mathbf{N}^{\geq 1}$ be such that $rL := L^{\otimes r}$ is very ample, i.e., $\iota : X \hookrightarrow \mathbf{P}(V^\vee)$ embedding, where $V = \mathrm{H}^0(X, rL)$. A group morphism $\rho : \mathbf{G}_m \to \mathrm{GL}(V)$ induces a test configuration $(\mathscr{X}_\rho, \mathscr{L}_\rho)$ where $\mathscr{X}_\rho$ is the Zariski closure in $\mathbf{P}(V^\vee) \times \mathbf{A}^1$ of the image of $X \times \mathbf{G}_m \hookrightarrow \mathbf{P}(V^\vee) \times \mathbf{G}_m$, $(x, a) \mapsto (\rho(a)x, a)$ and where $\mathscr{L}_\rho := \frac{1}{r}\mathscr{O}_{\mathscr{X}}(1)$. All test configurations of (X, L) are obtained in this way (Ross–Thomas, 2007).

Objective: Obtain "valuative" (i.e., numerical) criteria to study test configurations. We will see that:

$$\{\text{Test configurations of } (X, L)\} \leftrightsquigarrow \{\text{Filtrations of } R(X, L) := \oplus_{m \geq 0} \mathrm{H}^0(X, mL)\} \leftrightsquigarrow \{\text{Valuations in } \mathbf{C}(X)\}$$

More precisely, the (divisorial) valuations we construct in $\mathbf{C}(X)$ will allow us to study the Donaldson-Futaki invariant $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ using techniques from **birational geometry** (MMP) and **intersection theory**.

Recall 2.3. Given a vector space V, there is a bijection between linear actions of $\mathbf{G}_m$ on V and $\mathbf{Z}$-gradings on V: given an action of $\mathbf{G}_m$ on V, there is a **weight decomposition** $V = \oplus_{\lambda \in \mathbf{Z}} V_\lambda$ where

$$V_\lambda := \{v \in V,\ a \cdot v = a^\lambda v \text{ for all } a \in \mathbf{G}_m\}.$$

Conversely, the $\mathbf{Z}$-grading $V = \oplus_{\lambda \in \mathbf{Z}} V_\lambda$ allows us to define $a \cdot v := \sum a^\lambda v_\lambda$, for any $v = \sum v_\lambda$.

Definition 2.4. Let V be a finite-dimensional vector space (e.g., $\mathrm{H}^0(X, mL)$). A **Z-filtration** of V is a collection of subspaces $\{F^\lambda\}_{\lambda \in \mathbf{Z}} \subseteq V$ such that

1. $F^{\lambda+1} V \subseteq F^\lambda V$ for all $\lambda \in \mathbf{Z}$, i.e., it is a **decreasing** filtration,
2. $F^\lambda V = 0$ for all $\lambda \gg 0$, and
3. $F^\lambda = V$ for all $\lambda \ll 0$.

The **Rees algebra** associated with the filtration $F^\bullet$ is the finitely generated and torsion-free $k[t]$-module given by $\mathrm{Rees}(F^\bullet) := \oplus_{\lambda \in \mathbf{Z}} F^\lambda V t^{-\lambda}$, with $k[t]$-module structure given by $t \cdot (vt^{-\lambda}) := vt^{-\lambda+1}$.

Construction 2.5 (Rees correspondence)**.** There is a bijective correspondence between

$$\{\mathbf{G}_m\text{-linearized vector bundles } \mathscr{V} \to \mathbf{A}^1\} \longleftrightarrow \{\mathbf{Z}\text{-filtrations of finite-dimensional vector spaces } V\}.$$

Indeed, given a vector space V and a $\mathbf{Z}$-filtration $F^\bullet V$, we consider $R := \mathrm{Rees}(F^\bullet)$ and $\widetilde{R}$ is a locally free sheaf on $\mathbf{A}^1 = \mathrm{Spec}(k[t])$, with the vector bundle $\mathscr{V} := \mathbf{V}(\widetilde{R}) \to \mathbf{A}^1$. Since R admits a $\mathbf{Z}$-grading compatible with the $\mathbf{Z}$-grading of $k[t]$, we have that $\mathscr{V} \to \mathbf{A}^1$ is a $\mathbf{G}_m$-linearized vector bundle.

Given a $\mathbf{G}_m$-linearized vector bundle $\mathscr{V} \to \mathbf{A}^1$, we consider the induced $\mathbf{G}_m$ action on its global sections and the corresponding weight decomposition $\mathrm{H}^0(\mathbf{A}^1, \mathscr{V}) = \bigoplus_{\lambda \in \mathbf{Z}} \mathrm{H}^0(\mathbf{A}^1, \mathscr{V})_\lambda$.

[2] Known as the **deformation to the normal cone** in *Intersection Theory* (see Fulton §5.1 or Ravi Vakil's Class 14).

It is important to note that the $\mathbf{G}_m \cong \operatorname{Spec}(k[t,t^{-1}]) \stackrel{\text{def}}{=} \{t \neq 0\} \subseteq \mathbf{A}^1$ action on the global sections is given by the **dual** representation $t\cdot\sigma(x) := \sigma(t^{-1}\cdot x)$ and thus $t \in k[t]$ acts with weight -1 on the $k[t]$-module $\mathrm{H}^0(\mathbf{A}^1, \mathscr{V})$, i.e., $\mathrm{H}^0(\mathbf{A}^1,\mathscr{V})_\lambda \xrightarrow{\cdot t} \mathrm{H}^0(\mathbf{A}^1,\mathscr{V})_{\lambda-1}$ is an **injective** $k[t]$-module morphism.

We can construct $F^\lambda V$ geometrically as follows: Let $V := \mathscr{V}_1$ be the general fiber of $\mathscr{V} \to \mathbf{A}^1$ and

$$F^\lambda V := \operatorname{Im}\left(\mathrm{H}^0(\mathbf{A}^1,\mathscr{V})_\lambda \xrightarrow{\mathrm{ev}_1} V,\ s \mapsto s(1)\right)$$

where $F^\lambda V \subseteq F^{\lambda-1}V$ since $\cdot t$ is an injective morphism. Also, $F^\lambda V = 0$ (resp. $F^\lambda V = V$) for $\lambda \gg 0$ (resp. $\lambda \ll 0$) since $\mathrm{H}^0(\mathbf{A}^1,\mathscr{V})$ is a finitely generated $k[t]$-module (resp. $\operatorname{Im}(\mathrm{H}^0(\mathbf{A}^1,\mathscr{V})_\lambda \xrightarrow{\mathrm{ev}_1} V) = V$).

Remark 2.6. *The above construction has two consequences that will help us with calculations.*

1. *Since $\mathscr{V} \to \mathbf{A}^1$ is $\mathbf{G}_m$-equivariant, we have $\mathscr{V}_{\mathbf{A}^1\setminus\{0\}} \cong V \times (\mathbf{A}^1 \setminus \{0\})$. On the other hand, since $R \otimes_{k[t]} k[t]/\langle t\rangle \cong R/tR$, we have $\mathscr{V}_0 \cong \bigoplus_{\lambda\in\mathbf{Z}} F^\lambda V/F^{\lambda+1}V \stackrel{\text{def}}{=} \operatorname{gr}^\bullet_F V$.*
2. *The inclusion $\mathrm{H}^0(\mathbf{A}^1,\mathscr{V}) \cong \bigoplus_{\lambda\in\mathbf{Z}} F^\lambda V t^{-\lambda} \hookrightarrow \mathrm{H}^0(\mathbf{A}^1\setminus\{0\},\mathscr{V}) \cong \bigoplus_{\lambda\in\mathbf{Z}} V t^{-\lambda}$ implies that*

   $s \in F^\lambda V \Leftrightarrow \overline{s}t^{-\lambda} \in \mathrm{H}^0(\mathbf{A}^1,\mathscr{V})$, where $\overline{s} \in \mathrm{H}^0(\mathbf{A}^1\setminus\{0\},\mathscr{V})$ is a $\mathbf{G}_m$-invariant section such that $\mathrm{ev}_1(\overline{s}) = s$.

To use filtrations in the context of test configurations $(\mathscr{X} \xrightarrow{\pi} \mathbf{A}^1, \mathscr{L})$ of a polarized scheme (X, L), we consider $r \in \mathbf{N}^{\geq 1}$ such that $r\mathscr{L} \in \operatorname{Pic}(\mathscr{X})$ and $R := R(X, rL) := \bigoplus_{m\in\mathbf{N}} R_m$ with $R_m := \mathrm{H}^0(X, mrL)$ a finite-dimensional vector space. Let's see that we can construct a **graded Z-filtration $F^\bullet R$**, that is, a **Z**-filtration $F^\bullet R_m$ for all $m \in \mathbf{N}$ such that $F^\lambda R_m \cdot F^\mu R_n \subseteq F^{\lambda+\mu}F_{m+n}$. To do this, we note that:

1. By the projection formula, $\mathrm{H}^0(\mathscr{X}, mr\mathscr{L}) \cong \mathrm{H}^0(\mathbf{A}^1,\mathscr{V})$ where $\mathscr{V} := \pi_*(\mathscr{L}^{\otimes mr})$ is a $\mathbf{G}_m$-linearized vector bundle.
2. There is a canonical restriction morphism $\mathrm{ev}_1 : \mathrm{H}^0(\mathscr{X}, mr\mathscr{L}) \to \mathrm{H}^0(\mathscr{X}, mr\mathscr{L})_{t=1} \cong \mathrm{H}^0(X, mrL)$.

Thus, we can define $F^\lambda_{\mathscr{X},\mathscr{L}} \mathrm{H}^0(X, mrL) := \operatorname{Im}\left(\mathrm{H}^0(\mathscr{X}, mr\mathscr{L})_\lambda \xrightarrow{\mathrm{ev}_1} \mathrm{H}^0(X, mrL)\right)$, and this filtration is **finitely generated**. More precisely, the Rees correspondence gives us an isomorphism of $k[t]$-modules

$$\mathrm{H}^0(\mathscr{X}, mr\mathscr{L}) = \bigoplus_{\lambda\in\mathbf{Z}} \mathrm{H}^0(\mathscr{X}, mr\mathscr{L})_\lambda \xrightarrow[\text{Rees}]{\simeq} \bigoplus_{\lambda\in\mathbf{Z}} F^\lambda \mathrm{H}^0(X, mrL)t^{-\lambda} \text{ compatible with the grading, and then}$$

$$\bigoplus_{n\in\mathbf{N}} \mathrm{H}^0(\mathscr{X}, mr\mathscr{L}) \simeq \bigoplus_{n\in\mathbf{N}} \bigoplus_{\lambda\in\mathbf{Z}} F^\lambda_{\mathscr{X},\mathscr{L}} \mathrm{H}^0(X, mrL) \stackrel{\text{def}}{=} \operatorname{Rees}(F^\bullet_{\mathscr{X},\mathscr{L}} R(X, rL)),$$

where the latter is a finitely generated $k[t]$-algebra since $\mathscr{L}$ is relatively ample over $\mathbf{A}^1$.

Theorem 2.7. *There is a correspondence*[3] *between test configurations $(\mathscr{X},\mathscr{L})$ of the polarized variety (X, L) and graded **Z**-filtrations $F^\bullet$ of $R(X, L)$ for some $r > 0$.*

Proof. By the previous discussion, it suffices to note that $\operatorname{Rees}(F^\bullet_{\mathscr{X},\mathscr{L}} R(X, rL))$ induces the test configuration

$$\mathscr{X} := \operatorname{Proj}_{n\in\mathbf{N}}\left(\bigoplus_{n\in\mathbf{N}}\bigoplus_{\lambda\in\mathbf{Z}} F^\lambda_{\mathscr{X},\mathscr{L}} \mathrm{H}^0(X, mrL)\right) \xrightarrow{\pi} \mathbf{A}^1$$

where the morphism π is obtained since the degree $m = 0$ component of the Rees algebra is $k[t]$. By construction, π is projective and $\mathscr{L} := \frac{1}{k}\mathscr{O}_{\mathscr{X}}(k)$ is a **Q**-ample line bundle for $k \gg 0$. Moreover, the fact that there is a $\mathbf{G}_m$-equivariant isomorphism $(\mathscr{X},\mathscr{L})|_{\mathbf{A}^1\setminus\{0\}} \simeq (X, L) \times (\mathbf{A}^1 \times \{0\})$ follows from the above Remark. ☐

Corollary 2.8. *Let $(\mathscr{X},\mathscr{L})$ be a test configuration of the polarized variety (X, L). Then, if X is reduced and irreducible, then $\mathscr{X}$ also is.*

Proof. Let $F^\bullet R$, with $R = R(X, rL)$ for $r > 0$, be the **Z**-filtration associated with the test configuration $(\mathscr{X},\mathscr{L})$ such that $\mathscr{X} \simeq \operatorname{Proj}(\operatorname{Rees}(F^\bullet R))$. The result follows directly from the fact that $\operatorname{Rees}(F^\bullet R) \subseteq R[t, t^{-1}]$. ☐

Fact: An analogous analysis, using the characterization of normality by Serre's R_1 and S_2 conditions[4], implies that if X is **normal** and $\mathscr{X}_0$ is **reduced**, then $\mathscr{X}$ is normal.

[3] bijective, if we declare two filtrations equivalent if they coincide on $\mathrm{H}^0(X, mL)$ for all m sufficiently divisible.

[4] See Stack Project, Tag 031S.

3 Donaldson-Futaki Invariant

Definition 3.1. Consider V a finite-dimensional k-vector space with a $\mathbf{G}_m \curvearrowright V$ action. The weight of the action is defined as

$$\mathrm{wt}(V) = \sum_{\lambda \in \mathbf{Z}} \lambda \dim (V_\lambda),$$

where $V = \bigoplus_{\lambda\in\mathbf{Z}} V_\lambda$ is the weight decomposition of V with $V_\lambda := \{v \in V | \xi \cdot v = \xi^\lambda v \text{ for all } \xi \in \mathbf{G}_m(k)\}$

Remark 3.2. *If $\mathbf{G}_m$ acts on V, a vector space with $\dim(V) = n$, then $\mathrm{wt}(V) = \mathrm{wt}(\det(V))$ where $\det(V) := \bigwedge^n V$, which has an induced $\mathbf{G}_m$-action. If $s_1, \ldots, s_n \in V$ is a basis and $\lambda_1, \ldots, \lambda_n \in \mathbf{Z}$ are such that $\xi \cdot s_i = \xi^{\lambda_i} s_i$, then*

$$\xi \cdot s_1 \wedge \ldots \wedge s_n = \xi^{\sum_i \lambda_i} s_1 \wedge \ldots \wedge s_n$$

and thus $\mathrm{wt}(\det(V)) = \mathrm{wt}(V)$.

From now on, we consider X a projective variety over $\mathbf{C}$ with $\dim(X) = n$, and $(\mathscr{X}, \mathscr{L})$ a test configuration of a polarized pair (X, L). For $m \in \mathbf{N}$ such that $m\mathscr{L}$ is a line bundle, we define

$$N_m := \dim \mathrm{H}^0 (\mathscr{X}_0, m\mathscr{L}_0) \qquad \text{and} \qquad w_m := \mathrm{wt}\, \mathrm{H}^0 (\mathscr{X}_0, m\mathscr{L}_0)$$

It is known that the values N_m (the Hilbert polynomial of $\mathscr{X}_0$) are given by a polynomial with rational coefficients of degree n. Furthermore, it is possible to prove that the values of w_m are given by a rational polynomial of degree $n+1$, so we have an expansion

$$\frac{w_m}{mN_m} = F_0 + F_1 m^{-1} + F_2 m^{-2} + \ldots$$

for $m > 0$ sufficiently divisible.

Definition 3.3 (Futaki Invariant)**.** The *Donaldson-Futaki invariant* of $(\mathscr{X}, \mathscr{L})$ is defined as

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) := -2F_1$$

The goal is to express this number in terms of intersection numbers, for which we will need to compactify $(\mathscr{X}, \mathscr{L})$.

Construction 3.4 (Compactification)**.** Given a test configuration $(\mathscr{X}, \mathscr{L})$ of (X, L), we can consider the $\mathbf{G}_m$-equivariant families $(\mathscr{X}, \mathscr{L}) \to \mathbf{A}^1$ and $(X, L) \times \left(\mathbf{P}^1 \backslash \{0\}\right) \to \left(\mathbf{P}^1 \backslash \{0\}\right)$, where the $\mathbf{G}_m$ action on $(X, L) \times \left(\mathbf{P}^1 \backslash \{0\}\right)$ corresponds to the product of the trivial action on (X, L) and the standard action on $\mathbf{P}^1 \backslash \{0\}$. We have a $\mathbf{G}_m$-equivariant isomorphism

$$\begin{aligned}(\mathscr{X} \setminus \mathscr{X}_0, \mathscr{L}|_{\mathscr{X} \setminus \mathscr{X}_0}) &\cong (X, L) \times (\mathbf{A}^1 \setminus \{0\}) \\ (p, s) &\mapsto (a^{-1} \cdot p, a^{-1} \cdot s) \times \{a\}\end{aligned}$$

where $a = \pi(p)$, and therefore this isomorphism allows us to glue the two previous families, obtaining the *compactification* $\bar{\pi} : (\overline{\mathscr{X}}, \overline{\mathscr{L}}) \to \mathbf{P}^1$. This compactification has the following properties:

1. the morphism $\bar{\pi} : (\overline{\mathscr{X}}, \overline{\mathscr{L}}) \to \mathbf{P}^1$ is flat, proper, and $\mathbf{G}_m$-equivariant.
2. the $\mathbf{Q}$-line bundle $\overline{\mathscr{L}}$ is $\bar{\pi}$-ample and $\mathbf{G}_m$-linearized.
3. the fiber over ∞ corresponds to $\left(\overline{\mathscr{X}}_\infty, \overline{\mathscr{L}}_\infty\right) \cong (X, L)$.

Example 3.5. Consider the product test configuration $\mathscr{X} = \mathbf{P}^1 \times \mathbf{A}^1$ induced by the action

$$t \cdot [x : y] = [t^d x : y]$$

for some $d \in \mathbf{Z}$ and $\mathscr{L} = \mathscr{O}_{\mathbf{P}^1}(1) \times \mathbf{A}^1$. In this case, we see that $\mathrm{H}^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(m)) = \mathbf{C}[X, Y]_m$ with $t \cdot (x^k y^{m-k}) = t^{dk} x^k y^{m-k}$, so $w_m = \left(\frac{m(m+1)}{2}\right) d$, $N_m = m+1$, and $\mathrm{DF}(\mathscr{X}, \mathscr{L}) = 0$ (and thus $\mathbf{P}^1$ is **not** K-stable). Additionally, the previous construction tells us that

$$\overline{\mathscr{X}} = \mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(d))$$

Lemma 3.6. *Let $\mathbf{G}_m \curvearrowright \mathbf{P}^1$ be the action given by $t \cdot [x : y] = [tx : y]$. Given a $\mathbf{G}_m$-linearization of $\mathscr{O}_{\mathbf{P}^1}(m)$, then*

$$\mathrm{wt}\,(\mathscr{O}_{\mathbf{P}^1}(m)_0) - \mathrm{wt}\,(\mathscr{O}_{\mathbf{P}^1}(m)_\infty) = m$$

where $0 := [0 : 1]$ and $\infty := [1 : 0]$.

Proposition 3.7. *If $(\mathscr{X}, \mathscr{L})$ is a test configuration of (X, L) and $n = \dim(X)$, there exist $a_i, b_i \in \mathbf{Q}$ such that*

$$N_m := \dim \mathrm{H}^0\left(\mathscr{X}_0, m\mathscr{L}_0\right) = a_0 m^n + a_1 m^{n-1} + \ldots + a_n$$
$$w_m := \mathrm{wt}\, \mathrm{H}^0\left(\mathscr{X}_0, m\mathscr{L}_0\right) = b_0 m^{n+1} + b_1 m^n + \ldots + b_{n+1}$$

for all $m > 0$ sufficiently divisible. Moreover,

$$a_0 = \frac{L^n}{n!} \qquad \textit{and} \qquad b_0 = \frac{\overline{\mathscr{L}}^{n+1}}{(n+1)!}$$

and if $\mathscr{X}$ is normal, also

$$a_1 = -\frac{L^{n-1} \cdot K_X}{2(n-1)!} \qquad \textit{and} \qquad b_1 = -\frac{\overline{\mathscr{L}}^n \cdot K_{\overline{\mathscr{X}}/\mathbf{P}^1}}{2n!}$$

Proof. Serre's vanishing theorem implies that

$$\mathrm{H}^i(\overline{\mathscr{X}}_t, m\overline{\mathscr{L}}_t) = 0 \qquad \forall i > 0, m \gg 0, \forall t \in \mathbf{P}^1$$

The cohomology base change theorem (see Theorem II.12.11 in Hartshorne's book) implies that for an m such that the above holds, we also have that

1. $R^i \overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}}) = 0$ for all $i > 0$.
2. $\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})$ is locally free.
3. $\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}}) \otimes k(t) \to \mathrm{H}^0(\mathscr{X}_t, m\mathscr{L}_t)$ is an isomorphism for all $t \in \mathbf{P}^1$.

Conditions (2) and (3) allow us to state that

$$N_m := \dim \mathrm{H}^0\left(\mathscr{X}_0, m\mathscr{L}_0\right) = \dim \mathrm{H}^0\left(\mathscr{X}_1, m\mathscr{L}_1\right) = \dim \mathrm{H}^0(X, mL)$$

thus the statement about N_m and the formulas for a_0, a_1 are obtained from the Riemann-Roch theorem.
On the other hand, we can consider the line bundle $\det\left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right)$, which is a $\mathbf{G}_m$-linearized bundle over $\mathbf{P}^1$. From condition (3), we deduce

$$\mathrm{wt}\left(\det\left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right)_0\right) = \mathrm{wt}\left(\det \mathrm{H}^0\left(\mathscr{X}_0, m\mathscr{L}_0\right)\right) = w_m$$

and since $\mathbf{G}_m$ acts trivially on the fiber $(\overline{\mathscr{X}}, \overline{\mathscr{L}})_\infty$, we have $\mathrm{wt}\left(\det\left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right)_\infty\right) = 0$, and the previous lemma implies

$$\det\left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right) \simeq \mathscr{O}_{\mathbf{P}^1}\left(w_m\right)$$

The Hirzebruch-Riemann-Roch theorem for vector bundles on curves, combined with conditions (1), (2), (3) and Leray's direct image theorem, allows us to make the following calculation

$$\begin{aligned} w_m = \deg\left(\mathscr{O}_{\mathbf{P}^1}\left(w_m\right)\right) = \deg\left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right) &= \chi\left(\mathbf{P}^1, \left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right) - \mathrm{rk}\left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right)\right) \\ &= \chi\left(\mathbf{P}^1, \left(\overline{\pi}_* \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right)\right) - N_m \\ &= \chi\left(\overline{\mathscr{X}}, \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right) - N_m \end{aligned}$$

The above allows us to conclude that the values w_m are given by a polynomial with rational coefficients of degree $n+1$, so we only need to find the coefficients. The coefficient b_0 is obtained directly from the Riemann-Roch theorem, as $\deg(N_m) = n$. To finish, note that

$$2L^n = 2\overline{\mathscr{L}}^n \cdot \mathscr{O}_{\overline{\mathscr{X}}}\left(\mathscr{X}_1\right) = \overline{\mathscr{L}}^n \cdot \pi^* \mathscr{O}_{\mathbf{P}^1}(2) = -\overline{\mathscr{L}}^n \cdot \overline{\pi}^* K_{\mathbf{P}^2}$$

and from the Riemann-Roch theorem (assuming $\mathscr{X}$ is normal) we see that

$$w_m = \frac{\overline{\mathscr{L}}^{n+1}}{(n+1)!} m^{n+1} - \frac{\overline{\mathscr{L}}^n \cdot K_{\overline{\mathscr{X}}} + 2L^n}{2n!} m^n + O(m^{n-1})$$

The last two calculations allow us to deduce b_1. ☐

The previous proposition immediately allows us to obtain the following formula for the Donaldson-Futaki invariant.

Theorem 3.8 (Wang-Odaka)**.** *If $(\mathscr{X}, \mathscr{L})$ is a test configuration of (X, L) and $\mathscr{X}$ is normal, then*

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{\overline{\mathscr{L}}^n \cdot K_{\overline{\mathscr{X}}/\mathbf{P}^1}}{V} + \bar{S}\frac{\overline{\mathscr{L}}^{n+1}}{(n+1)V}$$

where $V = L^n$ and $\overline{S} = nV^{-1}(-K_X \cdot L^{n-1})$.

Proof. It suffices to note that $\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{2(b_0a_1 - b_1a_0)}{a_0^2}$. □

Since Wang-Odaka's formula is valid for normal tests, the goal will be to normalize test configurations.

Construction 3.9. Let $(\mathscr{X}, \mathscr{L})$ be a test configuration of a polarized variety (X, L) with X normal. The normalization of $(\mathscr{X}, \mathscr{L})$ is $(\widetilde{\mathscr{X}}, \widetilde{\mathscr{L}})$ where $\nu : \widetilde{\mathscr{X}} \to \mathscr{X}$ is the normalization and $\widetilde{\mathscr{L}} := \nu^*\mathscr{L}$. Indeed, this results in a test configuration since

$$\begin{array}{ccc} \mathbf{G}_m \times \widetilde{\mathscr{X}} & \overset{\exists!}{\dashrightarrow} & \widetilde{\mathscr{X}} \\ {\scriptstyle \mathrm{Id}\times\nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ \mathbf{G}_m \times \mathscr{X} & \longrightarrow & \mathscr{X} \end{array}$$

and $\widetilde{\mathscr{L}}$ is ample over $\mathbf{A}^1$ since ν is finite.
For $m > 0$ sufficiently divisible, we have $N'_m = \mathrm{H}^0(X, mL) = N_m$. On the other hand, denoting $\widetilde{\mathscr{X}} =: \mathscr{X}'$ for simplicity, note that there is an exact sequence

$$0 \to \mathscr{O}_{\overline{\mathscr{X}}} \to \nu_*\mathscr{O}_{\overline{\mathscr{X}'}} \to \mathscr{F} \to 0 \qquad / \otimes \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})$$

and we can calculate that

$$w'_m = w_m + \dim \mathrm{H}^0\left(\overline{\mathscr{X}}, \mathscr{F} \otimes \mathscr{O}_{\overline{\mathscr{X}}}(m\overline{\mathscr{L}})\right) \geq w_m$$

Thus,

$$\mathrm{DF}(\widetilde{\mathscr{X}}, \widetilde{\mathscr{L}}) \leq \mathrm{DF}(\mathscr{X}, \mathscr{L})$$

We now state the definition of K-stability.

Definition 3.10. Let X be a normal, proper variety and L an ample line bundle on X. We say that (X, L) is:

1. *K-semistable* if and only if $\mathrm{DF}(\mathscr{X}, \mathscr{L}) \geq 0$ for all normal test configurations of (X, L).
2. *K-polystable* if and only if (X, L) is K-semistable and $\mathrm{DF}(\mathscr{X}, \mathscr{L}) = 0$ only if $(\mathscr{X}, \mathscr{L})$ is a product test configuration.
3. *K-stable* if and only if (X, L) is K-semistable and $\mathrm{DF}(\mathscr{X}, \mathscr{L}) > 0$ when $(\mathscr{X}, \mathscr{L})$ is trivial.

In particular, (3) ⇒ (2) ⇒ (1).

Remark 3.11. *It is important to understand how these notions of stability are related.*

1. *If there is a $\mathbf{G}_m \curvearrowright X$ action and a $\mathbf{G}_m$-linearization of L, then $(\mathscr{X}, \mathscr{L})$ is not K-stable. Indeed, the action and its dual action give rise to two non-trivial test configurations $(\mathscr{X}, \mathscr{L}), (\mathscr{X}', \mathscr{L}')$ that satisfy $\mathrm{DF}(\mathscr{X}, \mathscr{L}) + \mathrm{DF}(\mathscr{X}', \mathscr{L}') = 0$, so one of these numbers is non-positive.*
2. *If X is a Fano variety over $\mathbf{C}$, then:*
   - *(a) $(X, -K_X)$ is K-polystable if and only if X admits a Kähler-Einstein metric.*
   - *(b) $(X, -K_X)$ is K-stable if and only if X admits a Kähler-Einstein metric and $\mathrm{Aut}(X)$ is finite.*
   - *(c) $(X, -K_X)$ is K-semistable if and only if there exists a test configuration $(\mathscr{X}, \mathscr{L})$ of $(X, -K_X)$ such that $\mathscr{X}_0$ is a (possibly singular) Fano variety that admits a Kähler-Einstein metric.*

4 K-stability and Singularities of MMP

Theorem 4.1 (Odaka)**.** *Let X be a projective normal variety and $L \in \mathrm{Pic}(X)_{\mathbf{Q}}$ ample. Then,*

1. *If $K_X \sim_{\mathbf{Q}} 0$, then X is klt (resp. lc) $\Leftrightarrow$ (X, L) is K-stable (resp. K-semistable).*
2. *If $L = K_X$, then X is lc $\Leftrightarrow$ (X, L) is K-stable $\Leftrightarrow$ (X, L) is K-semistable.*

We begin by modifying the intersection formula for the Donaldson-Futaki invariant. Let $(\mathscr{X}, \mathscr{L})$ be a normal test configuration of (X, L). We have

$$\begin{array}{ccc} & \mathscr{Y} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \overline{\mathscr{X}} & \dashrightarrow & X \times \mathbf{P}^1 \end{array}$$

where $\mathscr{Y}$ corresponds to the normalization of the graph of the birational map $\overline{\mathscr{X}} \dashrightarrow X \times \mathbf{P}^1$.

Proposition 4.2. *With the above notation,*

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{\overline{\mathscr{L}}^n \cdot f_*\left(K_{\mathscr{Y}/X\times\mathbf{P}^1} + g^*p_1^*K_X\right)}{V} + \bar{S}\frac{\overline{\mathscr{L}}^{n+1}}{(n+1)V}$$

where $\overline{S} := nV^{-1}\left(-K_X \cdot L^n\right)$.

Proof. By Wang-Odaka's formula, it suffices to prove that:

$$f_*\left(K_{\mathscr{Y}/X\times\mathbf{P}^1} + g^*p_1^*K_X\right) = K_{\mathscr{X}/\mathbf{P}^1}.$$

To do this, observe that:

$$\begin{aligned} K_{\mathscr{Y}/X\times\mathbf{P}^1} + g^*p_1^*K_X &= K_{\mathscr{Y}} - g^*\left(K_{X\times\mathbf{P}^1} - p_1^*K_X\right) \\ &= K_{\mathscr{Y}} - g^*p_2^*\left(K_{\mathbf{P}^1}\right) \\ &= K_{\mathscr{Y}} - f^*\overline{\pi}^*\left(K_{\mathbf{P}^1}\right). \end{aligned}$$

Finally,
$$f_*\left(K_{\mathscr{Y}/X\times\mathbf{P}^1} + g^*p_1^*K_X\right) = f_*K_{\mathscr{Y}} - f_*f^*\pi^*\left(K_{\mathbf{P}^1}\right) = K_{\overline{\mathscr{X}}} - \overline{\pi}^*K_{\mathbf{P}^1} = K_{\mathscr{X}/\mathbf{P}^1}.$$
□

Proposition 4.3.

1. *If X is lc, then $K_{\mathscr{Y}/\mathscr{X}\times\mathbf{P}}$ is effective.*
2. *If X is klt, then $K_{\mathscr{Y}/\mathscr{X}\times\mathbf{P}}$ is effective and has support on $\mathscr{Y}_0$.*

Proof. Let X be lc. Then $(X \times \mathbf{P}^1, X \times 0)$ is lc. Indeed, if $f : Y \to X$ is a log resolution of X, then $K_{Y/X} = \sum_i a_i E_i$ for certain prime divisors $E_i \subset Y$ such that $a_i \geq -1$. Now, $f_{\mathbf{P}^1} : Y \times \mathbf{P}^1 \to X \times \mathbf{P}^1$ is a log resolution of $(X \times \mathbf{P}^1, X \times 0)$ and

$$K_{Y\times\mathbf{P}^1/X\times\mathbf{P}^1} - f_{\mathbf{P}^1}^*(X \times 0) = \sum_{i=1}^{r} a_i E_i - Y \times 0$$

has coefficients ≥ -1. Thus, the pair $(X \times \mathbf{P}^1, X \times 0)$ is lc. Moreover, since

$$K_{\mathscr{Y}/X\times\mathbf{P}^1} - g^*(X \times 0) = K_{\mathscr{Y}/X\times\mathbf{P}^1} - \mathscr{Y}_0$$

has coefficients ≥ -1 and since $\mathrm{Supp}\left(K_{\mathscr{Y}/X\times\mathbf{P}^1}\right) \subset \mathrm{Exc}(g) = \mathscr{Y}_0$, it follows that $K_{\mathscr{Y}/X\times\mathbf{P}^1}$ has coefficients ≥ 0. If X is klt, note that no $E_i \subset \mathrm{Exc}(g)$ vanishes since $a(\mathscr{Y}_0, X \times \mathbf{P}^1) > -1$. □

Proof of (⇒) in Theorem 4.1.1. Fix $(\mathscr{X}, \mathscr{L})$ a non-trivial test configuration. Suppose X is lc with $K_X \sim_{\mathbf{Q}} 0$. Then, the Donaldson-Futaki invariant reduces to

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{\overline{\mathscr{L}}^n \cdot f_*\left(K_{\mathscr{Y}/X\times\mathbf{P}^1}\right)}{V}.$$

Now, since $f_*(K_{\mathscr{Y}/X\times\mathbf{P}^1})$ is effective with $\mathrm{Supp}\left(f_*K_{\mathscr{Y}/X\times\mathbf{P}^1}\right) \subset \mathscr{X}_{0,\mathrm{red}}$ and $\mathscr{L}|_{\mathscr{Y}_0}$ is ample, it follows from the Nakai-Moishezon criterion that $\overline{\mathscr{L}}^n \cdot f_*\left(K_{\mathscr{Y}/X\times\mathbf{P}^1}\right) \geq 0$, and hence X is K-semistable.
Now, suppose X is klt. Note that the condition $\mathrm{codim}\left(\mathrm{Exc}\left(X \dashrightarrow X_\times\mathbf{A}^1\right)\right) \geq 2$ implies that $(\mathscr{X}, \mathscr{L})$ is trivial. In this case, $\mathrm{codim}(\mathrm{Exc}(X \dashrightarrow X_\times\mathbf{A}^1)) = 1$, so $f_*(K_{\mathscr{Y}}/K_{X\times\mathbf{P}^1}) = \mathscr{X}_0$ and $f_*(K_{\mathscr{Y}}/K_{X\times\mathbf{P}^1}) \neq 0$. Thus, $\mathrm{DF}(\mathscr{X}, \mathscr{L}) > 0$ when $(\mathscr{X}, \mathscr{L})$ is non-trivial. This completes the case when $K_X \sim_{\mathbf{Q}} 0$.

Assume $L = K_X$ and that X is lc. Thus, the intersection formula results in:

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{\overline{\mathscr{L}}^n \cdot f_*\left(K_{\mathscr{Y}/X\times\mathbf{P}^1}\right)}{V} + \frac{\left(f^*\overline{\mathscr{L}}^n \cdot g^*L_{\mathbf{A}^1}\right) - \frac{n}{n+1}\overline{\mathscr{L}}^{n+1}}{(n+1)V}$$

By the same argument used earlier, the first term is ≥ 0. The fact that the second term is > 0 is a consequence of the works of Boucksom-Hisamoto-Jonsson (2017) and uses non-Archimedean methods. □

Theorem 4.4 (Odaka-Xu)**.** *If X is a normal variety such that K_X is $\mathbf{Q}$-Cartier, then there exists a proper birational morphism $f : Y \to X$ such that*

1. *$(Y, \Delta_Y := \mathrm{Exc}(f) = E_1 + \ldots + E_k)$ is lc.*
2. *$K_Y + \Delta_Y$ is f-ample.*

The pair (Y, Δ_Y) is known as the log canonical model of X, and it is unique up to isomorphism.

Example 4.5. Let $X = \{h = 0\} \subset \mathbf{A}^{n+1}$ with an isolated singularity at the origin, i.e., h is homogeneous. The blow-up of X at 0 gives a log resolution $Y \to X$ with $K_{Y/X} \sim (n - \deg(h))F$ with F an exceptional divisor. Thus, X is not lc when $\deg(h) > n + 1$, and then Y satisfies the conditions of Odaka-Xu's theorem.

Theorem 4.6 (Odaka, 2013)**.** *Let X be a normal variety such that K_X is $\mathbf{Q}$-Cartier, and let $L \in \mathrm{Pic}(X)_{\mathbf{Q}}$ be ample. If (X, L) is K-semistable, then it is lc.*

Proof Idea. We will assume that X is not lc and show that there is a test configuration $(\mathscr{X}, \mathscr{L})$ such that $\mathrm{DF}(\mathscr{X}, \mathscr{L}) < 0$. Consider Y the log canonical model of X and the divisor $E := K_{Y/X} + \Delta_Y$. Since E is nef, the *negativity lemma*[5] implies that $-E$ is effective. Consider the ideal sheaf $\mathscr{I} := f_*\mathscr{O}_Y(-mE)$ for $m > 0$ sufficiently divisible, and $Z \subset X$ the closed subscheme defined by the ideal $\mathscr{I}$. Then[6] $Y \cong \mathrm{Bl}_Z(X)$, so $E = \mathrm{Exc}(f)$, and hence $K_{Y/X} = -E - \Delta_Y$ has all its coefficients < -1. Given the closed $Z \subset X$ above, consider now the ideal:

$$\mathscr{I} = \mathscr{I}_{Z \times \mathbf{A}^1} + t^N \mathscr{O}_{X \times \mathbf{A}^1} \subset \mathscr{O}_{X \times \mathbf{A}^1}$$

where $N \in \mathbf{N}^{\geq 1}$, and we define $\mathscr{X}$ as the normalization of the blow-up of $X \times \mathbf{A}^1$ along $\mathscr{I}$:

$$\mathscr{X} := \widetilde{\mathrm{Bl}}_{\mathscr{I}} X \times \mathbf{A}^1 \xrightarrow{g} X \times \mathbf{A}^1.$$

We can write $\mathscr{I} \cdot \mathscr{O}_{\mathscr{X}} = \mathscr{O}_{\mathscr{X}}(-F)$ for some Cartier divisor $F \in \mathrm{Pic}(\mathscr{X})$, and we then have that $\mathscr{L}_\varepsilon := g^* L_{\mathbf{A}^1} - \varepsilon F$ is ample over $\mathbf{A}^1$ for $0 < \varepsilon \ll 1$ and thus $(\mathscr{X}, \mathscr{L}_\varepsilon)$ is a test configuration of (X, L), which indeed satisfies that $K_{\mathscr{X}/X \times \mathbf{A}^1}$ has (for N sufficiently large) only negative coefficients.

We then claim that $\mathrm{DF}(\mathscr{X}, \mathscr{L}_\varepsilon) < 0$ when $0 < \varepsilon \ll 1$. Indeed, the intersection formula implies

$$\mathrm{DF}\left(\mathscr{X}, \mathscr{L}_\varepsilon\right) = \frac{\overline{\mathscr{L}}_\varepsilon^n \cdot K_{\overline{\mathscr{X}}/X \times \mathbf{P}^1}}{V} + \frac{\overline{\mathscr{L}}_\varepsilon^n g^* p_1^* K_X}{V} + \frac{\bar{S}}{n+1} \frac{\overline{\mathscr{L}}_\varepsilon^{n+1}}{V},$$

resulting in $\mathrm{DF}\left(\mathscr{X}, \mathscr{L}_\varepsilon\right)$ being a polynomial in ε, and it suffices to analyze the lower order term. This last analysis is done by Odaka, who proves that

$$\mathrm{DF}\left(\mathscr{X}, \mathscr{L}_\varepsilon\right) = c\varepsilon^d + \text{higher order terms},$$

for some rational number $c < 0$. This proves that for $\varepsilon \ll 1$, the test configuration $(\mathscr{X}, \mathscr{L})$ is destabilizing, and hence (X, L) is **not** K-semistable. □

We conclude this section by stating a version for *klt* singularities of Theorem 4.1.

Theorem 4.7 (Odaka)**.**

1. *If X is Fano and $(X, -K_X)$ is K-semistable, then X is klt.*
2. *If X is Calabi-Yau and (X, L) is K-semistable, then X is klt.*

[5] See Lemma 3.39 in *Birational Geometry of Algebraic Varieties* by J. Kollár and S. Mori.
[6] For more details, see Lemma 1.13 in Boucksom-Hisamoto-Honsson (2017).

5 Valuations and Test Configurations

Throughout this section, k will be an algebraically closed field with $\mathrm{char}(k) = 0$.

Definition 5.1. Let K/k be a finitely generated field extension, i.e., its transcendence degree $\mathrm{tr.deg}\, K/k < +\infty$ is finite. A valuation (with real values) is a function $v : K^\times \to \mathbf{R}$ such that

1. $v(fg) = v(f) + v(g)$ for all $f, g \in K^\times$, i.e., $v : K^\times \to (\mathbf{R}, +)$ is a group homomorphism.
2. $v(f + g) \geq \min\{v(f), v(g)\}$ for all $f, g \in K^\times$.
3. $v|_{k^\times} = 0$.

Additionally, we define $v(0) = +\infty$.

Context: Given a normal algebraic variety X over k, we consider $K = K(X)$ the field of rational functions of X, which is a finitely generated extension of k with $\mathrm{tr.deg}\, K/k = \dim(X)$. We denote by Val_X the set of all valuations of the extension K/k.

Remark 5.2. *A valuation v of K/k has a list of associated invariants:*

1. *The valuation ring $\mathscr{O}_v := \{f \in K \mid v(f) \geq 0\}$, a local ring with maximal ideal $\mathfrak{m}_v := \{f \in K \mid v(f) > 0\}$.*
2. *The residue field $k(v) := \mathscr{O}_v/\mathfrak{m}_v$.*
3. *The transcendence degree* $\mathrm{tr.deg}(v) = \mathrm{tr.deg}_k\, k(v)$.
4. *The value group $\Gamma_v := v(K^\times) \subset \mathbf{R}$ and its* **rational rank** $\mathrm{rat.rk}(v) := \dim_{\mathbf{Q}}(\Gamma_v \otimes_{\mathbf{Z}} \mathbf{Q})$.

Example 5.3.

1. Let $x \in X$ be a smooth point of a variety of dimension n. We define the order of vanishing of a regular function $f \in \mathscr{O}_{X,x} \setminus \{0\}$ at x as

$$\mathrm{ord}_x(f) := \max\{d \in \mathbf{N} \mid f \in \mathfrak{m}_x^d\}$$

   We can extend this function to a valuation $K^\times \to \mathbf{R}$ by defining

$$\mathrm{ord}_x(f/g) := \mathrm{ord}_x(f) - \mathrm{ord}_x(g).$$

   In this case, we note that $\Gamma_v = \mathbf{Z}$ and therefore $\mathrm{rat.rk}(\mathrm{ord}_x) = 1$.

2. Consider $X = \mathbf{A}^2_{x,y}$. Given $f = \sum_{a,b \in \mathbf{N}} c_{a,b} x^a y^b$ where $c_{a,b} \in k$, we define the valuation v of $K(x, y)$ by

$$v(f) = \min\{a + b\sqrt{2} \mid c_{a,b} \neq 0\}$$

   The values $v(x) = 1, v(y) = \sqrt{2}$ are the weights of the action.

3. **Divisorial valuations.** A divisor E over X corresponds to a proper, birrational morphism $\mu : Y \to X$ with Y normal and $E \subset Y$ a prime divisor. In this case, the local ring $\mathscr{O}_{Y,E}$ of E is a discrete valuation ring (DVR), whose associated valuation is

$$\mathrm{ord}_E : K^\times \to \mathbf{Z}, \quad f \mapsto \mathrm{ord}_E(\mu^* f)$$

   i.e., it corresponds to computing the order of the pullback of regular functions along the subvariety E.

Definition 5.4 (Center of a valuation)**.** If $v \in \mathrm{Val}_X$ is a valuation, the *center of* v is the point $\xi \in X$ such that $v \geq 0$ on $\mathscr{O}_{X,\xi}$ and $v > 0$ on $\mathfrak{m}_\xi$. The center of the valuation v will be denoted $c_X(v)$.

Remark 5.5. *The fact that $c_X(v)$ exists is equivalent to $X \to \mathrm{Spec}(k)$ being proper, and if this center exists it is unique if X is a separated variety (cf. valuative criterions of properness and separatedness).*

Example 5.6.

1. The valuation ord_x associated with the order of vanishing at a smooth point $x \in X$ is divisorial. Indeed, $\operatorname{ord}_x = \operatorname{ord}_F$ where F corresponds to the exceptional divisor of the blowup of X at x. We can perform this calculation locally. Consider $u_1, \ldots, u_n \in \mathfrak{m}_x \subset \mathscr{O}_{X,x}$ local coordinates, and a function $f = \sum_{\alpha \in \mathbf{N}^n} c_\alpha u^\alpha \in \mathscr{O}_{X,x}$ where $u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$. By definition, $d := \operatorname{ord}_x(f) = \min\{|\alpha|, c_\alpha \neq 0\}$. Consider the blowup $\varepsilon : \widetilde{X} := \operatorname{Bl}_x X \to X$ given by:

$$\widetilde{X} \overset{\text{loc}}{=} \{(u, [y]) \in X \times \mathbf{P}^{n-1} \mid u_i y_j = u_j y_i \quad \forall i, j = 1, \ldots, n\}$$

   In the open set $y_i \neq 0$ we have coordinates $u_j = u_i y_j$, and the exceptional divisor is given by $F = \{u_i = 0\}$. We compute that:

$$\varepsilon^* f(x_1, \ldots, x_n) = u_i^d \widetilde{f}$$

   and $u_i \nmid \widetilde{f}$, so $\operatorname{ord}_F(f) = d$.

2. If $E \subset X$ is a prime divisor of X with generic point $\xi \in X$, the valuation $v = \operatorname{ord}_E$ is such that $v \geq 0$ on $\mathscr{O}_{X,\xi}$ and $v > 0$ on its maximal ideal.

3. More generally, if $E \subset Y \xrightarrow{\mu} X$ is a divisor over X then $\overline{c_X(v)} = \mu(E)$.

The example of divisorial valuations raises the question of how to characterize a valuation as divisorial. A theorem by Zariski shows that this can be done numerically in terms of transcendence degree and rational rank. The proof of this fact follows from Lemma 2.45 in the book *Birational Geometry of Algebraic Varieties* by J. Kollár and S. Mori.

Theorem 5.7. *Let v be a valuation of K. Then v is divisorial if and only if $\operatorname{tr.deg}(v) = n-1$ and $\operatorname{rat.rk}(v) = 1$.*

Construction 5.8. Let $v \in \operatorname{Val}_X$. Given a line bundle $L \in \operatorname{Pic}(X)$ we can make sense of $v(s)$ for sections $s \in \mathrm{H}^0(X, L)$:

In a neighborhood U of the center $\xi = c_X(v)$ we can trivialize L, i.e., fix an isomorphism $L|_U \cong U \times \mathbf{A}^1$ in which a local section $s \in \mathrm{H}^0(U, L|_U)$ is represented by a regular function $s : U \to \mathbf{A}^1$. In this way we can define $v(s)$ by evaluating this local representation, which is well-defined since two trivializations of L differ by a unit $a \in k^\times$, and therefore if s', s'' are two local representations of a section s we have $s' = as''$ for some $a \in k^\times$, and then $v(s') = v(as') = v(s'')$. Furthermore, $v(s) > 0$ if and only if $s(\xi) = 0$.

Similarly, we can evaluate v on a Cartier divisor D by considering the valuation of the local equation of D around $c_X(v)$.

Since the function field is a birational invariant, any test configuration $(\mathscr{X}, \mathscr{L})$ of a polarized pair (X, L) has function field $k(\mathscr{X}) \cong K(X)(t)$ since $\mathscr{X} \setminus \mathscr{X}_0 \cong X \times (\mathbf{A}^1 \setminus \{0\})$. Thus, it is natural to study the valuations of $K(X)(t)$.

Theorem 5.9 (Generalized Abhyankar's Inequality)**.** *Let $k \subset K' \subset K$ be field extensions. Then*

$$\operatorname{tr.deg}(v) + \operatorname{rat} \cdot \operatorname{rk}(v) \leq \operatorname{tr.deg}(v') + \operatorname{rat.rk}(v') + \operatorname{tr.deg} K/K'$$

where $v' = v|_{K'}$ is the restriction of the valuation v to K'.

Proposition 5.10. *Let v be a valuation of $K(X)(t)$. If v is divisorial, its restriction $r(v) = v|_{K(X)}$ to $K(X)$ is divisorial or trivial.*

Proof. Abhyankar's inequality implies that

$$\operatorname{tr.deg}(v) + \operatorname{rat.rk}(v) \leq \operatorname{tr.deg}(r(v)) + \operatorname{rat.rk}(r(v)) + 1 \leq n + 1$$

Since v is divisorial, in particular $\operatorname{tr.deg}(v) + \operatorname{rat.rk}(v) = n + 1$, so

$$\operatorname{tr.deg}(r(v)) + \operatorname{rat.rk}(r(v)) = n$$

It is clear that $\operatorname{rat.rk}(r(v)) \leq \operatorname{rat.rk}(v) = 1$, from which we conclude. □

Remark 5.11. *There is a natural action $\mathbf{G}_m \curvearrowright K(X)(t)$ given by*

$$a \cdot f = \sum_{\lambda \in \mathbf{Z}} a^{-\lambda} f_\lambda t^\lambda$$

where $f = \sum_{\lambda \in \mathbf{Z}} f_\lambda t^\lambda$ with $f_\lambda \in K(X)$.

Definition 5.12. A valuation of $K(X)(t)$ is $\mathbf{G}_m$-equivariant if $v(f) = v(a \cdot f)$ for every $f \in K(X)(t), a \in \mathbf{G}_m$. We denote by $\mathrm{Val}^{\mathbf{G}_m}_{X \times \mathbf{A}^1}$ the set of equivariant valuations of $X \times \mathbf{A}^1$.

Example 5.13. Let w be a valuation of $K(X)$ and $s \in \mathbf{R}^{\geq 0}$. We can define a valuation w_s of $K(X)(t)$ by

$$w_s(f) := \min\{w(f_\lambda) + \lambda s\}$$

where $f = \sum_{\lambda \in \mathbf{Z}} f_\lambda t^\lambda$.

1. This valuation is $\mathbf{G}_m$-equivariant, given that

$$w(a^{-\lambda} f_\lambda) = w(a^{-\lambda}) + w(f_\lambda) = w(f_\lambda) \quad \forall a \in \mathbf{G}_m$$

2. If w has a center in X then
$$c_{X \times \mathbf{A}^1}(w_s) = \begin{cases} c_X(w) \times 0 & \text{if } s > 0 \\ c_X(w) \times \mathbf{A}^1 & \text{if } s = 0 \end{cases}$$

Note that there is a bijection between the valuations of X and the $\mathbf{G}_m$-equivariant valuations of $X \times \mathbf{A}^1$, given explicitly by

$$\begin{aligned} \mathrm{Val}_X \times \mathbf{R} &\longleftrightarrow \mathrm{Val}^{\mathbf{G}_m}_{X \times \mathbf{A}^1} \\ (w, s) &\longmapsto w_s \\ (v|_{K(X)}, v(t)) &\longleftarrow\!\shortmid v \end{aligned}$$

Let $(\mathscr{X}, \mathscr{L})$ be a normal test configuration of (X, L). We have a birational map $\mathscr{X} \to X \times \mathbf{A}^1$, and defining $\mathscr{Y}$ as the normalization of the graph of this map, we have a diagram

$$\begin{array}{ccc} & \mathscr{Y} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathscr{X} & \dashrightarrow & X \times \mathbf{A}^1 \end{array}$$

and given $E \subset \mathscr{X}_0$ an irreducible component of $\mathscr{X}_0$, this induces a divisorial valuation ord_E of the field $K(X)(t)$, whose restriction to $K(X)$ will be denoted $v_E = r(\mathrm{ord}_E)$.

Proposition 5.14. *For $m > 0$ sufficiently large*

$$F^\lambda_{\mathscr{X},\mathscr{L}} \mathrm{H}^0(X, mL) = \bigcap_{E \subset \mathscr{X}_0} \left\{ s \in \mathrm{H}^0(X, mL) \mid v_E(s) + m \,\mathrm{ord}_E(D) \geq \lambda \,\mathrm{ord}_E(t) \right\}$$

where D denotes the $\mathbf{Q}$-divisor on $\mathscr{Y}$ supported on $\mathscr{Y}_0$ such that $f^*\mathscr{L} \cong g^*(L \times \mathbf{A}^1) + D$.

Proof. Recall that the filtration of $\mathrm{H}^0(X, mL)$ defined earlier corresponds by definition to

$$F^\lambda_{\mathscr{X},\mathscr{L}} \mathrm{H}^0(X, mL) = \left\{ s \in \mathrm{H}^0(X, mL) \mid t^{-\lambda} \bar{s} \in \mathrm{H}^0(\mathscr{X}, m\mathscr{L}) \right\}$$

where $\bar{s} \in \mathrm{H}^0(\mathscr{X} \backslash \mathscr{X}_0, m\mathscr{L})$ denotes the $\mathbf{G}_m$-invariant section such that its restriction to $t = 1$ is $\bar{s}_1 = s$. At the same time, it defines a rational section of $L_{\mathbf{A}^1} = L \times \mathbf{A}^1$, which we denote $\bar{s}_{m\mathscr{L}}$ and $\bar{s}_{mL_{\mathbf{A}^1}}$. Now, since X is normal, $\bar{s}t^{-\lambda} \in \mathrm{H}^0(\mathscr{X}, \mathscr{L})$ if and only if $\mathrm{ord}_E(\bar{s}t^{-\lambda}) \geq 0$ for all E irreducible components of $\mathscr{X}_0$. We calculate that

$$\begin{aligned} \mathrm{ord}_E\left(\bar{s}t^{-\lambda}_{m\mathscr{L}}\right) = \mathrm{ord}_E\left(\bar{s}_{m\mathscr{L}}\right) - \lambda \,\mathrm{ord}_E(t) &= \mathrm{ord}_E(\bar{s}_{f^*m\mathscr{L}}) - \lambda \,\mathrm{ord}_E(t) \\ &= \mathrm{ord}_E(\bar{s}_{g^*mL_{\mathbf{A}^1}(D)}) - \lambda \,\mathrm{ord}_E(t) \\ &= \mathrm{ord}_E(\bar{s}_{g^*mL_{\mathbf{A}^1}}) + m \,\mathrm{ord}_E(D) - \lambda \,\mathrm{ord}_E(t) \\ &\overset{\text{def}}{=} v_E(s) + m \,\mathrm{ord}_E(D) - \lambda \,\mathrm{ord}_E(t) \end{aligned}$$

□

6 Numerical Criteria for K-Stability

Let (X,L) be a polarized normal projective variety of $\dim(X)=n$, and let $(\mathscr{X},\mathscr{L})\xrightarrow{\pi}\mathbf{A}^1$ be a test configuration.

Recall 6.1. We saw that an irreducible component $F\subseteq\mathscr{X}_0$ defines a $\mathbf{G}_m$-equivariant divisorial valuation $\operatorname{ord}_E\in\operatorname{Val}^{\mathbf{G}_m}_{X\times\mathbf{A}^1}$, and we defined $v_F:=\operatorname{ord}_F|_{K(X)}$. Additionally, by normalizing the graph of the natural rational map between $\mathscr{X}$ and the trivial test configuration, we obtain a diagram

$$\begin{array}{ccc} & \mathscr{Y} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathscr{X} & \dashrightarrow & X\times\mathbf{A}^1 \end{array} \qquad \text{where } f^*\mathscr{L}\simeq g^*L_{\mathbf{A}^1}+D \text{ with } \operatorname{Supp}(D)\subseteq\mathscr{Y}_0.$$

Thus, $F^{\lambda}_{\mathscr{X},\mathscr{L}}\operatorname{H}^0(X,mL)=\bigcap_{E\subset\mathscr{X}_0}\{s\in\operatorname{H}^0(X,mL),\ \operatorname{ord}_F(\overline{s}t^{-\lambda})\overset{\text{def}}{=}v_F(s)+\overbrace{m\operatorname{ord}_F(D)}^{=\lambda_1\text{ fixed}}-\overbrace{\lambda\operatorname{ord}_F(t)}^{=\lambda_2\text{ fixed}}\geq 0\}$.

Definition 6.2. Let $v=\operatorname{ord}_E:K(X)^*\to\mathbf{Z}$ be a divisorial valuation induced by a prime divisor $E\subseteq Y\xrightarrow{\mu}X$. Then, v filters the algebra $R:=R(X,L)\overset{\text{def}}{=}\bigoplus_{m\geq 0}\operatorname{H}^0(X,mL)$ by defining

$$F^{\lambda}_v\operatorname{H}^0(X,mL):=\{s\in\operatorname{H}^0(X,mL),\ v(s)\geq\lambda\}.$$

Warning. The numerical characterization of Zariski divisorial valuations **does not** only consider surjective valuations. More precisely, if $v:K(X)^*\to\mathbf{Z}$ is divisorial and $\operatorname{Im}(v)=c\mathbf{Z}$ with $c\in\mathbf{N}^{\geq 1}$ then $v=c\operatorname{ord}_E$. Thus, in the previous definition, $v(s)\geq\lambda$ if and only if $\operatorname{ord}_E(s)\geq\lceil\frac{\lambda}{c}\rceil$.

Definition 6.3 (K. Fujita)**.** We say that $v=c\operatorname{ord}_E$ (or that the divisor E) is **dreamy** if $F^{\bullet}_vR(X,L)$ is finitely generated, i.e., the Rees algebra $\operatorname{Rees}(F^{\bullet}_vR)\cong\bigoplus_{m\in\mathbf{N},\,\lambda\in\mathbf{Z}}\operatorname{H}^0(Y,m\mu^*L-\lceil\frac{\lambda}{c}\rceil E)$ is finitely generated.

Example 6.4 (BCHM, 2010)**.** If Y is **log-Fano** (i.e., there exists $\Delta_Y\geq 0$ effective with coefficients ≤ 1 such that (Y,Δ_Y) is klt[7] and $-(K_Y+\Delta_Y)$ is ample), then every divisor $E\subseteq Y$ is dreamy (Y is a **Mori Dream Space**).

Theorem 6.5. *Let (X,L) with X Fano klt and $L=-K_X$. Then, there is a bijection between:*

1. *Normal test configurations $(\mathscr{X},\mathscr{L})$ of (X,L) with $\mathscr{L}=-K_{\mathscr{X}/\mathbf{A}^1}$ and $\mathscr{X}_0$* **reduced and irreducible***.*
2. *$v:K(X)^*\to\mathbf{Z}$ dreamy divisorial valuation.*

Proof. (1) $\mapsto$ (2) is given by $\mathscr{X}_0\mapsto v_{\mathscr{X}_0}\overset{\text{def}}{=}\operatorname{ord}_{\mathscr{X}_0}|_{K(X)}$. Here, $\mathscr{L}\cong -K_{\mathscr{X}/\mathbf{A}^1}$ and the filtration induced in $R=R(X,L)$ is given by $F^{\lambda}_{\mathscr{X},\mathscr{L}}\operatorname{H}^0(X,mL)=\{s\in\operatorname{H}^0(X,mL),\ v_{\mathscr{X}_0}(s)+m\operatorname{ord}_{\mathscr{X}_0}(D)\geq\lambda\operatorname{ord}_{\mathscr{X}_0}(t)\}$ with $\operatorname{ord}_{\mathscr{X}_0}(D):=-A$, $\operatorname{ord}_{\mathscr{X}_0}(t)\overset{\text{def}}{=}1$, i.e., $F^{\lambda}_{\mathscr{X},\mathscr{L}}\operatorname{H}^0(X,mL)=F^{\lambda+mA}_{v_{\mathscr{X}_0}}\operatorname{H}^0(X,mL)$ and $R_{(\mathscr{X},\mathscr{L})}:=\operatorname{Rees}(F^{\bullet}_{\mathscr{X},\mathscr{L}}R)\cong\operatorname{Rees}(F^{\bullet}_{v_{\mathscr{X}_0}}R)$ as $k[t]$-algebras. Since $R_{(\mathscr{X},\mathscr{L})}$ is finitely generated, $v_{\mathscr{X}_0}$ is a dreamy valuation.

(2) $\mapsto$ (1) is given by $v\mapsto\mathscr{X}:=\operatorname{Proj}_{\mathbf{A}^1}(\operatorname{Rees}(F^{\bullet}_vR))$. Here, $\mathscr{X}_0$ is given by the Proj of the algebra

$$\operatorname{Rees}(F^{\bullet}_vR)\otimes_{k[t]}k[t]/\langle t\rangle\cong\frac{\operatorname{Rees}(F^{\bullet}_vR)}{t\cdot\operatorname{Rees}(F^{\bullet}_vR)}\overset{\text{def}}{=}\bigoplus_{m\in\mathbf{N},\,\lambda\in\mathbf{Z}}\frac{F^{\lambda}_v\operatorname{H}^0(X,mL)}{F^{\lambda+1}_v\operatorname{H}^0(X,mL)}\overset{\text{def}}{=}\bigoplus_{m\in\mathbf{N},\,\lambda\in\mathbf{Z}}\operatorname{gr}^{\lambda}_{F_v}\operatorname{H}^0(X,mL).$$

Given that if $\overline{s},\overline{t}\neq 0$ have degrees λ and μ, respectively, then $\overline{s}\overline{t}$ is nonzero of degree $\lambda+\mu$, we deduce that $\mathscr{X}_0$ is irreducible and reduced. In particular, since X is normal, $\mathscr{X}$ is irreducible and normal. Finally, the previous construction implies that $(\mathscr{X},\mathscr{L})$ satisfies $v_{\mathscr{X}_0}=v$. ☐

Warning. In the previous context:

1. Since $\mathscr{L}=-K_{\mathscr{X}/\mathbf{A}^1}$ and $K_{\mathbf{A}^1}=0$ then $D\overset{\text{def}}{=}-g^*(L_{\mathbf{A}^1})+f^*\mathscr{L}\overset{\text{def}}{=}g^*(K_{X\times\mathbf{A}^1})-f^*(K_{\mathscr{X}})\pm K_{\mathscr{Y}}\overset{\text{def}}{=}K_{\mathscr{Y}/\mathscr{X}}-K_{\mathscr{Y}/(X\times\mathbf{A}^1)}$. Then, if $\widetilde{\mathscr{X}_0}=f^*\mathscr{X}_0$ it follows that by definition of (log-)discrepancy

$$-A\overset{\text{def}}{=}\operatorname{ord}_{\mathscr{X}_0}(D)=\overbrace{\operatorname{coeff}_{\widetilde{\mathscr{X}_0}}(K_{\mathscr{Y}/\mathscr{X}})}^{\overset{\text{def}}{=}0}-\operatorname{coeff}_{\widetilde{\mathscr{X}_0}}(K_{\mathscr{Y}/(X\times\mathbf{A}^1)})\overset{\text{def}}{=}-(A_{X\times\mathbf{A}^1}(\widetilde{\mathscr{X}_0})-1)=1-(cA_X(E)+\overbrace{\operatorname{ord}_{\mathscr{X}_0}(t)}^{\overset{\text{def}}{=}1}),$$

   and thus $A=cA_X(E)\overset{\text{def}}{=}A_X(v_{\mathscr{X}_0})$, where $v_{\mathscr{X}_0}=c\operatorname{ord}_E$ is a divisorial valuation on X induced by $\mathscr{X}_0$.

2. Li and Xu (2014) proved that it suffices to check K-stability of Fano varieties by considering **special test configurations** $(\mathscr{X},\mathscr{L})$, i.e., those with $\mathscr{L}=-K_{\mathscr{X}/\mathbf{A}^1}$ and $\mathscr{X}_0$ a klt Fano variety.

[7] i.e., the pair (Y,Δ_Y) is dlt.

Definition 6.6 (β-invariant)**.** Let X be a klt Fano variety and $r \in \mathbf{N}^{\geq 1}$ such that $-rK_X$ is Cartier. For $v := c\,\mathrm{ord}_E : K(X)^* \to \mathbf{Z}$ a divisorial valuation, we define the invariant $\beta(v) := A_X(v) - S_X(v)$, where

$$S_X(v) = \limsup_{m\to+\infty} \frac{\sum_{\lambda\in\mathbf{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{F_v} \mathrm{H}^0(X, -mrK_X)}{m \dim \mathrm{H}^0(X, -mrK_X)}$$

and where $A_X(v) = cA_X(E)$, with $A_X(E)$ the log-discrepancy of the divisor $E \subseteq Y \xrightarrow{\mu} X$.

Proposition 6.7. *Let $v = c\,\mathrm{ord}_E$ (i.e., $v = v_{\mathscr{X}_0} \overset{\text{def}}{=} \mathrm{ord}_{\mathscr{X}_0}|_{K(X)}$) a dreamy valuation, and $(\mathscr{X}, \mathscr{L})$ the associated test configuration, with $\mathscr{X}_0$ reduced and irreducible. Then,* $\mathrm{DF}(\mathscr{X},\mathscr{L}) = A_X(v) - S_X(v) = c(A_X(E) - S_X(E))$.

Proof. Let $(\overline{\mathscr{X}}, \overline{\mathscr{L}}) \xrightarrow{\overline{\pi}} \mathbf{P}^1$ be the associated projective test configuration. Considering $L = -K_X$ and $\overline{\mathscr{L}} = -K_{\overline{\mathscr{X}}/\mathbf{P}^1}$ the formula for the Donaldson-Futaki invariant using $w_m/mN_m = F_0 + F_1 m^{-1} + \cdots$ reduces to

$$\mathrm{DF}(\mathscr{X},\mathscr{L}) \overset{\text{def}}{=} -2F_1 = -\frac{1}{(n+1)(-K_X)^n}(-K_{\overline{\mathscr{X}}/\mathbf{P}^1})^{n+1} \overset{\text{def}}{=} -\frac{b_0}{a_0} \overset{\text{def}}{=} -F_0.$$

And the term F_0 is simply calculated by observing that if $v = v_{\mathscr{X}_0} = c\,\mathrm{ord}_E$ then

$$w_m \overset{\text{def}}{=} \mathrm{wt}\,\mathrm{H}^0(\mathscr{X}_0, -mK_{\mathscr{X}/\mathbf{A}^1}|_{\mathscr{X}_0}) \overset{\text{def}}{=} \sum_{\lambda\in\mathbf{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{F_{\mathscr{X},\mathscr{L}}} \mathrm{H}^0(X, -mK_X) = \sum_{\lambda\in\mathbf{Z}} \lambda \dim \mathrm{gr}^{\lambda+mA}_{F_v} \mathrm{H}^0(X, -mK_X)$$

$$= \sum_{\lambda\in\mathbf{Z}} (\lambda - mA) \dim \mathrm{gr}^{\lambda}_{F_v} \mathrm{H}^0(X, -mK_X) = -mA_X(v) \underbrace{\dim \mathrm{H}^0(X, -mK_X)}_{\overset{\text{def}}{=} N_m} + \sum_{\lambda\in\mathbf{Z}} \lambda \dim \mathrm{gr}^{\lambda}_{F_v} \mathrm{H}^0(X, -mK_X)$$

and then $-F_0 = -\limsup_{m\to+\infty} \frac{w_m}{mN_m} = A_X(v) - S_X(v)$. □

Theorem 6.8. *If X is a klt Fano variety of* $\dim(X) = n$ *and $E \subseteq Y \xrightarrow{\mu} X$ is a prime divisor over X then*

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^{\tau} \mathrm{vol}(-\mu^* K_X - tE)\,\mathrm{d}t \text{ where } \tau = \sup\{t \in \mathbf{R}^{\geq 0},\ -\mu^*K_X - tE \text{ big divisor}\}.$$

Proof. Let $v = \mathrm{ord}_E$ and assume that $-K_X$ is Cartier (to avoid writing $-rK_X$ throughout the proof). If $v_\lambda := \dim F^{\lambda}_v \mathrm{H}^0(X, -mK_X)$, we obtain a telescoping sum that calculates $\sum_{\lambda\in\mathbf{Z}} \lambda\, \mathrm{gr}^{\lambda}_{F_v} \mathrm{H}^0(X, -mK_X)$

$$\sum_{\lambda\in\mathbf{Z}} \lambda(v_\lambda - v_{\lambda+1}) = \sum_{\lambda=0}^{+\infty} v_\lambda \overset{\text{def}}{=} \sum_{\lambda=1}^{+\infty} h^0(Y, -m\mu^*K_X - \lambda E) \overset{\text{def}}{=} \int_0^{+\infty} h^0(Y, -m\mu^*K_X - \lceil t \rceil E)\,\mathrm{d}t,$$

and thus $\sum_{\lambda\in\mathbf{Z}} \lambda\, \mathrm{gr}^{\lambda}_{F_v} \mathrm{H}^0(X, -mK_X) = m\int_0^{+\infty} h^0(Y, -m\mu^*K_X - \lceil mt \rceil E)\,\mathrm{d}t$. Then, $S_X(E)$ is given by

$$\limsup_{m\to+\infty} \int_0^{+\infty} \frac{h^0(Y, -m\mu^*K_X - \lceil mt\rceil E)/(m^n/n!)}{h^0(X, -mK_X)/(m^n/n!)}\,\mathrm{d}t = \int_0^{+\infty} \frac{\mathrm{vol}(-\mu^*K_X - tE)}{\mathrm{vol}(-K_X)}\,\mathrm{d}t \overset{\text{def}}{=} \int_0^{\tau} \frac{\mathrm{vol}(-\mu^*K_X - tE)}{(-K_X)^n}\,\mathrm{d}t$$

by the Dominated Convergence Theorem. □

The above can be summarized in the following fundamental result[8], by Chi Li (2017) and Kento Fujita (2019).

Theorem 6.9 (Valuative Criterion for K-Stability)**.** *Let X be a klt Fano variety. Then, X is*

K-stable (resp. K-semistable) $\Leftrightarrow$ $\beta_X(E) > 0$ (resp ≥ 0) for every (dreamy) divisor E over X.

Example 6.10. Let $X := \mathrm{Bl}_p(\mathbf{P}^2) \xrightarrow{\varepsilon} \mathbf{P}^2$ with exceptional divisor $E \subseteq X$ and let L be the pullback of a line. Then, $K_X = \varepsilon^* K_{\mathbf{P}^2} + E = -3L + E$ and we then calculate $S_X(E)$ as

$$S_X(E) = \frac{1}{(-K_X)^2}\int_0^{+\infty} \mathrm{vol}(-K_X - tE)\,\mathrm{d}t = \frac{1}{8}\int_0^{\tau} \mathrm{vol}(3L - E - tE)\,\mathrm{d}\,t = \int_0^2 (9 - (1+t)^2)\,\mathrm{d}t = \frac{7}{6}.$$

Since E and X are smooth, $A_X(E) = 1$ and thus $\beta_X(E) = 1 - \frac{7}{6} = -\frac{1}{6} < 0$. Hence, X **is not** K-semistable (and therefore not K-polystable either) and thus $\mathrm{Bl}_p(\mathbf{P}^2)$ **does not admit** Kähler-Einstein metrics.

Example 6.11 (K. Fujita, 2015)**.** Let X be a K-semistable klt Fano variety and let $p \in X$ be a smooth point. Let $\varepsilon : Y := \mathrm{Bl}_p(X) \to X$ with exceptional divisor $E \subseteq Y$, where $A_X(E) = (n-1)+1 = n$ and where it holds[9] that $\mathrm{vol}_Y(\varepsilon^*(-K_X) - tE) \geq (-K_X)^n - t^n$ and thus $\beta_X(E) = A_X(E) - S_X(E) = n - S_X(E) \geq 0$ is equivalent to

$$n \geq \frac{1}{(-K_X)^n}\int_0^{\infty} \mathrm{vol}_Y(\varepsilon^*(-K_X) - tE)\,\mathrm{d}t \geq \frac{1}{(-K_X)^n}\int_0^{\sqrt[n]{(-K_X)^n}} ((-K_X)^n - t^n)\,\mathrm{d}t = \frac{n}{n+1}\sqrt[n]{(-K_X)^n}$$

and thus we have that X satisfies the inequality $(-K_X)^n \leq (n+1)^n$.

Remark 6.12. *The inequality $(-K_X)^n \leq (n+1)^n$ is true for every smooth Fano variety of* $\dim(X) = n \leq 3$, *but $X = \mathbf{P}(\mathscr{O}_{\mathbf{P}^{n-1}} \oplus \mathscr{O}_{\mathbf{P}^{n-1}}(n))$ does not satisfy it for $n \geq 4$ (and thus is not K-semistable). Furthermore, using results by Yuchen Liu and Ziquan Zhuang (2018), it can be proved that in the K-semistable case the equality $(-K_X)^n = (n+1)^n$ is equivalent to $X \cong \mathbf{P}^n$.*

[8] Our calculations, along with the Li-Xu Theorem (2014), allow proving it for dreamy divisors. Moreover, Blum, Liu, Xu, and Zhou proved in 2019 that if $\beta_X(E) < 0$ for an arbitrary divisor E, then this inequality holds for a dreamy divisor.

[9] It suffices to compare dimensions in the exact sequence $0 \to \mathrm{H}^0(X, \mathscr{O}_X(-mK_X)\cdot \mathfrak{m}_p^{mt}) \to \mathrm{H}^0(X, -mK_X) \to \mathscr{O}_X/\mathfrak{m}_p^{mt} \to 0$.

7 Invariants α and δ

In this section, two different techniques for proving K-stability will be illustrated, for which the invariants α and δ will be introduced.

Construction 7.1 (Fujita–Odaka 2016, Blum–Jonsson 2020)**.** The valuative criterion for K-stability states that a klt Fano variety is K-stable (resp. K-semistable) $\Leftrightarrow \delta(X) > 1$ (resp. ≥ 1), where:

$$\delta(X) := \inf_{E \subseteq Y \xrightarrow{\mu} X} \frac{A_X(E)}{S_X(E)},$$

i.e., the infimum is taken over all divisors over X. The **δ-invariant** was originally defined by Fujita-Okada as a certain limit of **log-canonical thresholds** of *m-basis type divisors* and then Blum-Jonsson proved that it coincided with the previous expression, and additionally it satisfies:

$$\frac{n+1}{n}\alpha(X) \leq \delta(X) \leq (n+1)\alpha(X),$$

where $n = \dim(X)$ and where

$$\alpha(X) = \inf\{\operatorname{lct}(X, D),\ 0 \leq D \sim_{\mathbf{Q}} -K_X\}$$

is the **α-invariant** of Tian, where

$$\operatorname{lct}(X, D) = \sup\{c \in \mathbf{R}^{\geq 0},\ (X, cD) \text{ is lc}\}.$$

Theorem 7.2 (Tian, 1987)**.** *Let X be a klt Fano variety of dimension $n = \dim(X)$. If*

$$\alpha(X) > (\geq)\ \frac{n}{n+1}$$

then X is K-(semi)stable.

Example 7.3. We will use Tian's criterion to prove that a degree 1 del Pezzo surface X is K-stable. Recall that $X \cong \mathrm{Bl}_{p_1,\ldots,p_8}(\mathbf{P}^2)$ is the blow-up of $\mathbf{P}^2$ at 8 points in general position[10]. Denoting $\varepsilon : X \to \mathbf{P}^2$ the blow-up, the canonical divisor corresponds to:

$$-K_X = \varepsilon^*(-K_{\mathbf{P}^2}) - \sum_{i=1}^{8} E_i$$

where E_i are the exceptional divisors. Thus, the linear system $|-K_X|$ corresponds to the linear system of (strict transforms of) cubics passing through $p_1, \ldots, p_8$.

Consider $D \sim_{\mathbf{Q}} -K_X$ (i.e., there exists $m \in \mathbf{N}$ such that $mD \sim (-K_X)$ are linearly equivalent) and note that D is reduced (i.e., it can only have multiplicities of 1). Indeed, if $\operatorname{Supp}(D) \in |-K_X|$ this is directly true since $D \sim \operatorname{Supp}(D)$ (considering $\operatorname{Supp}(D)$ as a cycle). If $\operatorname{Supp}(D) \notin |-K_X|$, since $-K_X$ defines a pencil, for any $x \in D$ there exists $C \in |-K_X|$ with $x \in D$. Then

$$D \cdot C = (-K_X)^2 = 1$$

and therefore D must be reduced. This fact implies that it suffices to compute the lct when D is a curve. The condition of passing through $p_1, \ldots, p_8$ implies that D is irreducible, and then the possibilities are reduced to:

$$D \text{ smooth: } \operatorname{lct}(X, D) \overset{\text{def}}{=} \sup\{c \in \mathbf{R}^{\geq 0},\ 1 - \operatorname{ord}_E(cD) \geq 0\} = 1$$

$$D \text{ nodal: } \operatorname{lct}(X, D) = 1$$

$$D \text{ cusp: } \operatorname{lct}(X, D) = \frac{5}{6}$$

The case of the cusp is obtained by resolving the pair $(\mathbf{A}^2, D)$ where $D = \{y^2 - x^3 = 0\} \subset \mathbf{A}^2$. If $f : \widetilde{X} \to X$ is this resolution (which corresponds to 3 blow-ups) then

$$f^*(cD) - K_{\widetilde{X}/X} = c\widetilde{D} + (2c-1)E_1 + (3c-2)E_2 + (6c-4)E_3$$

and the lct is obtained from the condition $6c - 4 \leq 1$. In any case, $\alpha(X) > 2/3$ and therefore X is K-stable.

Remark 7.4. *Cheltsov (2008) shows that $\alpha(X) \geq 2/3$ for every del Pezzo surface of degree ≤ 4. Fujita (2019) shows that $\alpha(X) = \frac{n}{n+1}$ implies K-stability for* smooth *Fano varieties.*

Using the language of filtrations, valuations, and Newton-Okounkov bodies, Abban and Zhuang (2022) prove one of the most currently used methods to estimate the δ invariant via *adjunction*. The first observation is that the valuative criterion allows extending the definition of K-stability to log Fano pairs.

[10]This means that there are no 3 collinear points, no 6 points lying on a conic, and there is no nodal or cuspidal cubic passing through the 8 points such that one of them is exactly the singular point.

Definition 7.5. Given a log Fano pair (X, D) of dimension $n = \dim(X)$ and a divisor $E \subset Y \xrightarrow{\mu} X$ over X, we define

$$\delta_{(X,D)}(E) = \frac{A_{(X,D)}(E)}{S_{(X,D)}(E)}$$

where

$$S_{(X,D)}(E) = \frac{1}{(-K_X - D)^n} \int_0^\infty \operatorname{vol}(\mu^*(-K_X - D) - tE)\, \mathrm{d}t$$

We say that (X, D) is K-stable (resp. K-semistable) if

$$\delta(X, D; V_\bullet) = \inf\{\delta_{(X,D)}(E), E \subset Y \xrightarrow{\mu} X \text{ divisor over } X\} > 1 \ (\geq 1).$$

Remark 7.6. *If X is an algebraic variety and $L \in \operatorname{Pic}(X)$ is ample, $V_\bullet := \{V_m = \mathrm{H}^0(X, mL)\}_{m\geq 0}$ is the associated linear series, and if $E \subset Y \to X$, the filtration $(\mathscr{F}_E V_m)_t := \{s \in V_m, \operatorname{ord}_E(s) \geq mt\}$ is defined and*

$$\operatorname{vol}(\mathscr{F}_E V_m)_t = \lim_{m\to+\infty} \frac{\dim((\mathscr{F}_E V_m)_t)}{m^n/n!}$$

Then

$$S(V_\bullet, E) := \frac{1}{\operatorname{vol}(V_\bullet)} \int_0^{+\infty} \operatorname{vol}(\mathscr{F}_E V_m)_t \,\mathrm{d}t = S_{(X,D)}(E),$$

considering $L = -K_X - D$.

Construction 7.7 (Abban-Zhuang, 2022)**.** Let (X, Δ) be a klt pair with $\Delta \geq 0$, and let $E \subseteq Y \xrightarrow{\mu} X$ be a divisor over X of **plt type**, i.e., $-E$ is μ-ample and $(Y, \Delta_Y + E)$ is a plt pair[11], where Δ_Y is defined by the condition

$$K_Y + \Delta_Y = \mu^*(K_X + \Delta) + (A_{(X,\Delta)}(E) - 1)E.$$

If Δ_E is the **different** of Δ_Y on E (that is, $K_E + \Delta_E = (K_Y + \Delta_Y + E)|_E$) then

$$\delta_Z(X, \Delta; V_\bullet) = \inf_{F,\ Z\subseteq c_X(F)} \frac{A_{(X,\Delta)}(F)}{S(V_\bullet, F)} \ \text{ verifies } \ \delta_Z(X, \Delta; V_\bullet) \geq \min\left\{\frac{A_{(X,\Delta)}(E)}{S(V_\bullet, E)}, \inf_{Z'} \delta_{Z'}(E, \Delta_E; \mathbf{W}^E_{\bullet,\bullet})\right\}$$

with $Z' \subset Y$ ranging over the subvarieties of Y such that $\mu(Z') = Z$, and where

$$\delta_{Z'}(E, \Delta_E; \mathbf{W}^E_{\bullet,\bullet}) = \inf_{F,\ Z'\subseteq c_E(F)} \frac{A_{(E,\Delta_E)}(F)}{S(\mathbf{W}^E_{\bullet,\bullet}; F)}.$$

The term $S(\mathbf{W}^E_{\bullet,\bullet}; F)$ is obtained analogously to $S(V_\bullet, E)$ but considering the refinement

$$W^E_{m,j} := \operatorname{Im}(\mathrm{H}^0(Y, -m(K_X + \Delta) - jE) \to \mathrm{H}^0(E, -m(K_E + \Delta_E) - jE|_E)).$$

In practice, this volume can be calculated or estimated using the notion of *restricted volume* defined by Lazarsfeld and collaborators, which in turn is shown to be calculable using *slices* of Newton-Okounkov bodies. The latter, in the case of surfaces, is calculated using the *Zariski decomposition.*

Remark 7.8. *By definition, $\delta(X, D; V_\bullet) = \inf_{Z\subset X} \delta_Z(X, D; V_\bullet)$. In particular, the condition $\delta_p(X, D; V_\bullet) \geq 1$ for every $p \in X$ implies that X is K-semistable.*

From now on, X will be a **surface** and $E \subset Y \to X$ will be a smooth curve[12] fixed on X. In this case, Z' (which is a subvariety in E) will be a point $Z' = p$ such that $p \in \overline{c_E(F)}$, i.e., $p = F$. We need to calculate

$$\delta_p(E, D_E, W^E_{\bullet,\bullet}) = \frac{A_{(E,D_E)}(p)}{S(W^E_{\bullet,\bullet}; p)} = \frac{1 - \operatorname{ord}_p(D_E)}{S(W^E_{\bullet,\bullet}; p)}$$

where we have used that E is smooth. Let $\tau = \sup\{u \in \mathbf{R}^{\geq 0}, \mu^*(-K_X - D) - uE \text{ is pseudo-effective}\}$, and consider the Zariski decomposition

$$\mu^*(-K_X - D) - uE = \underbrace{P(u)}_{\text{nef}} + \underbrace{N(u)}_{\text{negative}} .$$

We will assume that $\operatorname{Supp}(E) \not\subseteq N(u)$ for every u (for simplicity). In such a case, we have a *flag* $\{p\} \subset E \subset Y$, whose *Newton-Okounkov body* allows calculating the volume of the divisor[13]:

$$S(W^E_{\bullet,\bullet}; p) = \frac{\dim(X)}{\operatorname{vol}(L)} \int_0^\tau \int_0^{+\infty} \operatorname{vol}(P(u)|_E - vp)\, \mathrm{d}v\, \mathrm{d}u = \frac{2}{(-K_X - D)^2} \int_0^\tau \int_0^{t(u)} \max\{\operatorname{ord}_p(P(u)|_E) - v, 0\}\, \mathrm{d}v\, \mathrm{d}u$$

[11]Recall that (X, Δ) is plt (resp. klt) if for $A_{X,\Delta}(E) > 0$ (resp. $A_{X,\Delta}(E) > 0$ and $\lfloor \Delta \rfloor \leq 0$) for every divisor E over X.

[12]It suffices to consider smooth curves due to the plt hypothesis.

[13]See Corollary 1.109 in *The Calabi Problem for Fano Threefolds, Araújo et al., 2023.*

Example 7.9. Using the Abban-Zhuang method, we will prove that every cubic surface is K-semistable. Let X be a cubic surface, $p \in X$ and $E \in |-K_X|$ an elliptic curve (smooth) such that $p \in E$ and $E|_E = 3p$. Here, $D = 0$ and due to smoothness $A_X(E) = 1$. We calculate

$$S_X(E) = \frac{1}{(-K_X)^2} \int_0^{+\infty} \mathrm{vol}(-K_X - tE)\,\mathrm{d}t = \frac{1}{(-K_X)^2} \int_0^1 (-K_X)^2(1-t)^2\,\mathrm{d}t = \frac{1}{3}.$$

Now note that

$$-K_X - uE \sim (1-u)(-K_X) \text{ is nef} \iff (1-u)(-K_X) \text{ is pseudo-effective} \iff 0 \le u \le 1.$$

In this case $P(u) = (1-u)(-K_X)$ and $N(u) = 0$, and so $P(u)|_E = 3(1-u)p, \mathrm{ord}_p(P(u)|_E) = 3(1-u)$. Then,

$$S(W^E_{\bullet,\bullet}; p) = \frac{2}{\mathrm{vol}(L)} \int_0^{\tau} \int_0^{t(u)} \max\{\mathrm{ord}_p(P(u)|_E) - v, 0\}\,\mathrm{d}v\,\mathrm{d}u = \frac{2}{3} \int_0^1 \int_0^{3(1-u)} (3(1-u) - v)\,\mathrm{d}v\,\mathrm{d}u = 1.$$

We calculate that $\delta_p(E, D_E, W^E_{\bullet,\bullet}) = 1$, and then

$$\delta_p(X; V_\bullet) \ge \min\left\{ \frac{A_X(E)}{S_X(E)}, \delta_p(E, \underbrace{\Delta_E}_{=0}; W^E_{\bullet,\bullet}) \right\} = \min\{3, 1\} = 1.$$

The previous calculation concludes that X is K-semistable.

Remark 7.10. *In fact, Abban-Zhuang verify that $\delta(X) \ge 3/2$, and every cubic surface is K-stable.*